Mathematics Education Innovation

Lesson 15: Volume of prisms

Goals

- Apply dividing by fractions to calculate one edge length of a cuboid, given its volume and the other two edge lengths.
- Explain (orally, in writing, and using other representations) how to solve a problem involving the volume of a cuboid with fractional edge lengths.
- Generalise that it takes more smaller cubes or fewer larger cubes to fill the same volume.

Learning Targets

- I can solve volume problems that involve fractions.

Lesson Narrative

In this lesson, students complete their understanding of why the method of multiplying the edge lengths works for finding the volume of a cuboid, or rectangular prism, with fractional edge lengths, just as it did for cuboids with whole-number edge lengths. They use this understanding to find the volume of cuboids given the edge lengths, and to find unknown edge lengths given the volume and other edge lengths.

Problems about rectangles and triangles in the previous two lessons involved three quantities: length, width, and area; or base, height, and area. Problems in this lesson involve four quantities: length, width, height, and volume. So finding an unknown quantity might involve an extra step, for example, multiplying two known lengths first and then dividing the volume by this product, or dividing the volume twice, once by each known length.

In tackling problems with increasing complexity and less scaffolding, students must make sense of problems and persevere in solving them.

Addressing

- Find the volume of a right cuboid with fractional edge lengths by packing it with unit cubes of the appropriate unit fraction edge lengths, and show that the volume is the same as would be found by multiplying the edge lengths of the cuboid. Apply the formulas $V = lwh$ and $V = bh$ to find volumes of right cuboids with fractional edge lengths in the context of solving real-world and mathematical problems.

Instructional Routines

- Discussion Supports

Student Learning Goals

Let’s look at the volume of cuboids that have fractional measurements.

15.1 A Box of Cubes

Warm Up: 5 minutes

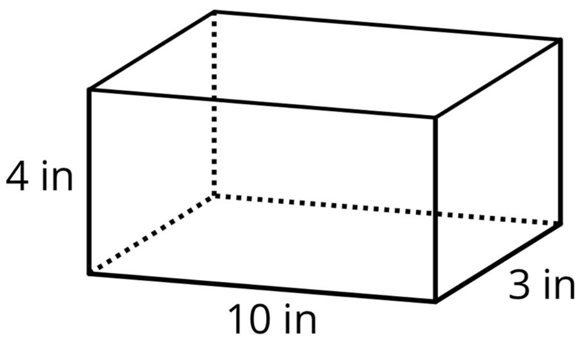
This warm-up reviews the volume work students had done previously to prepare for the work in this lesson. It reinforces the idea of using unit cubes and fractional-unit cubes as a way to measure the volume of a cuboid.

Launch

Give students 2–3 minutes of quiet work time. Follow with a class discussion.

Student Task Statement

1. How many cubes with an edge length of 1 inch fill this box?



2. If the cubes had an edge length of 2 inches, would you need more or fewer cubes to fill the box? Explain your reasoning.

3. If the cubes had an edge length of $\frac{1}{2}$ inch, would you need more or fewer cubes to fill the box? Explain your reasoning.

Student Response

1. 120 cubes with edge length of 1 inch fill the box since $10 \times 3 \times 4 = 120$.

2. Fewer. Sample explanation: 8 cubes with edge length of 1 inch fit into a cube with edge length of 2 inches, so there will be fewer cubes.

3. More. Sample explanation: Each cube with edge length of 1 inch can be packed with 8 cubes with edge length of $\frac{1}{2}$ inch, so there will be more cubes.

Activity Synthesis

Select several students to share their responses and reasoning. After each person explains, ask students to indicate whether they agree. To involve more students in the discussion, consider asking:

- “Who can restate ___’s reasoning in a different way?”
- “Does anyone want to add on to _____’s reasoning?”

- "Do you agree or disagree with the reasoning? Why?"

Tell students that they will use their understanding of the volume of cuboids to solve other geometric problems.

15.2 Cubes with Fractional Edge Lengths

20 minutes

In this activity, students continue the work on finding the volume of a right cuboid with fractional edge lengths. This time, they do so by packing it with unit cubes of different unit fractions for their edge lengths— $\frac{1}{3}$, $\frac{1}{2}$, and $\frac{1}{4}$ of an inch. They use these cubes to find the volume of the cuboid in cubic inches, decide which unit fraction works better to accomplish this goal and why, and explain whether cubes of different fractional edge lengths would lead to the same volume in cubic inches.

As students work, notice those who are able to clearly explain why cubes with a particular fractional edge length are preferable as a unit of measurement and why the volume in cubic inches will be the same regardless of the cubes used. Invite them to share later.

Launch

Arrange students in groups of 3–4. Give them 8–10 minutes of quiet work time, and then 5 minutes to discuss their responses with their group. Ask groups to be sure to discuss the third question. Encourage students to draw a sketch to help with reasoning, if needed.

Representation: Internalise Comprehension. Activate or supply background knowledge about measurement. Students may benefit from watching a quick demonstration or video of packing cubes into a box. Review terms such as dimensions, volume, and cubic units.
Supports accessibility for: Memory; Conceptual processing

Anticipated Misconceptions

If students have trouble getting started, ask them to revisit their work with $\frac{1}{2}$ inch cubes from a previous lesson. Remind them that a cube with $\frac{1}{2}$ inch edge length has a volume of $\frac{1}{8}$ in^3 (because we can fit 8 of such cubes in a 1 inch cube). Ask them to think about how many $\frac{1}{3}$ inch cubes can fit into a 1 inch cube, or think about what the volume of a $\frac{1}{3}$ inch cube is in cubic inches.

Some students may not be able to visualise and keep track of the measurements of the boxes in this task. Encourage students to draw and label the measurements of the boxes described in these questions.

Student Task Statement

1. Diego says that 108 cubes with an edge length of $\frac{1}{3}$ inch are needed to fill a cuboid that is 3 inches by 1 inch by $1\frac{1}{3}$ inch.

a. Explain or show how this is true. If you get stuck, consider drawing a diagram.

b. What is the volume, in cubic inches, of the cuboid? Explain or show your reasoning.

2. Lin and Noah are packing small cubes into a larger cube with an edge length of $1\frac{1}{2}$ inches. Lin is using cubes with an edge length of $\frac{1}{2}$ inch, and Noah is using cubes with an edge length of $\frac{1}{4}$ inch.

   a. Who would need more cubes to fill the $1\frac{1}{2}$ inch cube? Be prepared to explain your reasoning.

   b. If Lin and Noah each use their small cubes to find the volume of the larger $1\frac{1}{2}$ inch cube, will they get the same answer? Explain or show your reasoning.

Student Response

1.

   a. Sample reasoning: There are 9 groups of $\frac{1}{3}$ inch in 3 inches, 3 groups in 1 inch, and 4 groups in $\frac{4}{3}$ inches. So it would take $9 \times 3 \times 4$ (or 108 cubes) to pack the cuboid.

   b. 4 in^3. Sample reasoning:

      - Each cube with edge length of $\frac{1}{3}$ inch has a volume of $\frac{1}{27}$ in^3. $108 \times \frac{1}{27} = 4$
      - $3 \times 1 \times \frac{4}{3} = 4$

2.

   a. Noah would need more cubes, because his small cubes are smaller.

   b. Yes, they should get the same volume. Sample reasoning:

      - There are three $\frac{1}{2}$ inches in $1\frac{1}{2}$ inches, so Lin would need $3 \times 3 \times 3$ or 27 cubes with edge length of $\frac{1}{2}$ inch. The volume of each $\frac{1}{2}$ inch cube is $\frac{1}{8}$ in^3, so the volume of the $1\frac{1}{2}$ inch cube is $27 \times \frac{1}{8}$ (or $\frac{27}{8}$ in^3).
      - There are six $\frac{1}{4}$ inches in $1\frac{1}{2}$ inches, so Noah would need $6 \times 6 \times 6$ or 216 cubes with edge length of $\frac{1}{4}$ inch. The volume of each cube with edge length of $\frac{1}{4}$ inch is $\frac{1}{64}$ in^3, so the volume of the $1\frac{1}{2}$ inch cube is $216 \times \frac{1}{64}$ (or $\frac{27}{8}$ in^3).

- Both sets of cubes can be packed into the $1\frac{1}{2}$-inch cube. In both cases, the volume can be calculated using $\left(1\frac{1}{2}\right) \times \left(1\frac{1}{2}\right) \times \left(1\frac{1}{2}\right)$, which equals $\frac{27}{8}$ (or $3\frac{3}{8}$ in^3).

Activity Synthesis

Select several students to share their responses and articulate their reasoning. Compare the different strategies students used for finding the volume of the cuboid. Ask students:

- "Does it matter which fractional-unit cubes we use to find the volume? Why or why not?" (As long as the unit fraction can fit evenly into all three edge lengths of the cuboid, it doesn't matter what unit fraction we use.)
- "Do certain unit fractions work better as edge lengths of the small cubes than others?" (It helps to use as large a unit fraction as possible, since it means using fewer cubes and working with fractions that are closer to 1.)
- "Is there another way of finding the volume of a cuboid with fractional edge length besides using these small cubes?" (Multiply the fractional edge lengths.)

Point out that it is helpful to use a unit fraction that is a common factor of the fractional edge lengths of the cuboid. Make sure students also recognise that multiplying the edge lengths of the cuboid is a practical way to find the volume of such a cuboid.

15.3 Fish Tank and Baking tin

Optional: 20 minutes

In this activity, students solve word problems that involve finding the volume of cuboids that have fractional edge lengths, and calculate unknown edge lengths given other measurements. The last question in the activity requires students to interpret how the same volume of liquid would fit in two different containers in the shape of cuboids.

As they work, monitor for different representations students use to solve the problems.

Instructional Routines

- Discussion Supports

Launch

Keep students in groups of 3–4. Give students 5 minutes of quiet work time for the first question and 2–3 minutes to discuss their responses with their group. Then, give students time to complete the second question either individually or with their group. Encourage students to draw a sketch to help with reasoning, if needed.

Representation: Access for Perception. Read all problems aloud. Provide appropriate reading accommodations and supports to ensure students access to written directions, word

problems, and other text-based content.
Supports accessibility for: Language; Conceptual processing

Student Task Statement

1. A nature centre has a fish tank in the shape of a cuboid. The tank is 10 feet long, $8\frac{1}{4}$ feet wide, and 6 feet tall.

   a. What is the volume of the tank in cubic feet? Explain or show your reasoning.

   b. The nature centre's caretaker filled $\frac{4}{5}$ of the tank with water. What was the volume of the water in the tank, in cubic feet? What was the height of the water in the tank? Explain or show your reasoning.

   c. Another day, the tank was filled with 330 cubic feet of water. The height of the water was what fraction of the height of the tank? Show your reasoning.

2. Clare's recipe for banana bread won't fit in her favourite baking tin. The baking tin is $8\frac{1}{2}$ inches by 11 inches by 2 inches. The batter fills the baking tin to the very top, and when baking, the batter spills over the sides. To avoid spills, there should be about an inch between the top of the batter and the rim of the baking tin.

   Clare has another baking tin that is 9 inches by 9 inches by $2\frac{1}{2}$ inches. If she uses this baking tin, will the batter spill over during baking?

Student Response

1.

   a. 495 ft^3, because $10 \times \left(8\frac{1}{4}\right) \times (6) = 495$.

   b. 396 ft^3, because $(10) \times \left(8\frac{1}{4}\right) \times \left(\frac{4}{5} \times 6\right) = 396$. The water is $4\frac{4}{5}$ feet deep.

c. $\frac{2}{3}$ of the height of the tank. $330 \div \left(10 \times 8\frac{1}{4}\right) = 4$ and $4 = \frac{2}{3} \times 6$.

2. Yes, it would spill. Sample reasoning: The volume of the batter is 187 inch, since $\left(8\frac{1}{2}\right) \cdot 11 \times 2 = 187$. In the second baking tin, the batter will have a height of a little over 2 inches, because $187 \div (9 \times 9) \approx 2\frac{1}{3}$. Since the baking tin is only $2\frac{1}{2}$ inches deep, there would not be at least 1 inch between the top of batter and the rim of the baking tin.

Are You Ready for More?

1. Find the area of a rectangle with side lengths $\frac{1}{2}$ and $\frac{2}{3}$.
2. Find the volume of a cuboid with side lengths $\frac{1}{2}, \frac{2}{3}$, and $\frac{3}{4}$.
3. What do you think happens if we keep multiplying fractions $\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \frac{4}{5} \times \frac{5}{6} \ldots$?
4. Find the area of a rectangle with side lengths $\frac{1}{1}$ and $\frac{2}{1}$.
5. Find the volume of a cuboid with side lengths $\frac{1}{1}, \frac{2}{1}$, and $\frac{1}{3}$.
6. What do you think happens if we keep multiplying fractions $\frac{1}{1} \times \frac{2}{1} \times \frac{1}{3} \times \frac{4}{1} \times \frac{1}{5} \ldots$?

Student Response

1. $\frac{1}{3}$
2. $\frac{1}{4}$
3. Approach the value of 0
4. 2
5. $\frac{2}{3}$
6. Approach the value of 1

Activity Synthesis

Invite a few students to share their solutions, explanations, and drawings (if any). Record and display their solutions for all to see. To involve more students in the discussion, ask students to indicate whether they agree or disagree with their classmate's reasoning, if they approached it the same way but could explain it differently, or if they have an alternative path.

Speaking: Discussion Supports. Use this routine to support whole-class discussion. After a student explains their solution to the class, call on students to use mathematical language to restate and/or revoice the strategy presented (volume, cubic feet, cubic units, cuboid

edge lengths). Consider providing students time to restate what they hear to a partner, before selecting one or two students to share with the class. This will provide additional opportunities for all students to produce language that describes strategies for finding the volume of cuboids.
Design Principle(s): Support sense-making; Maximise meta-awareness

Lesson Synthesis

In this lesson, we used fraction multiplication and division to solve several kinds of problems about the volume of cuboids. Consider using this time to help students reflect on their problem-solving process and asking questions such as:

- "How was finding the volume of a cuboid with fractional edge lengths like finding the volume of a cuboid with whole-number edge lengths? How is it different?"
- "When calculating volume, did you find it harder to work with mixed numbers than with fractions less than 1? Why or why not?" (Working with mixed numbers is a little harder since it often involves an extra step of converting them into fractions. If an error is made then, the work that follows is affected. It is easier, however, to make sense of the size of a quantity when it is written as a mixed number.)
- "How was the process of finding an unknown length of a rectangle the same or different than finding an unknown length of a cuboid?" (In both cases, there is one missing factor. When working with area, there are 3 quantities to keep track of: area, base, and height. When working with volume, there are 4 quantities to consider: volume, length, width, and height.)
- "Were there certain parts of calculating a volume or an unknown length that you found challenging or were prone to making mistakes? If so, which parts?"

15.4 Storage Box

Cool Down: 5 minutes

Launch

Encourage students to draw a sketch to help with reasoning, if needed.

Student Task Statement

A storage box has a volume of 56 cubic inches,. The base of the box is 4 inches by 4 inches.

1. What is the height of the box?
2. Lin's teacher uses the box to store her set of cubes with an edge length of $\frac{1}{2}$ inch. If the box is completely full, how many cubes are in the set?

Student Response

1. $3\frac{1}{2}$ (or equivalent) inches. $56 \div (4 \times 4) = \frac{56}{16} = 3\frac{8}{16}$

2. 448 cubes. Each cubic inch fits 8 cubes with edge length of $\frac{1}{2}$ inch. $56 \times 8 = 448$

Student Lesson Summary

If a cuboid has edge lengths a units, b units, and c units, the volume is the product of a, b, and c. $V = a \times b \times c$

This means that if we know the *volume* and *two edge lengths*, we can divide to find the *third* edge length.

Suppose the volume of a cuboid is $400\frac{1}{2}$ cm^3, one edge length is $\frac{11}{2}$ cm, another is 6 cm, and the third edge length is unknown. We can write a multiplication equation to represent the situation: $\frac{11}{2} \times 6 \times ? = 400\frac{1}{2}$

We can find the third edge length by dividing: $400\frac{1}{2} \div \left(\frac{11}{2} \times 6\right) = ?$

Lesson 15 Practice Problems

1. **Problem 1 Statement**

   A pool in the shape of a cuboid is being filled with water. The length and width of the pool is 24 feet and 15 feet. If the height of the water in the pool is $1\frac{1}{3}$ feet, what is the volume of the water in cubic feet?

   Solution

   480 cubic feet. ($24 \times 15 = 360$, and $360 \times \frac{4}{3} = 480$.)

2. **Problem 2 Statement**

   A cuboid measures $2\frac{2}{5}$ inches by $3\frac{1}{5}$ inches by 2 inches.

   a. Priya said, "It takes more cubes with edge length $\frac{2}{5}$ inch than cubes with edge length $\frac{1}{5}$ inch to pack the cuboid." Do you agree with Priya? Explain or show your reasoning.

   b. How many cubes with edge length $\frac{1}{5}$ inch fit in the cuboid? Show your reasoning.

   c. Explain how you can use your answer in the previous question to find the volume of the cuboid in cubic inches.

Solution

a. Disagree. Sample reasoning: Cubes with side lengths $\frac{2}{5}$ inch are larger than cubes with side lengths $\frac{1}{5}$ inch, so it would take fewer of the former to pack the same cuboid.

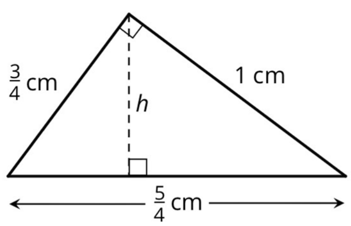
b. 1 920 cubes. Reasoning varies. Sample reasoning: $2\frac{2}{5} \div \frac{1}{5} = 12$, $3\frac{1}{5} \div \frac{1}{5} = 16$, and $2 \div \frac{1}{5} = 10$. We can fit 12 cubes along the length of the cuboid, 16 cubes along the width, and 10 cubes along the height, so the number of cubes is: $12 \times 16 \times 10 = 1\,920$.

c. Each unit cube (edge length $\frac{1}{5}$ inch) has a volume of $\frac{1}{5} \times \frac{1}{5} \times \frac{1}{5}$ or $\frac{1}{125}$ cubic inch. There are 1 920 of these unit cubes, so the volume is $1\,920 \times \frac{1}{125}$ or 15.36 cubic inches.

3. **Problem 3 Statement**

a. Here is a right-angled triangle. What is its area?

b. What is the height h for the base that is $\frac{5}{4}$ units long? Show your reasoning.



Solution

a. $\frac{3}{8}$ cm^2. Sample reasoning: The area of a triangle is found with $\frac{1}{2} \times b \times h$. We can use the two perpendicular sides as the base and the height. $\frac{1}{2} \times \frac{3}{4} \times 1 = \frac{3}{8}$

b. $\frac{3}{5}$ cm. Sample reasoning: The area of the triangle is $\frac{3}{8}$ cm^2 and we can also write the area using the $\frac{5}{4}$ side and h. $\frac{1}{2} \times \frac{5}{4} h = \frac{3}{8}$, so $\frac{5}{8} h = \frac{3}{8}$. To find what we could multiply by $\frac{5}{8}$ to get $\frac{3}{8}$, we can divide $\frac{3}{8} \div \frac{5}{8}$, which is $\frac{3}{8} \times \frac{8}{5}$, which is $\frac{3}{5}$.

4. **Problem 4 Statement**

To give their animals essential minerals and nutrients, farmers and ranchers often have a block of salt—called “salt lick”—available for their animals to lick.

a. A rancher is ordering a box of cube-shaped salt licks. The edge lengths of each salt lick are $\frac{5}{12}$ foot. Is the volume of one salt lick greater or less than 1 cubic foot? Explain your reasoning.

b. The box that contains the salt lick is $1\frac{1}{4}$ feet by $1\frac{2}{3}$ feet by $\frac{5}{6}$ feet. How many cubes of salt lick fit in the box? Explain or show your reasoning.

Solution

a. Less than 1 cubic foot. Reasoning varies. Sample reasoning: A cube with edge length 1 foot has a volume of 1 cubic foot. A salt-lick cube has edge length $\frac{5}{12}$ foot, which is less than 1 foot, so its volume $\left(\frac{5}{12} \times \frac{5}{12} \times \frac{5}{12}\right)$ is less than 1 cubic foot.

b. 24 cubes. Reasoning varies. Sample reasoning: The length of the box can fit $\frac{5}{4} \div \frac{5}{12}$ or 3 cubes. The width of the box can fit $\frac{5}{3} \div \frac{5}{12}$ or 4 cubes. The height of the box can fit $\frac{5}{6} \div \frac{5}{12}$ or 2 cubes. The box can fit $(3 \times 4 \times 2)$ or 24 cubes.

5. **Problem 5 Statement**

a. How many groups of $\frac{1}{3}$ inch are in $\frac{3}{4}$ inch?

b. How many inches are in $1\frac{2}{5}$ groups of $1\frac{2}{3}$ inches?

Solution

a. $2\frac{1}{4}$. Sample reasoning: To find "how many groups," compute $\frac{3}{4} \div \frac{1}{3}$, which is $\frac{3}{4} \times \frac{3}{1}$, which is $\frac{9}{4}$ or $2\frac{1}{4}$

b. $2\frac{1}{3}$. Sample reasoning: To find "how many inches," compute $1\frac{2}{5} \times 1\frac{2}{3}$, which is $\frac{7}{5} \cdot \frac{5}{3}$, which is $\frac{7}{3}$ or $2\frac{1}{3}$.

6. **Problem 6 Statement**

Here is a table that shows the ratio of flour to water in an art paste. Complete the table with values in equivalent ratios.

cups of flour	cups of water
1	$\frac{1}{2}$
4	
	3
$\frac{1}{2}$	

Solution

cups of flour	cups of water
1	$\frac{1}{2}$
4	2
6	3
$\frac{1}{2}$	$\frac{1}{4}$