Lesson 8: Measurement error (Part 1)

Goals

- Estimate a measurement and determine the largest possible percentage error of the estimate.
- Generalise (orally) a process for calculating the maximum percentage error of measurements that are added together.

Lesson Narrative

This lesson is optional. The activities in this lesson and the next all address the concept of measurement error. Any of these activities can stand on its own, although activities in the next lesson are more challenging, and students would likely benefit from doing the earlier ones first. Note that the first activity in this lesson gives students an opportunity to make measurements and analyse the size of the error. The second activity provides the percentage error in the measurement in an addition context. In addition to examining accuracy of measurements carefully, students will work through examples and look for patterns in order to hypothesise, and eventually show, how percentage error behaves when measurements with error are added to one another.

As with all lessons in this unit, all related topics have been addressed in prior units. This lesson provides an optional opportunity to go deeper and make connections between domains.

Addressing

- Solve real-life and mathematical problems using numerical and algebraic expressions and equations.
- Use proportional relationships to solve multistep ratio and percentage age problems. Examples: simple interest, tax, markups and markdowns, gratuities and commissions, fees, percentage age increase and decrease, percentage age error.

Instructional Routines

- Discussion Supports
- Think Pair Share

Required Materials

Four-function calculators

Required Preparation

Since the calculations for each of these tasks are involved, students will need access to calculators.

Student Learning Goals

Let's check how accurate our measurements are.

8.1 How Long Are These Pencils?

Optional: 20 minutes

In this activity, students measure lengths and determine possibilities for actual lengths. There are two layers of attending to precision involved in this task:

- Deciding how accurately the pencils can be measured, probably to the nearest mm or to the nearest 2 mm, but this depends on the eyesight and confidence of the student
- Finding the possible percentage error in the measurement chosen

Instructional Routines

- Discussion Supports
- Think Pair Share

Launch

Arrange students in groups of 2. Provide access to calculators. Give students 4–5 minutes of quiet work time, followed by partner and whole-class discussion.

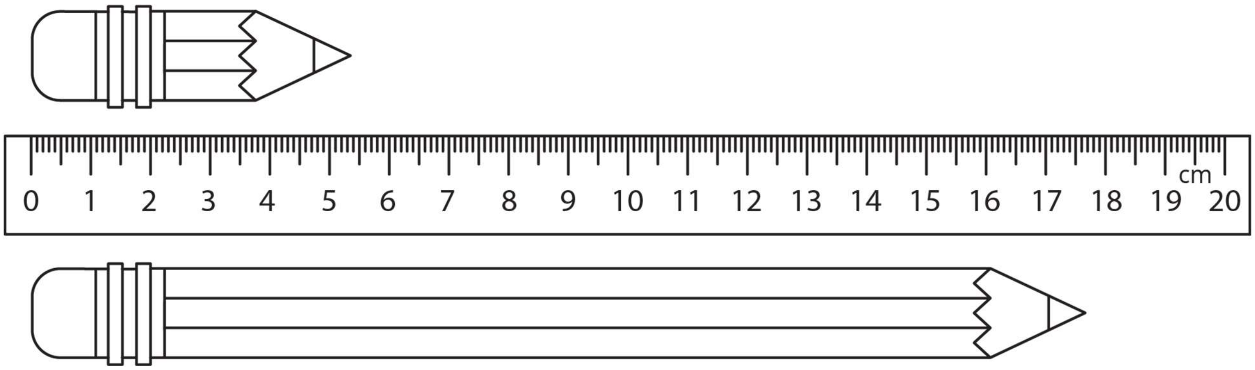
Writing, Speaking: Discussion Supports. To support students as they respond to "How accurate are your estimates?", provide a sentence frame such as: "My estimate is within ___ mm of the actual length because . . . ." Encourage students to consider what details are important to share and to think about how they will explain their reasoning using mathematical language. This will help students use mathematical language as they justify the accuracy of their estimates.
Design Principle(s): Optimise output (for explanation)

Anticipated Misconceptions

Some students may think that they can find an exact value for the length of each pencil. Because the pictures of the pencils are far enough away from the ruler, it requires a lot of care just to identify the "nearest" millimetre (or which two millimetre markings the length lies between). Prompt them to consider the error in their measurements by asking questions like, "Assuming you measured the pencil accurately to the nearest millimetre, what is the longest the actual length of the pencil could be? What is the shortest it could be?"

Some students may not remember how to calculate percentage error. Ask them, "What is the biggest difference possible between the estimated and actual lengths? What percentage age of the actual length would that be when the difference is as big as possible?"

Student Task Statement



1. Estimate the length of each pencil.

2. How accurate are your estimates?

3. For each estimate, what is the largest possible percentage error?

Student Response

1. Answers vary. Sample response: The shorter pencil appears to be between 5.3 and 5.5 cm, perhaps 5.4 to the nearest mm, while the longer pencil appears to be 17.7 cm to the nearest mm (it is between 17.6 and 17.8 cm).

2. Answers vary. Sample response: The estimate is accurate to within 1 mm. For the short pencil, it is more than 5.3 cm and less than 5.5 cm, but it is not possible to tell which is closer. Similarly, the longer pencil is more than 17.6 cm and less than 17.8 cm.

3. For the shorter pencil, taking 5.4 cm as the measured length, the actual length x is at least 5.3 cm and at most 5.5 cm. The percentage error if x is as small as possible is $\frac{0.1}{5.3} \approx 2\%$, and if x is as big as possible, then the error is $\frac{0.1}{5.5} \approx 2\%$. The first of these gives the greatest percentage error, although they are close. For the longer pencil, the percentage error is smaller. The biggest it can be is $\frac{0.1}{17.7} \approx 0.6\%$. This makes sense because 0.1 cm is a bigger percentage age of the length of the small pencil.

Activity Synthesis

The goal of this discussion is for students to practise how they talk about precision.

Discussion questions include:

- “How did you decide how accurately you can measure the pencils?” (I looked for a value that I was certain was less than the length of the pencil and a value that I was certain was bigger. My estimate was halfway in between.)

- “Were you sure which mm measurement the length is closest to?” (Answers vary. Possible responses: Yes, I could tell that the short pencil is closest to 5.4 cm. No, the long pencil looks to be closest to 17.7 mm, but I'm not sure. I am sure it is between 17.6 cm and 17.8 cm.)

- "Were the percentage errors the same for the small pencil and for the long pencil? Why or why not?" (No. I was able to measure each pencil to within 1 mm. This is a *smaller* percentage age of the longer pencil length than it is of the smaller pencil length.)

Other possible topics of conversation include noting that the level of accuracy of a measurement depends on the measuring device. If the ruler were marked in sixteenths of an inch, we would only be able to measure to the nearest sixteenth of an inch. If it were only marked in cm, we would only be able to measure to the nearest cm.

8.2 How Long Are These Floor Boards?

Optional: 20 minutes

This activity examines how measurement errors behave when they are added together. In other words, if I have a measurement m with a maximum error of 1% and a measurement n with a maximum error of 1%, what percentage age error can $m + n$ have? In addition to examining accuracy of measurements carefully, students work through examples and look for patterns in order to hypothesise, and eventually show, how percentage error behaves when measurements with error are added to one another.

Monitor for students who look for patterns, recognise the usefulness of the distributive property, or formulate the problem abstractly with variables.

Launch

Read the problem out loud and ask students what information they would need to know to be able to solve the problem. Students may say that they need to know what length the boards are supposed to be, because it is likely that they haven't realised that they can solve the problem without this information. Explain that floor boards come in many possible lengths, that 18-inch and 36-inch lengths are both common, but the boards can be anywhere between 12 and 84 inches. Ask students to pick values for two actual lengths and figure out the error in that case. Then they can pick two different examples, make the calculations again, and look for patterns.

Provide access to calculators.

Representation: Internalise Comprehension. Represent the same information through different modalities by using diagrams. If students are unsure where to begin, suggest that they draw a diagram to help illustrate the information provided.
Supports accessibility for: Conceptual processing; Visual-spatial processing

Anticipated Misconceptions

Some students may pick some example lengths but then struggle with knowing what to do with them. Ask them “what would be the maximum measured lengths? The minimum? What would be the error if both measurements were maximum? What if they were both minimum?”

Some students may pick numbers that make the calculations more complicated, leading to arithmetic errors. Suggest that they choose simple, round numbers for lengths, like 50 inches or 100 centimetres.

Student Task Statement

A wood floor is made by laying multiple boards end to end. Each board is measured with a maximum percentage error of 5%. What is the maximum percentage error for the total length of the floor?

Student Response

The maximum percentage error is 5%. Sample explanation: If x is the actual length and m is the measured length of one board, then $0.95x < m$ and $m < 1.05x$: I know this because the measurement m has a maximum error of 5%. If y is the actual length and n is the measured length of a second board, then $0.95y < n$ and $n < 1.05y$. If both boards have maximum length, the total length would be $1.05x + 1.05y = 1.05(x + y)$. If they are both minimum, the total length would be $0.95x + 0.95y = 0.95(x + y)$. So the maximum percentage error would be 5%. This same argument works for any number of boards because the distributive property works for any number of addends.

Activity Synthesis

The goal of this discussion is for students to generalise from their specific examples of measurements to understand the general pattern and express it algebraically.

Poll the class on the measurements they tried and the maximum percentage error they calculated. Invite students to share any patterns they noticed, especially students who recognised the usefulness of the distributive property for making sense of the general pattern.

Guide students to use variables to talk about the patterns more generally.

- If a board is supposed to have length x with a maximum percentage error of 5%, then the shortest it could be is $0.95x$ and the longest it could be is $1.05x$.
- If another board is supposed to have length y, it could be between $0.95y$ and $1.05y$.
- When the boards are laid end-to-end, the shortest the total length could be is $0.95x + 0.95y$, which is equivalent to $0.95(x + y)$.
- The longest the total length could be is $1.05x + 1.05y$, or $1.05(x + y)$.

- Because of the distributive property, we can see that the maximum percentage error is still 5% after the board lengths are added together.

One interesting point to make, if students have also done the previous activity about measuring pencils, is that you could *measure* the sum of the board lengths with a lower percentage error than you could measure each individual board (assuming your tape measure is long enough), just like an error of 1 mm was a smaller percentage age of the length of the longer pencil.