Lesson 13: Expressions with rational numbers

Goals

- Evaluate an expression for given values of the variable, including negative values, and compare (orally) the resulting values of the expression.
- Generalise (orally) about the relationship between additive inverses and about the relationship between multiplicative inverses.
- Identify numerical expressions that are equal, and justify (orally) that they are equal.

Learning Targets

- I can add, subtract, multiply, and divide rational numbers.
- I can evaluate expressions that involve rational numbers.

Lesson Narrative

As students start to gain fluency with rational number arithmetic, they encounter complicated numerical expressions, and algebraic expressions with variables, and there is a danger that they might lose the connection between those expressions and numbers on the number line. The purpose of this lesson is to help students make sense of expressions, and reason about their position on the number line, for example whether the number is positive or negative, which of two numbers is larger, or whether two expressions represent the same number. They work through common misconceptions that can arise about expressions involving variables, for example the misconception that $-x$ must always be a negative number. (It is positive if x is negative.) In the last activity they reason about expressions in a and b given the positions of a and b on a number line without a given scale, in order to develop the idea that you can always think of the letters in an algebraic expression as numbers and deduce, for example, that $\frac{1}{4}a$ is a quarter of the way from 0 to a on the number line, even if you don't know the value of a.

When students look at a numerical expression and see without calculation that it must be positive because it is a product of two negative numbers, they are making use of structure.

Addressing

- Apply and extend previous understandings of operations with fractions to add, subtract, multiply, and divide rational numbers.
- Solve real-world and mathematical problems involving the four operations with rational numbers. Calculations with rational numbers extend the rules for manipulating fractions to complex fractions.

Instructional Routines

- Collect and Display

- Compare and Connect
- Discussion Supports
- Notice and Wonder
- Take Turns
- True or False

Required Materials

Pre-printed slips, cut from copies of the blackline master

Card Sort: The Same But Different $1+2$	Card Sort: The Same But Different $1-(-2)$	Card Sort: The Same But Different $(1)\times(4)$	Card Sort: The Same But Different $1\div\frac{1}{4}$
Card Sort: The Same But Different $1-2$	Card Sort: The Same But Different $1+-2$	Card Sort: The Same But Different -1×4	Card Sort: The Same But Different $1\div\left(-\frac{1}{4}\right)$
Card Sort: The Same But Different $-10+7$	Card Sort: The Same But Different $-10--7$	Card Sort: The Same But Different $8\div 4$	Card Sort: The Same But Different $(8)\left(\frac{1}{4}\right)$
Card Sort: The Same But Different $-10+(-7)$	Card Sort: The Same But Different $-10-7$	Card Sort: The Same But Different $8\div(-4)$	Card Sort: The Same But Different $(8)\times(-\frac{1}{4})$
Card Sort: The Same But Different $-15\div -6$	Card Sort: The Same But Different $15\times\frac{1}{6}$	Card Sort: The Same But Different $15\div(-6)$	Card Sort: The Same But Different $-15\times\frac{1}{6}$

Required Preparation

Print and cut up slips from the Card Sort: The Same But Different blackline master. Prepare 1 copy for every 2 students. If possible, copy each complete set on a different colour of paper, so that a stray slip can quickly be put back.

Student Learning Goals

Let's develop our directed number sense.

13.1 True or False: Rational Numbers

Warm Up: 5 minutes

The purpose of this warm-up is for students to reason about numeric expressions using what they know about operations with negative and positive numbers without actually calculating anything.

Instructional Routines

- True or False

Launch

Explain to students that we sometimes leave out the symbol for multiplication that tells us that two numbers are being multiplied. So when a number, or a number and a variable, are next to each other without a symbol between them, that means we are finding their product. For example, (-5) × (-10) can be written (-5)(-10) and -5 × x can be written -5x.

Display one problem at a time. Give students 1 minute of quiet think time per problem and ask them to give a signal when they have an answer. Follow with a whole-class discussion.

Student Task Statement

Decide if each statement is true or false. Be prepared to explain your reasoning.

1. (-38.76)(-15.6) is negative
2. $10\,000 - 99\,999 < 0$
3. $\left(\frac{3}{4}\right)\left(-\frac{4}{3}\right) = 0$
4. $(30)(-80) - 50 = 50 - (30)(-80)$

Student Response

1. false
2. true
3. false
4. false

Activity Synthesis

Ask students to share their reasoning.

13.2 Card Sort: The Same But Different

10 minutes

In this activity students continue to build fluency operating with directed numbers as they match different expressions that have the same value. Students look for and use the relationship between inverse operations.

As students work, identify groups that make connections between the operations, for example they notice that subtracting a number is the same as adding its opposite, or that dividing is the same as multiplying by the multiplicative inverse.

Instructional Routines

- Compare and Connect
- Take Turns

Launch

Arrange students in groups of 2. Distribute pre-cut slips from the blackline master.

Anticipated Misconceptions

If students struggle to find matches, encourage them to think of the operations in different ways. You might ask:

- How else can you think of ____ (subtraction, addition, multiplication, division)?

Student Task Statement

Your teacher will give you a set of cards. Group them into pairs of expressions that have the same value.

Student Response

answer	one expression	other expression
-17	$-10 + (-7)$	$-10 - 7$
-4	-1×4	$1 \div (-\frac{1}{4})$
-3	$-10 + 7$	$-10 - -7$
-2.5	$15 \div (-6)$	$-15 \times \frac{1}{6}$
-2	$8 \div (-4)$	$8 \times (-\frac{1}{4})$
-1	$1 - 2$	$1 + -2$
2	$8 \div 4$	$(8)(\frac{1}{4})$

2.5	$15 \times \frac{1}{6}$	-15 ÷ -6
3	$1 + 2$	$1 - (-2)$
4	$(1)(4)$	$1 \div \frac{1}{4}$

Activity Synthesis

Select students to share their strategies. Highlight strategies that compared the structure of the expressions instead of just the final answers.

Discuss:

- Subtracting a number is equivalent to adding the additive inverse.
- A number and its additive inverse have opposite signs (but the same magnitude).
- Dividing by a number is equivalent to multiplying by the multiplicative inverse.
- A number and its multiplicative inverse have the same sign (but different magnitudes).

If desired, have students order their pairs of equivalent expressions from least to greatest to review ordering rational numbers.

Engagement: Develop Effort and Persistence. Break the class into small discussion groups and then invite a representative from each group to report back to the whole class. *Supports accessibility for: Attention; Social-emotional skills Speaking, Representing: Compare and Connect.* Use this routine to support whole-class discussion. Ask students to consider what is the same and what is different about the structure of each expression. Draw students' attention to the association between quantities (e.g., adding the additive inverse is subtracting; multiplying by the multiplicative inverse is dividing). These exchanges strengthen students' mathematical language use and reasoning to describe when comparing inverse-operation relationships involving rational numbers.
Design Principle(s): Support sense-making; Maximise meta-awareness

13.3 Near and Far From Zero

15 minutes

In the previous activity, students interpreted the meaning of $-x$ when x represented a positive value and when x represented a negative value. The purpose of this activity is to understand that variables can have negative values, but if we compare two expressions containing the same variable, it is not possible to know which expression is larger or smaller (without knowing the values of the variables). For example, if we know that a is positive, then we know that $5a$ is greater than $4a$; however, if a can be any rational number, then it is possible for $4a$ to be greater than $5a$, or equal to $5a$.

Instructional Routines

- Discussion Supports

Launch

a, b,-a,-4b,-$a + b$, $a \div$ -b, a^2, b^3

Display the list of expressions for all to see. Ask students: Which expression do you think has the largest value? Which has the smallest? Which is closest to zero? It is reasonable to guess that -4b is the smallest and b^3 is the largest. (This guess depends on supposing that b is positive. But this activity is designed to elicit the understanding that it is necessary to know the value of variables to be able to compare expressions. For the launch, you are just looking for students to register their initial suspicions.)

If you know from students' previous work that they struggle with basic operations on rational numbers, it might be helpful to demonstrate a few of the computations that will come up as they work on this activity. Tell students, "Say that a represents 10, and b represents -2. What would be the value of . . .

- -b
- b^3
- $a \times \frac{1}{b}$
- $\frac{a}{b} \div a$
- $a + \frac{1}{b}$
- $\left(\frac{1}{b}\right)^2$

Arrange students in groups of 2. Encourage them to check in with their partner as they evaluate each expression, and work together to resolve any discrepancies.

Action and Expression: Internalise Executive Functions. Chunk this task into more manageable parts to support students who benefit from support with organisational skills in problem solving. For example, present one question at a time and monitor students to ensure they are making progress throughout the activity.
Supports accessibility for: Organisation; Attention

Anticipated Misconceptions

If students struggle to find the largest value, smallest value, or value closest to zero in the set, encourage them to create a number line to help them reason about the positions of different candidates.

Student Task Statement

a	b	$-a$	$-4b$	$-a+b$	$a \div -b$	a^2	b^3
$-\frac{1}{2}$	6						
$\frac{1}{2}$	-6						
-6	$-\frac{1}{2}$						

1. For each set of values for a and b, evaluate the given expressions and record your answers in the table.

2. When $a = -\frac{1}{2}$ and $b = 6$, which expression:

   has the largest value?

   has the smallest value?

   is the closest to zero?

3. When $a = \frac{1}{2}$ and $b = -6$, which expression:

   has the largest value?

   has the smallest value?

   is the closest to zero?

4. When $a = -6$ and $b = -\frac{1}{2}$, which expression:

   has the largest value?

   has the smallest value?

   is the closest to zero?

Student Response

1.

a	b	$-a$	$-4b$	$-a+b$	$a \div -b$	a^2	b^3
$-\frac{1}{2}$	6	$\frac{1}{2}$	-24	$6\frac{1}{2}$	$\frac{1}{12}$	$\frac{1}{4}$	216
$\frac{1}{2}$	-6	$-\frac{1}{2}$	24	$-6\frac{1}{2}$	$\frac{1}{12}$	$\frac{1}{4}$	-216
-6	$-\frac{1}{2}$	6	2	$5\frac{1}{2}$	-12	36	$-\frac{1}{8}$

2.

a. $b^3 = 216$

b. $-4b = -24$

c. $a \div -b = \frac{1}{12}$

3.

a. $-4b = 24$

b. $b^3 = -216$

c. $a \div -b = \frac{1}{12}$

4.

a. $a^2 = 36$

b. $a \div -b = -12$

c. $b^3 = -\frac{1}{8}$

Are You Ready for More?

Are there any values could you use for a and b that would make all of these expressions have the same value? Explain your reasoning.

Student Response

No. If $a = 0$ and $b = 0$, then all of the expressions will also have a value of 0, except for $a \div (-b)$, because $0 \div 0$ is undefined.

Activity Synthesis

Display the completed table for all to see. Ask selected students to share their values that are largest, smallest, and closest to zero from each set and explain their reasoning.

Ask students if any of these results were surprising? What caused the surprising result? Some possible observations are:

- b^3 was both the largest and smallest value at different times. 6 and -6 are both relatively far from 0, so b^3 is a large number when b is positive and a small number when b is negative.
- $a \div -b$ was often the closest to zero, because it had the same absolute value no matter the sign of a and b.

Speaking, Representing: Discussion Supports. Use this routine to support whole-class discussion. After each student shares, provide the class with the following sentence frames to help them respond: "I agree because ….” or "I disagree because ….” If necessary, revoice student ideas to demonstrate mathematical language use by restating a statement as a question in order to clarify, apply appropriate language, and involve more students. *Design Principle(s): Support sense-making*

13.4 Seagulls and Sharks Again

Optional: 10 minutes (there is a digital version of this activity)

The purpose of this activity is for students to interpret an expression in terms of the position it represents on a number line and to interpret the meaning of an associated equation. Expressions are equal when they represent the same position on a number line. This activity uses a familiar context that students have encountered before, so that they can more quickly engage with the meaning of the expressions. In this activity, students use the structure of the number line to reason about the relative values of expressions.

Instructional Routines

- Collect and Display
- Notice and Wonder

Launch

Display the diagram for all to see. Ask students to share anything they notice and wonder about the diagram. Some things to notice are that

- The seagull is above the water and the shark is below the water.
- a and b represent the vertical position of the seagull and shark, respectively.
- a represents a positive value and b represents a negative value.
- a is farther from 0 than b, so $|a| > |b|$.

To facilitate engagement with the task, consider demonstrating the placement of one animal before students start working. For example, ask students, “If there were a minnow with vertical position m, and $m = \frac{1}{3}b$, where is the minnow?” Help students interpret the equation. Explain that the value of m only gives the vertical position. The *horizontal* position of the minnow is unknown. Encourage students to plot all of the new points *on* the vertical axis (as specified in the task statement).

Keep students in the same groups as the previous activity. Encourage students to check in with their partner periodically, and work together to resolve any discrepancies.

Classes using the digital version have an applet to use. Students can choose their own vertical positions for a and b. Teachers may allow students to turn on a grid to facilitate the placement of the other animals.

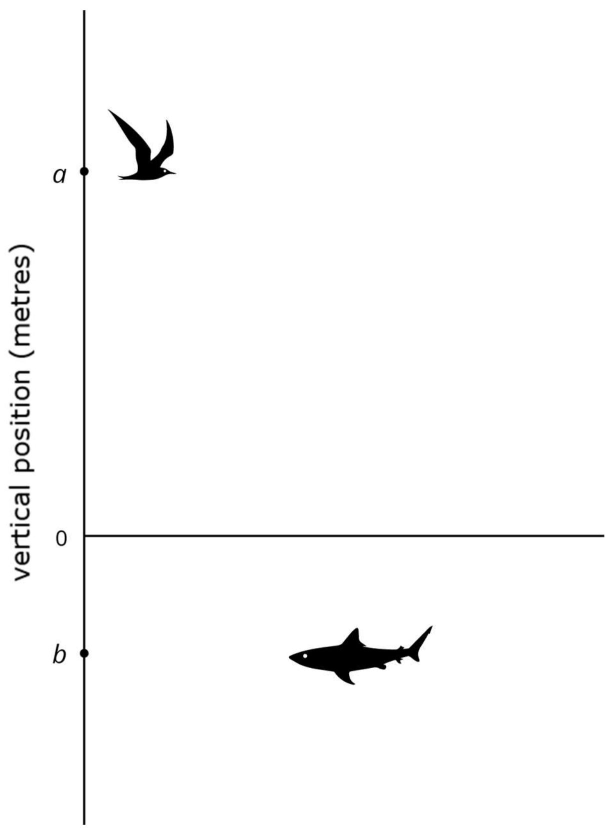
Conversing, Representing, Writing: Collect and Display. As students work, circulate and listen to students' conversations as they decide where on the vertical axis to show the position of each new animal. Write down common or important phrases you hear students say as they reason about the relative value of each expression (e.g., twice as far, opposite direction, half the distance, etc.). Write the students' words and expressions on a whole-class display of the task's diagram. This will help students interpret and use mathematical language during their paired and whole-group discussions.

Design Principle(s): Support sense-making

Anticipated Misconceptions

For students who are struggling to measure out a length of a or b or a sum, difference, or multiple of them, suggest that they measure and cut strips of paper for the lengths of a and b to help guide them. Ask how they could use the strips to find other distances such as $a - b$ and $\frac{a}{2}$.

Student Task Statement



A seagull has a vertical position a, and a shark has a vertical position b. Draw and label a point on the vertical axis to show the vertical position of each new animal.

1. A dragonfly at d, where $d = -b$

2. A jellyfish at j, where $j = 2b$

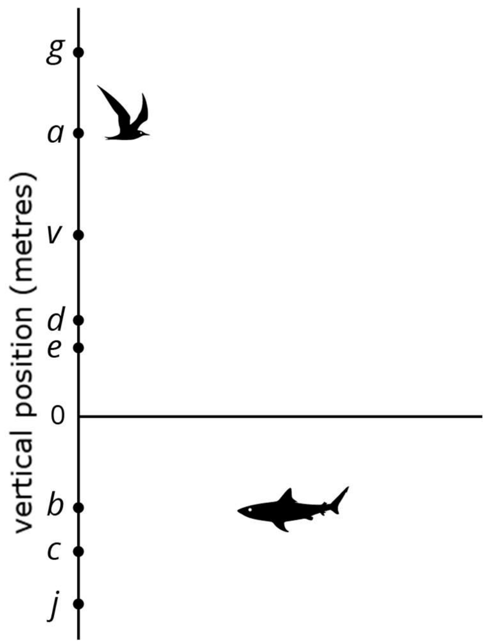
3. An eagle at e, where $e = \frac{1}{4}a$.

4. A clownfish at c, where $c = \frac{-a}{2}$

5. A vulture at v, where $v = a + b$

6. A goose at g, where $g = a - b$

Student Response



Activity Synthesis

Display the diagram from the task statement. Ask selected students to share their responses and the reasoning behind them, and record them on the diagram. As the discussion proceeds, illustrate the meaning of the equal sign by saying, for example, "We could either label *this point* with v, or we could label it with $a + b$, since we know $v = a + b$. Since these expressions are equal, they represent the same position on the number line."

If not brought up by students, consider drawing attention to the equation $c = \frac{-a}{2}$ and discuss how this relates to what they have learned about dividing directed numbers. It is important for students to understand that $\frac{-a}{2} = -\frac{a}{2}$ as well as $\frac{a}{-2}$, but these are not equal to $\frac{-a}{-2}$.

Lesson Synthesis

Key takeaways:

- Develop flexible thinking with all four operations across the rational numbers.
- Recognise multiplicative and additive inverses and the difference between them.

Discussion questions:

- Explain the rules for arithmetic with negative numbers.
- Can you give an example of a number whose additive inverse is the same as its multiplicative inverse? Why not?

13.5 Make Them True

Cool Down: 5 minutes

Student Task Statement

In each equation, select an operation to make the equation true.

1. $24 \_\_ \frac{3}{4} = 18$
2. $24 \_\_ \text{-}\frac{3}{4} = \text{-}32$
3. $12 \_\_ 15 = \text{-}3$
4. $12 \_\_ \text{-}15 = 27$
5. $\text{-}18 \_\_ \text{-}\frac{3}{4} = 24$

Student Response

1. $24 \times \frac{3}{4} = 18$
2. $24 \div \text{-}\frac{3}{4} = \text{-}32$
3. $12 - 15 = \text{-}3$
4. $12 - \text{-}15 = 27$
5. $\text{-}18 \div \text{-}\frac{3}{4} = 24$

Student Lesson Summary

We can represent sums, differences, products, and quotients of **rational numbers**, and combinations of these, with numerical and algebraic expressions.

Sums:

$\frac{1}{2} + \text{-}9$

$\text{-}8.5 + x$

Differences:

$\frac{1}{2} - -9$

$-8.5 - x$

Products:

$\left(\frac{1}{2}\right)(-9)$

$-8.5x$

Quotients:

$\frac{1}{2} \div -9$

$\frac{-8.5}{x}$

We can write the product of two numbers in different ways.

- By putting a multiplication symbol between the factors, like this: $-8.5 \times x$.
- By putting the factors next to each other without any symbol between them at all, like this: $-8.5x$.

We can write the quotient of two numbers in different ways as well.

- By writing the division symbol between the numbers, like this: $-8.5 \div x$.
- By writing a fraction bar between the numbers like this: $\frac{-8.5}{x}$.

When we have an algebraic expression like $\frac{-8.5}{x}$ and are given a value for the variable, we can find the value of the expression. For example, if x is 2, then the value of the expression is -4.25, because $-8.5 \div 2 = -4.25$.

Glossary

- rational number

Lesson 13 Practice Problems

1. **Problem 1 Statement**

   The value of x is $\frac{-1}{4}$. Order these expressions from least to greatest:

   x

   $1 - x$

   $x - 1$

$-1 \div x$

Solution

$x - 1, x, 1 - x, -1 \div x$.

The expressions' values are $\frac{-5}{4}, \frac{-1}{4}, \frac{5}{4}$, and 4.

2. **Problem 2 Statement**

Here are four expressions that have the value $\frac{-1}{2}$:

$\frac{-1}{4} + \left(\frac{-1}{4}\right)$

$\frac{1}{2} - 1$

$-2 \times \frac{1}{4}$

$-1 \div 2$

Write five expressions: a sum, a difference, a product, a quotient, and one that involves at least two operations that have the value $\frac{-3}{4}$.

Solution

Answers vary. Sample response: $\frac{-1}{4} + (\frac{-1}{2}), \frac{1}{4} - 1, -3 \times \frac{1}{4}, -3 \div 4, 1 \div 4 - 1$

3. **Problem 3 Statement**

Find the value of each expression.

a. $-22 + 5$

b. $-22 - (-5)$

c. $(-22)(-5)$

d. $-22 \div 5$

Solution

a. -17

b. -17

c. 110

d. -4.4

4. **Problem 4 Statement**

The price of an ice cream cone is £3.25, but it costs £3.51 with sugar tax. What is the sugar tax rate?

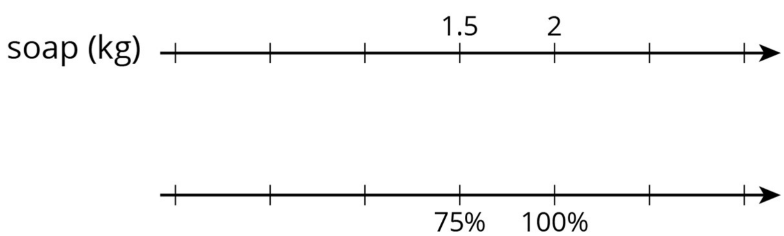
Solution

a. 8% (Any answer between 7.85% and 8.15% is acceptable.)

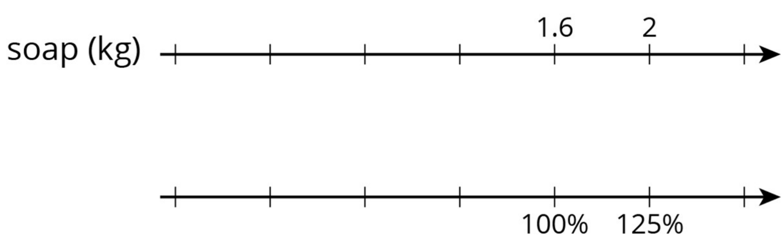
5. **Problem 5 Statement**

Two students are both working on the same problem: A box of laundry soap has 25% more soap in its new box. The new box holds 2 kg. How much soap did the old box hold?

– Here is how Jada set up her double number line.



– Here is how Lin set up her double number line.



Do you agree with either of them? Explain or show your reasoning.

Solution

Answers vary. Sample response: I agree with Lin. The soap in the old box represents 100% and the new box now holds 125% which is 2 kg.

6. **Problem 6 Statement**

a. A coffee maker's directions say to use 2 tablespoons of ground coffee for every 6 ounces of water. How much coffee should you use for 33 ounces of water?

b. A runner is running a 10 km race. It takes her 17.5 minutes to reach the 2.5 km mark. At that rate, how long will it take her to run the whole race?

Solution

a. 11 tablespoons; from the information, one can determine that 1 tablespoon of coffee will be used for every 3 ounces of water, therefore for 33 ounces of water one would need 11 tablespoons of coffee.

b. 70 minutes; from the information, one can determine that it would take the runner 7 minutes to run 1 km, therefore it would take the runner 70 minutes to run 10 km.

© These materials were derived and adapted from Illustrative Mathematics's IM 6–8 Math™. IM 6–8 Math was originally developed by Open Up Resources and authored by Illustrative Mathematics®, and is copyright 2017–2019 by Open Up Resources. It is licensed under the Creative Commons Attribution 4.0 International License (CC BY 4.0) https://creativecommons.org/licenses/by/4.0/. OUR's 6–8 Math Curriculum is available at https://openupresources.org/math-curriculum/. Adaptations and updates to IM 6–8 Math™ are copyright 2019 by Illustrative Mathematics® , and are licensed under the Creative Commons Attribution 4.0 International License (CC BY 4.0). Further adaptations have been made by MEI.