Lesson 12: Solutions to linear equations

Goals

- Comprehend that the points that lie on the graph of an equation represent exactly the solution set of the equation of the line (i.e., that every point on the line is a solution, and any point not on the line is not a solution).
- Create a graph and an equation in the form $Ax + By = C$ that represent a linear relationship.
- Determine pairs of values that satisfy or do not satisfy a linear relationship using an equation or graph.

Learning Targets

- I know that the graph of an equation is a visual representation of all the solutions to the equation.
- I understand what the solution to an equation in two variables is.

Lesson Narrative

The goal of this lesson and the next is to start getting students to think about linear equations in two variables in a different way in preparation for their work on systems of linear equations in a future unit. Until now, students have mostly been working with contexts where one variable depends on another, for example, distance depending on time. The linear equation representing such a situation is often written in the form $y = mx + c$. In this lesson, they look at contexts where both variables have to satisfy a constraint, and a natural way to write the constraint is with an equation of the form $Ax + By = C$. For example, the first activity gets students to think about different ways of spending a fixed sum of money on two differently priced items, and the second activity gets them to write an equation expressing a numerical constraint on two numbers (twice the first number plus the second number adds up to 10).

Pairs of numbers that make the equation true are **solutions to the equation (with two variables)**; they are the coordinates of points that lie on the graph. Students also consider pairs of numbers that do not make the equation true, and notice that they do not lie on the graph. This insight is developed in the next lesson, where students look for points that are simultaneously the solution to two different equations.

Building On

- Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems.

Addressing

- Understand the connections between proportional relationships, lines, and linear equations.
- Analyse and solve linear equations and pairs of simultaneous linear equations.

Building Towards

- Analyse and solve linear equations and pairs of simultaneous linear equations.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Stronger and Clearer Each Time
- Three Reads
- Think Pair Share

Required Materials

Geometry toolkits

tracing paper, graph paper, coloured pencils, scissors, and an index card to use as a straightedge or to mark right angles, a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Student Learning Goals

Let's think about what it means to be a solution to a linear equation with two variables in it.

12.1 Estimate Area

Warm Up: 5 minutes

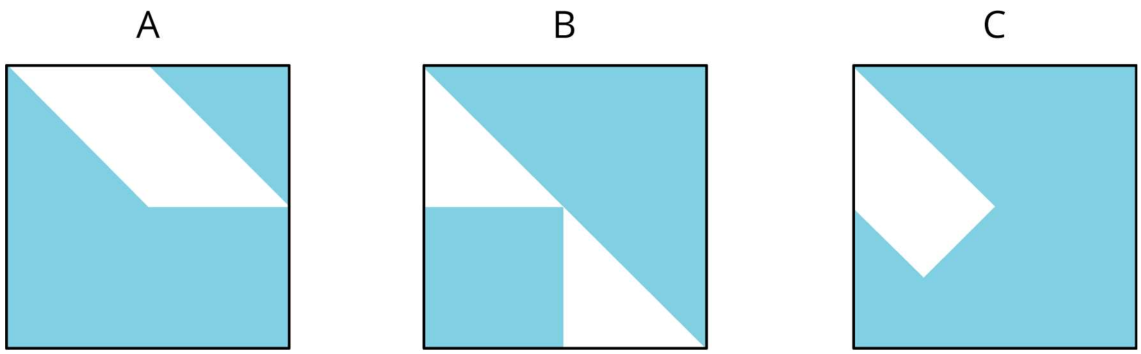
This warm-up prompts students to reason about the area of triangles and quadrilaterals. If there is time after sharing the estimates and reasoning, ask the students for the information they would need to get an exact answer to this question and how they would use that information.

Launch

Ensure students know what is meant by the shaded region. Display the figures. Give students 1 minute of quiet think time and ask them to give a signal when they have an answer and reasoning. Encourage students to have a reasoning to defend their choice beyond, "It looks like more." Follow with a whole-class discussion.

Student Task Statement

Which figure has the largest shaded region?



Student Response

Figure C. Figures A and B have the same amount shaded, $\frac{3}{4}$ of the whole square, and Figure C has $\frac{3}{4}+\frac{1}{16}$ or $\frac{13}{16}$ of the whole square shaded. These calculations assume that all distances that appear equal are equal (which can be verified by folding).

Activity Synthesis

Invite students to share how they visualised the shaded region of each figure. Record and display their explanations for all to see. Solicit from the class alternative ways of quantifying the shaded portion and alternative ways of naming the size of the shaded portion (to elicit representations as fractions, decimals, or percentages). To involve more students in the conversation, consider asking:

- “Who can restate ___’s reasoning in a different way?”
- “Did anyone solve the shaded portion the same way but would explain it differently?”
- “Did anyone solve the shaded portion in a different way?”
- “Does anyone want to add on to _____’s strategy?”
- “Do you agree or disagree? Why?”

12.2 Apples and Oranges

15 minutes

In previous lessons in this unit, students have analysed multiple situations that lead to equations of the form $y=mx+c$, including positive, negative, and 0 gradient. In the previous lesson, they saw that vertical lines cannot be described by equations of this form and saw a geometric situation that could be represented by an equation of the form $Ax+By=C$. This form of a linear equation is examined in greater detail here as students consider combinations of numbers that keep a total cost constant and write an equation that will be used again in the next activity. In this problem, the solutions are limited to non-negative integers and the set of all solutions is finite. In the next activity, the solutions comprise all real numbers and the set of solutions is infinite.

There are many equations students could use to describe the cost of apples in pounds, a, and the cost of oranges in pounds, r. Identify students who write $a + 2r = 10$, and ask them to share during the discussion. There is no reason to solve for one variable in terms of the other, in this case, because a graph has not been requested and neither variable is a "natural" candidate for dependent versus independent. In this activity, students need to explain their reasoning.

Instructional Routines

- Three Reads
- Think Pair Share

Launch

Arrange students in groups of 2. Give students 3–5 minutes of quiet think time to answer the first question and think about the others. Have partners compare solutions and discuss the remaining questions. Follow with a whole-class discussion.

Engagement: Develop Effort and Persistence. Encourage and support opportunities for peer interactions. Invite students to talk about their ideas with a partner before writing them down. Display sentence frames to support students when they explain their strategy. For example, "First I ____ because . . ." or "That could/couldn't be true because . . ."
Supports accessibility for: Language; Social-emotional skills Reading: Three Reads. To support reading comprehension for students, use this routine with the corner produce market problem. In the first read, students read the problem with the goal of comprehending the situation (e.g., finding the cost of different combinations of apples and oranges, determining how many apples and oranges can be bought with £10, and writing an equation). In the second read, ask students to look for quantities that can be used or measured (e.g., apples cost £1, oranges cost £2, Noah has £10). In the third read, ask students to brainstorm possible strategies to answer the question, "What combinations of apples and oranges can Noah buy if he spends all his £10? Use two variables to write an equation that represents £10-combinations of apples and oranges." This will help students make sense of the problem by looking for patterns and strategies to come up with an equation before being asked to do so.
Design Principle(s): Support sense-making; Maximise meta-awareness

Anticipated Misconceptions

Students may write an equation like $a + r = 10$ where a is the *cost* of the apples and r is the *cost* of the oranges. While this is technically correct, it leaves extra work figuring out whether or not a solution to this corresponds to some actual number of apples and oranges. Ask these students what they are trying to figure out (numbers of apples and numbers of oranges that cost £10). Then suggest that they use their variables to represent the number of apples and the number of oranges rather than their cost.

Student Task Statement

At the corner produce market, apples cost £1 each and oranges cost £2 each.

1. Find the cost of:
    a. 6 apples and 3 oranges
    b. 4 apples and 4 oranges
    c. 5 apples and 4 oranges
    d. 8 apples and 2 oranges
2. Noah has £10 to spend at the produce market. Can he buy 7 apples and 2 oranges? Explain or show your reasoning.
3. What combinations of apples and oranges can Noah buy if he spends all of his £10?
4. Use two variables to write an equation that represents £10-combinations of apples and oranges. Be sure to say what each variable means.
5. What are 3 combinations of apples and oranges that make your equation true? What are three combinations of apples and oranges that make it false?

Student Response

1. £12, £12, £13, £12.
2. No. 7 apples and 2 oranges would cost £11.
3. 0 apples and 5 oranges, 2 apples and 4 oranges, 4 apples and 3 oranges, 6 apples and 2 oranges, 8 apples and 1 orange, 10 apples and 0 oranges.
4. Answers vary. Sample response: $a + 2r = 10$ where a represents the number of apples and r represents the number of oranges.
5. The combinations that add up to £10 make the equation true. The combinations that add up to other amounts make it false.

Are You Ready for More?

1. Graph the equation you wrote relating the number of apples and the number of oranges.
2. What is the gradient of the graph? What is the meaning of the gradient in terms of the context?
3. Suppose Noah has £20 to spend. Graph the equation describing this situation. What do you notice about the relationship between this graph and the earlier one?

Student Response

1. The graph of the equation $x + 2y = 10$ should show horizontal and vertical intercepts, first quadrant only, with the horizontal axis labelled "number of apples" and the vertical axis labelled "number of oranges". Note that a graph with axis labels reversed is also correct.

2. The gradient is -0.5. (-2 is a possible answer if the axes are swapped.) This is the number of oranges per apple. Since oranges cost £2 and apples cost £1, you have to give up 0.5 oranges to buy an apple (or, for a gradient of -2, you have to give up two apples to buy one orange).

3. The graph is the same as in the first question except that the vertical and horizontal intercepts are at 20 and 10 instead of 10 and 5. The new line has the same gradient, because the trade-off between apples and oranges is the same.

Activity Synthesis

The equation $a + 2r = 10$ (or some equivalent) is an example of a naturally arising equation of the form $Ax + By = C$, introduced in the previous lesson. In a graphical representation of this situation, either variable (oranges or apples) could be plotted on either axis. If students mention gradient, make sure to ask them how they are plotting the variables. This will be taken up in greater detail in the next activity where the graphical representation will be central.

Invite students from different groups to share what they discovered with their partners by asking:

- "How many combinations that cost £10 did you find in total? How do you know you have found them all?" (6 combinations, these are all the possibilities of whole numbers that would make the sum 10.)
- "Did you notice any patterns that helped you find the combination?" (There are many patterns; the one we would like students to notice is that buying 1 less orange means you can buy 2 more apples.)
- "What would the graph of the equation you wrote look like?" (Some may realise that the graph would contain the six points that represent the six combinations for £10. Help them to see that these points would all lie on a line, given by the equation $a + 2r = 10$. The non-integer points on the line would represent a fractional number of pieces of each fruit.)
- "According to the equation you wrote, if you bought $\frac{1}{2}$ orange and 9 apples you would spend £10. Do you think this situation is realistic when buying fruit in a store?" (Probably not, you would only buy whole pieces of fruit.)

12.3 Solutions and Everything Else

15 minutes

Students write an equation representing a stated relationship between two quantities, and use the equation to find pairs of numbers that make it true and pairs of numbers for which it is not true. By graphing both sets of points, students see that the graph of a linear equation is the set of its solutions, that is, the points whose coordinates make the equation true.

Students should be encouraged to use rational number coordinates and points in all four quadrants and on the two axes in their graphs, and to see that solutions exist all along the line, between and beyond the pairs they explicitly choose, making for an infinite number of solutions.

Some students might notice that the points where $x = 0$ and $y = 0$ are easy to quickly find and that connecting these two points yields the line containing all the other solutions. They may then try to read solutions from the graph instead of using the equation to more accurately calculate the coordinates of solutions. While this is a valuable observation in terms of quickly drawing the graph of the linear equation, students should be guided to realise that the equation enables a level of precision that reading from the graph often lacks. Precision can be improved through the use of technology in the form of graphing calculators and applications.

As students find points, both satisfying the relationship from the first question and not satisfying the relationship, monitor for these choices and invite these students to share during the discussion.

- Points in the first quadrant
- Points on the axes (there are only two of these!)
- Points in the first and/or fourth quadrants
- Points with non-integer coordinate values

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Stronger and Clearer Each Time

Launch

Students should have access to geometry toolkits, especially rulers and graph paper.

Display for all to see the following statement: “You have two numbers. If you double the first number and add it to the second number, the sum is 10.”

Poll the class for "predict how many pairs of numbers make this statement true?" Display the answers for all to see.

Instruct students to pause their work after creating their graph and then to check their graph with you. Identify students who struggle with writing the equation or with finding and graphing solutions. Check that the graph and the equation match since some students might write the equation as $x + 2y = 10$ instead of $2x + y = 10$. Also check that students are not restricting x and y values to non-negative integers.

Representation: Internalise Comprehension. Display and discuss a range of examples and counterexamples of pairs of numbers that satisfy the given statement. For example, the pair $(1, 8)$ makes the statement true, whereas $(2, 4)$ does not.
Supports accessibility for: Conceptual processing

Anticipated Misconceptions

Students might write $x + 2y = 10$. Ask these students what x and y represent and to pick a couple of values of x and y so that the double of x and y sum to 10. Ask them to see if these are solutions to their equation. They most likely will not be. Suggest that these students reconsider their equation.

Student Task Statement

You have two numbers. If you double the first number and add it to the second number, the sum is 10.

1. Let x represent the first number and let y represent the second number. Write an equation showing the relationship between x, y, and 10.

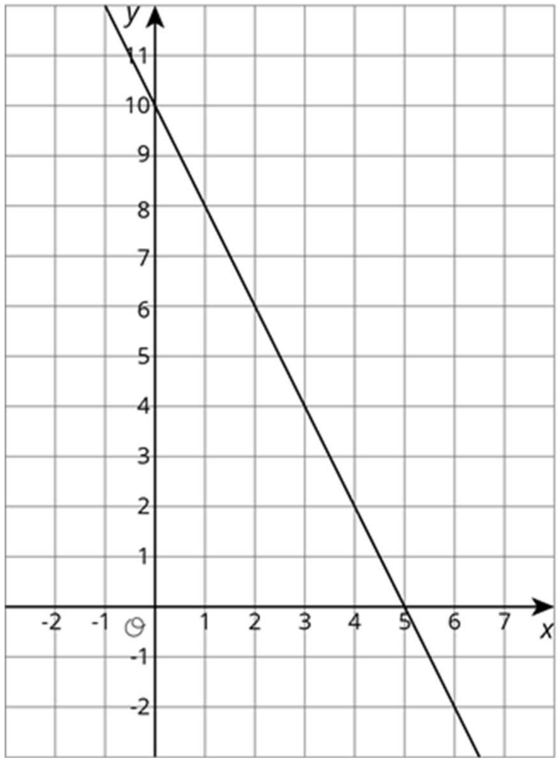
2. Draw and label a set of x- and y-axes. Plot at least five points on this coordinate plane that make the statement and your equation true. What do you notice about the points you have plotted?

3. List ten points that do *not* make the statement true. Using a different colour, plot each point in the same coordinate plane. What do you notice about these points compared to your first set of points?

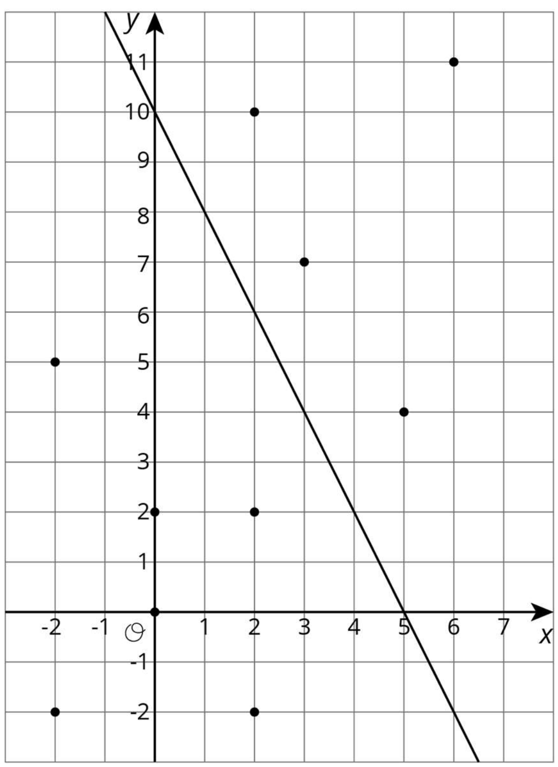
Student Response

1. $2x + y = 10$.

2. Answers vary. Sample response: All the points lie on a line.



3. Answers vary. Sample responses: These points all lie off the line formed by the set of points that make the statement true.



Activity Synthesis

Select students to share their examples of points in the coordinate plane that make their equation true, sequenced in the following order:

- Points in the first quadrant
- Points on the axes (there are only two of these!)
- Points in the first and/or fourth quadrants
- Points with non-integer coordinate values

If it does not come up in student work, ask them if the equation could be true if x = -3 (yes, y = 16) or if y = -4 (yes, x = 7). What if $x = 3\frac{1}{2}$? (Yes, y = 3.) Through the discussion questions, bring out that there are an infinite number of solutions to this equation and that, taken collectively and plotted in the plane, they make up the line of all solutions to the equation $2x + y = 10$.

Connect student solutions to the graphical representation of the scenario by asking:

- "For an equation with two variables, a pair of numbers that makes the equation true is called a *solution* of the equation. Where in your graph do you see solutions of the equation that you wrote? Are all possible solutions on your graph?" Bring out here that the set of *all* solutions to the equation is a line with y-intercept 10 and gradient -2. So an alternative way to write its equation, other than $2x + y = 10$, is $y = -2x + 10$.
- "Where in your graph do you see pairs that are *not* solutions of the equation you wrote?" (All points not on the line.)
- "Based on your observations, what is the relationship between the solutions of an equation and its graph?" (For this equation, the set of solutions is a line.)
- "What does the graph tell you about the number of solutions of your equation?" (There are an infinite number.)
- "Does the line contain all possible solutions to your equation? How do you know?" (Yes, for each value of x, there is exactly one value of y giving a solution.)

In summary, articulate the idea that the solutions are precisely the points on the line, and no more.

Ask students to compare solutions to the apple and orange activity with solutions in this activity. "Would a graph of all solutions to the apples and oranges problem look the same as the graph in this activity?" It would not because you can only purchase non-negative whole number values of apples and oranges. Make sure students understand that in this activity it makes sense to draw a line connecting all points because the coordinates of all the points on the line are solutions to the equation.

A **solution to an equation with two variables** is an ordered pair of values that makes an equation true, and the graph of the equation is the set of *all* solution pairs (x, y) plotted as points in coordinate plane: for $2x + y = 10$, this set of solutions is a line with gradient -2 and y-intercept 10.

Writing, Conversing: Stronger and Clearer Each Time. Use this routine for students to respond in writing to the prompt: "What does a graph tell you about the solutions to an equation with two variables?" Give students time to meet with 2–3 partners, to share and get feedback on their responses. Encourage the listener to press for supporting details and evidence by asking, "Could you give an example from your graph?" or "Could you make a generalisation about the solutions to an equation from the specific case you mentioned?" Have the students write a second draft based on their peer feedback. This will help students articulate their understanding of the solution to an equation and clearly define it using a graph.
Design Principle(s): Optimise output ; Cultivate conversation

Lesson Synthesis

Students explored several big ideas in this lesson:

1. A solution to a linear equation is a pair of values that makes the equation true.
2. Solutions can be found by substituting a value for one of the variables and solving the equation for the other.
3. The set of all the solutions to a linear equation can be shown visually in the coordinate plane and is called the graph of the equation.
4. The graph of a linear equation is a line.
5. Any points in the coordinate plane that do not lie on the line that is on the graph of the linear equation are not solutions to the equation.
6. The number of solutions might be limited in a real-world situation even though the equation has an infinite number of solutions.

Invite students to describe how they found solutions to linear equations and to explain how they knew they had found a solution.

Ask what difference there may be between reading solutions from a graph and calculating them using the equation. Listen for students to identify a possible lack of precision when reading from a graph. If you are using graphing software this will be less of an issue.

Finally, ask if points that are not on the line can be solutions to the equation represented by the line. Listen for students to understand that the line represents *all* pairs that make the equation true so a point not on the line cannot be a solution.

12.4 Identify the Points

Cool Down: 5 minutes

Students verify whether or not certain points in the x-y plane make a linear equation true.

Student Task Statement

Which of the following coordinate pairs make the equation $x - 9y = 12$ true?

1. (12,0)
2. (0,12)
3. (3,-1)
4. $\left(0,-\frac{4}{3}\right)$

Student Response

1. Yes.
2. No.
3. Yes.
4. Yes.

Student Lesson Summary

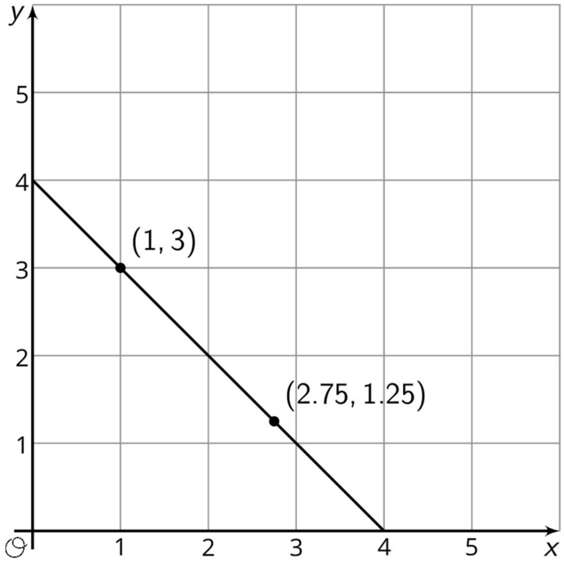
Think of all the rectangles whose perimeters are 8 units. If x represents the width and y represents the length, then $2x + 2y = 8$ expresses the relationship between the width and length for all such rectangles.

For example, the width and length could be 1 and 3, since $2 \times 1 + 2 \times 3 = 8$ or the width and length could be 2.75 and 1.25, since $2 \times (2.75) + 2 \times (1.25) = 8$.

We could find many other possible pairs of width and length, (x, y), that make the equation true—that is, pairs (x, y) that when substituted into the equation make the left side and the right side equal.

A **solution to an equation with two variables** is any pair of values (x, y) that make the equation true.

We can think of the pairs of numbers that are solutions of an equation as points on the coordinate plane. Here is a line created by all the points (x, y) that are solutions to $2x + 2y = 8$. Every point on the line represents a rectangle whose perimeter is 8 units. All points not on the line are not solutions to $2x + 2y = 8$.



Glossary

- solution to an equation with two variables

Lesson 12 Practice Problems

1. **Problem 1 Statement**

   Select **all** of the ordered pairs (x, y) that are solutions to the linear equation $2x + 3y = 6$.

   a. (0,2)

   b. (0,6)

   c. (2,3)

   d. (3,-2)

   e. (3,0)

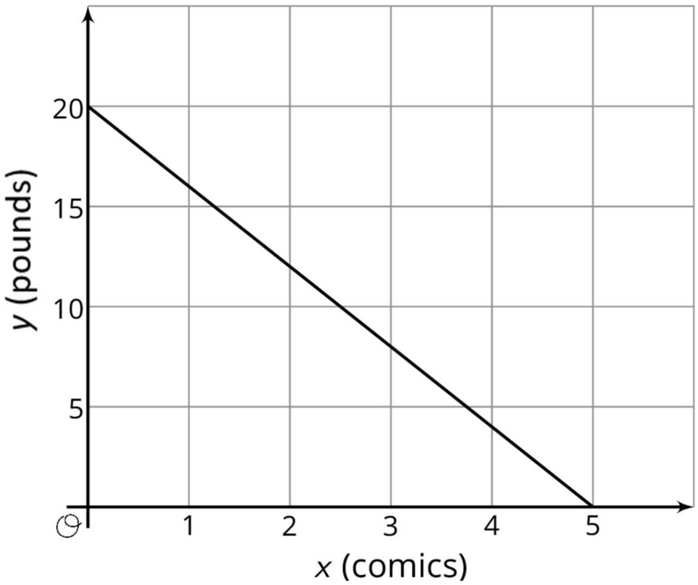
   f. (6,-2)

   Solution ["A", "E", "F"]

2. **Problem 2 Statement**

The graph shows a linear relationship between x and y.

x represents the number of comic books Priya buys at the store, all at the same price, and y represents the amount of money (in pounds) Priya has after buying the comic books.



a. Find and interpret the x- and y-intercepts of this line.

b. Find and interpret the gradient of this line.

c. Find an equation for this line.

d. If Priya buys 3 comics, how much money will she have remaining?

Solution

a. Priya has £20 before buying any comics, and if she buys 5 comics, Priya will have no money left.

b. The gradient is -4. The amount of money left goes does down by 4 with each comic book; each comic book costs £4.

c. $y = 20 - 4x$

d. £8

3. **Problem 3 Statement**

Match each equation with its three solutions.

a. $y = 1.5x$

b. $2x + 3y = 7$

c. $x - y = 4$

d. $3x = \frac{y}{2}$

e. $y = -x + 1$

1. (14,21), (2,3), (8,12)

2. (-3,-7), (0,-4), (-1,-5)

3. $\left(\frac{1}{8},\frac{7}{8}\right),\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{4},\frac{3}{4}\right)$

4. $\left(1,1\frac{2}{3}\right)$, (-1,3), $\left(0,2\frac{1}{3}\right)$

5. (0.5,3), (1,6), (1.2,7.2)

Solution

- A: 1
- B: 4
- C: 2
- D: 5
- E: 3

4. **Problem 4 Statement**

A container of fuel dispenses fuel at the rate of 5 gallons per second. If y represents the amount of fuel remaining in the container, and x represents the number of seconds that have passed since the fuel started dispensing, then x and y satisfy a linear relationship.

In the coordinate plane, will the gradient of the line representing that relationship have a positive, negative, or zero gradient? Explain how you know.

Solution

Negative because the amount of fuel in the tank is decreasing.

5. **Problem 5 Statement**

A sandwich store charges a delivery fee to bring lunch to an office building. One office pays £33 for 4 turkey sandwiches. Another office pays £61 for 8 turkey sandwiches. How much does each turkey sandwich add to the cost of the delivery? Explain how you know.

Solution

£7. Explanations vary. Sample response: The second office pays $61 - 33$, or 28 pounds more, for $8 - 4$, or 4, more sandwiches. So each sandwich adds $28 \div 4$, or 7 pounds, to the cost.