Lesson 4: Reasoning about equations and bar models (Part 1)

Goals

- Coordinate bar models, equations of the form $px + q = r$, and verbal descriptions of the situations.
- Explain (orally and in writing) how to use a bar model to determine the value of an unknown quantity in an equation of the form $px + q = r$.
- Interpret (in writing) the solution to an equation in the context of the situation it represents.

Learning Targets

- I can draw a bar model to represent a situation where there is a known amount and several copies of an unknown amount and explain what the parts of the diagram represent.
- I can find a solution to an equation by reasoning about a bar model or about what value would make the equation true.

Lesson Narrative

The focus of this lesson is situations that lead to equations of the form $px + q = r$. Bar models are used to help students understand why these situations can be represented with equations of this form, and to help them reason about solving equations of this form. Students also attend to the meaning of the equation's solution in the context. Note that we are not generalising solution methods yet; just using diagrams as a tool to reason about solving equations.

Building On

- Understand solving an equation or inequality as a process of answering a question: which values from a specified set, if any, make the equation or inequality true? Use substitution to determine whether a given number in a specified set makes an equation or inequality true.

Addressing

- Solve multi-step real-life and mathematical problems posed with positive and negative rational numbers in any form (whole numbers, fractions, and decimals), using tools strategically. Apply properties of operations to calculate with numbers in any form; convert between forms as appropriate; and assess the reasonableness of answers using mental computation and estimation strategies.
- Solve word problems leading to equations of the form $px + q = r$ and $p(x + q) = r$, where p, q, and r are specific rational numbers. Solve equations of these forms fluently. Compare an algebraic solution to an arithmetic solution, identifying the

sequence of the operations used in each approach. For example, the perimeter of a rectangle is 54 cm. Its length is 6 cm. What is its width?

Building Towards

- Solve word problems leading to equations of the form $px + q = r$ and $p(x + q) = r$, where p, q, and r are specific rational numbers. Solve equations of these forms fluently. Compare an algebraic solution to an arithmetic solution, identifying the sequence of the operations used in each approach. For example, the perimeter of a rectangle is 54 cm. Its length is 6 cm. What is its width?

Instructional Routines

- Algebra Talk
- Clarify, Critique, Correct
- Compare and Connect
- Discussion Supports

Student Learning Goals

Let’s see how bar models can help us answer questions about unknown amounts in stories.

4.1 Algebra Talk: Seeing Structure

Warm Up: 10 minutes

The purpose of this Algebra Talk is to elicit strategies and understandings students have for solving equations. These understandings help students develop fluency and will be helpful later in this unit when students will need to be able to come up with ways to solve equations of this form. While four equations are given, it may not be possible to share every strategy. Consider gathering only two or three different strategies per problem, saving most of the time for the final question.

Students should understand the meaning of *solution to an equation* from earlier work in KS3 as well as from work earlier in this unit, but this is a good opportunity to re-emphasise the idea.

In this string of equations, each equation has the same solution. Digging into why this is the case requires noticing and using the structure of the equations. Noticing and using the structure of an equation is an important part of fluency at solving equations.

Instructional Routines

- Algebra Talk
- Discussion Supports

Launch

Display one equation at a time. Give students 30 seconds of quiet think time for each equation and ask them to give a signal when they have an answer and a strategy. Keep all equations displayed throughout the talk. Follow with a whole-class discussion.

Representation: Internalise Comprehension. To support working memory, provide students with sticky notes or mini whiteboards.
Supports accessibility for: Memory; Organisation

Student Task Statement

Find a solution to each equation without writing anything down.

$x + 1 = 5$

$2(x + 1) = 10$

$3(x + 1) = 15$

$500 = 100(x + 1)$

Student Response

- 4 is the solution because $4 + 1 = 5$.
- 4. Possible strategies: Trial and error to arrive at $2(4 + 1) = 10$, noticing that the equation is 2 times something is 10, so the something must be a 5, applying the distributive property to get $2x + 2 = 10$, and reasoning from there.
- 4

$x = 4$ is a solution to each equation because $x + 1$ has to equal 5.

Activity Synthesis

Ask students to share their strategies for each problem. Record and display their responses for all to see. To involve more students in the conversation, consider asking:

- “Who can restate ___’s reasoning in a different way?”
- “Did anyone have the same strategy but would explain it differently?”
- “Did anyone solve the equation in a different way?”
- “Does anyone want to add on to _____’s strategy?”
- “Do you agree or disagree? Why?”

An important idea to highlight is the meaning of a solution to an equation; a solution is a value that makes the equation true.

For the second and third equations, some students may first think about applying the distributive property before reasoning about the solution. As students see the third and fourth equations, they are likely to notice commonalities among the equations that can support solving them. One likely observation to highlight is that each equation has the same solution. It is worth asking why each equation has the same solution. A satisfying answer to this question requires attending to the structure of the equations.

Speaking: Discussion Supports.: Display sentence frames to support students when they explain their strategy. For example, "First, I _____ because . . ." or "I noticed _____ so I . . . ." Some students may benefit from the opportunity to rehearse what they will say with a partner before they share with the whole class.
Design Principle(s): Optimise output (for explanation)

4.2 Situations and Diagrams

15 minutes (there is a digital version of this activity)

The purpose of this activity is to work toward showing students that some situations can be represented by an equation of the form $px + q = r$ (or equivalent). In this activity, students are simply tasked with drawing a bar model to represent each situation. In the following activity, they will work with corresponding equations.

The last question is tough to represent with a bar model, because you would have to divide the diagram into 30 equal pieces. This is intentional, and can be used to make the point that we are trying to develop more efficient ways of solving problems than drawing a diagram every time.

For each question, monitor for one student with a correct diagram. Press students to explain what any variables used to label the diagram represent in the situation.

Instructional Routines

- Compare and Connect

Launch

Ensure students understand that the work of this task is to draw a bar model to represent each situation. There is no requirement to write an equation or solve a problem yet.

Arrange students in groups of 2. Give 5–10 minutes to work individually or with their partner, followed by a whole-class discussion.

Engagement: Internalise Self Regulation. Chunk this task into more manageable parts to differentiate the degree of difficulty or complexity. Invite students to choose and respond to 3 out of the 5 situations. Require students to choose the last situation as it is different from the others.
Supports accessibility for: Organisation; Attention

Student Task Statement

Draw a bar model to represent each situation. For some of the situations, you need to decide what to represent with a variable.

1. Diego has 7 packs of markers. Each pack has x markers in it. After Lin gives him 9 more markers, he has a total of 30 markers.

2. Elena is cutting a 30-foot piece of ribbon for a craft project. She cuts off 7 feet, and then cuts the remaining piece into 9 equal lengths of x feet each.

3. A construction manager weighs a bundle of 9 identical bricks and a 7-pound concrete block. The bundle weighs 30 pounds.

4. A skating rink charges a group rate of £9 plus a fee to rent each pair of skates. A family rents 7 pairs of skates and pays a total of £30.

5. Andre bakes 9 pans of brownies. He donates 7 pans to the school bake sale and keeps the rest to divide equally among his class of 30 students.

Student Response

Answers vary. Sample responses are rectangles with the following features:

1. 7 same-sized boxes each marked x, one box marked 9, bracket showing total is 30

2. 9 same-sized boxes each marked x, one box marked 7, bracket showing total is 30

3. 9 same-sized boxes each marked x, one box marked 7, bracket showing total is 30

4. 7 same-sized boxes each marked x, one box marked 9, bracket showing total is 30

5. 30 same-sized boxes each marked x (or one marked $30x$ or equivalent), one box marked 7, bracket showing total is 9

Activity Synthesis

Select one student for each situation to present their correct diagram. Ensure that students explain the meaning of any variables used to label their diagram. Possible questions for discussion:

- "For the situations with no x, how did you decide what quantity to represent with a variable?" (Think about which amount is unknown but has a relationship to one or more other amounts in the story.)

- "Did any situations have the same diagrams? How can you tell from the story that the diagrams would be the same?" (Same number of equal parts, same amount for unequal parts, same amount for the total.)

- "How is the last situation different from the others?" (It's the only one where 30 is the coefficient of x rather than the total.)
- "Why was it tough to draw a diagram for the last question?" (You would have to divide the diagram into 30 equal pieces.)

Conversing, Representing: Compare and Connect. Use this routine to support student understanding of the connections between bar models and the situations they represent. Because all situations in this activity share the same quantities of 7, 9, and 30, students can compare how each situation affects how quantities appear in a bar model. Invite students to compare their diagrams with their partner. Display the following sentence frames: "One thing our diagrams have in common is …." and "One thing that is different about our diagrams is …". This routine will help students identify, explain, and verbally respond to correspondences between context and mathematical representations.
Design Principle(s): Optimise output (for comparison); Cultivate conversation

4.3 Situations, Diagrams, and Equations

10 minutes

This activity is a continuation of the previous one. Students match each situation from the previous activity with an equation, solve the equation by any method that makes sense to them, and interpret the meaning of the solution. Students are still using any method that makes sense to them to reason about a solution. In later lessons, a balance diagram representation will be used to justify more efficient methods for solving. For example, when they are using bar models, they could just say "I subtracted the 9 extra markers and then divided the remaining 21 markers by 7." Later, when working with balance diagrams, we can press them to say "I can subtract 9 from each side and then divide each side by 7."

For each equation, monitor for a student using their diagram to reason about the solution and a student using the structure of the equation to reason about the solution.

Instructional Routines

- Clarify, Critique, Correct

Launch

Keep students in the same groups. 5 minutes to work individually or with a partner, followed by a whole-class discussion.

Engagement: Develop Effort and Persistence. Encourage and support opportunities for peer interactions. Prior to the whole-class discussion, invite students to share their work with a partner. Display sentence frames to support student conversation such as "To find the solution, first I ____ because...", "I made this match because I noticed...", "Why did you...?", or "I agree/disagree because…"
Supports accessibility for: Language; Social-emotional skills

Student Task Statement

Each situation in the previous activity is represented by one of the equations.

- $7x + 9 = 30$
- $30 = 9x + 7$
- $30x + 7 = 9$

1. Match each situation to an equation.
2. Find the solution to each equation. Use your diagrams to help you reason.
3. What does each solution tell you about its situation?

Student Response

- $7x + 9 = 30$: Situations 1 (markers) and 4 (skating rink)
- $9x + 7 = 30$: Situations 2 (ribbons) and 3 (bricks)
- $30x + 7 = 9$: Situation 5 (pans of brownies)
- $7x + 9 = 30$: 3
- $9x + 7 = 30$: $2\frac{5}{9}$ or about 2.6
- $30x + 7 = 9$: $\frac{1}{15}$

a. There are 3 markers in each pack.

b. Each of the 9 pieces is about 2.6 feet long.

c. Each brick weighs about 2.6 pounds.

d. It costs £3 to rent a pair of skates.

e. Each student receives $\frac{1}{15}$ of a pan of brownies.

Are You Ready for More?

While in New York City, is it a better deal for a group of friends to take a taxi or the subway to get from the Empire State Building to the Metropolitan Museum of Art? Explain your reasoning.

Student Response

Answers vary. Sample response: If there are 4 people in the group of friends, then they should take the subway. The subway fare for 4 people is $4(2.75) = 11$. The taxi fare is $2.50 initially and $2.50 for each mile they drive. The distance between the 2 landmarks is

2.9 miles, so the total taxi fare is $2.5 + 2.9 \times 2.5 = 9.75$. After paying a tip to the taxi driver, it costs more than the subway.

Activity Synthesis

For each equation, ask one student who reasoned with the diagram and one who reasoned only about the equation to explain their solutions. Display the diagram and the equation side by side, drawing connections between the two representations. If no students bring up one or both of these approaches, demonstrate manoeuvres on a diagram side by side with manoeuvres on the corresponding equation. For example, "I subtracted the 9 extra markers and then divided the remaining 21 markers by 7," can be shown on a bar model and on a corresponding equation. It is not necessary to invoke the more abstract language of "doing the same thing to each side" of an equation yet.

Speaking, Representing: Clarify, Critique, Correct. Present an incorrect match of an equation with a situation and/or diagram (e.g., $30 = 9x + 7$ with Situation 1). Invite students to clarify and then critique the error (e.g., there are 7 packs of an unknown amount of markers in Situation 1; not 9 as illustrated in the equation). Press for details, if needed. For example, students can reference "markers" in Situation 1, not "x", as the unknown value represent by the variable x. This will help students reflect on the quantity of the unknown values that would be a reasonable solution and improve on their reasoning when matching an equation to a situation.
Design Principle(s): Optimise output (for justification)

Lesson Synthesis

Display one of the situations from the lesson and its corresponding equation. Ask students to explain:

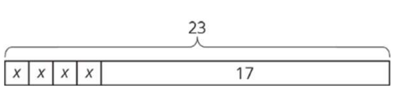
- "What does each number and letter in the equation represent in the situation?"
- "What is the reason for each operation (multiplication or addition) used in the equation?"
- "What is the solution to the equation? What does it mean to be a solution to an equation? What does the solution represent in the situation?"

4.4 Finding Solutions

Cool Down: 5 minutes

Student Task Statement

Here is a diagram and its corresponding equation. Find the solution to the equation and explain your reasoning.



$$4x + 17 = 23$$

Student Response

$x = 1\frac{1}{2}$. Explanations vary.

Student Lesson Summary

Many situations can be represented by equations. Writing an equation to represent a situation can help us express how quantities in the situation are related to each other, and can help us reason about unknown quantities whose value we want to know. Here are three situations:

1. An architect is drafting plans for a new supermarket. There will be a space 144 inches long for rows of nested shopping carts. The first cart is 34 inches long and each nested cart adds another 10 inches. The architect wants to know how many shopping carts will fit in each row.

2. A bakery buys a large bag of sugar that has 34 cups. They use 10 cups to make some cookies. Then they use the rest of the bag to make 144 giant muffins. Their customers want to know how much sugar is in each muffin.

3. Kiran is trying to save £144 to buy a new guitar. He has £34 and is going to save £10 a week from money he earns mowing lawns. He wants to know how many weeks it will take him to have enough money to buy the guitar.

We see the same three numbers in these situations: 10, 34, and 144. How could we represent each situation with an equation?

In the first situation, there is one shopping cart with length 34 and then an unknown number of carts with length 10. Similarly, Kiran has 34 dollars saved and then will save 10 each week for an unknown number of weeks. Both situations have one part of 34 and then equal parts of size 10 that all add together to 144. They each have the equation $34 + 10x = 144$.

Since it takes 11 groups of 10 to get from 34 to 144, the value of x in these two situations is $(144 - 34) \div 10$ or 11. There will be 11 nested shopping carts in each row, and it will take Kiran 11 weeks to raise the money for the guitar.

In the bakery situation, there is one part of 10 and then 144 equal parts of unknown size that all add together to 34. The equation is $10 + 144x = 34$. Since 24 is needed to get from 10 to 34, the value of x is $(34 - 10) \div 144$ or $\frac{1}{6}$. There is $\frac{1}{6}$ cup of sugar in each giant muffin.

Lesson 4 Practice Problems

1. **Problem 1 Statement**

   Draw a square with side length 7 cm.

   a. Predict the perimeter and the length of the diagonal of the square.

b. Measure the perimeter and the length of the diagonal of the square.

c. Describe how close the predictions and measurements are.

Solution

a. Perimeter: 28 cm. Length of diagonal: Approximately 9.9 cm.

b. Answers vary.

c. Answers vary.

2. **Problem 2 Statement**

Find the products.

a. (100) × (-0.09)

b. (-7) × (-1.1)

c. (-7.3) × (5)

d. (-0.2) × (-0.3)

Solution

a. -9

b. 7.7

c. -36.5

d. 0.06

3. **Problem 3 Statement**

Here are three stories:

- A family buys 6 tickets to a show. They also pay a £3 parking fee. They spend £27 to see the show.

- Diego has 27 ounces of juice. He pours equal amounts for each of his 3 friends and has 6 ounces left for himself.

- Jada works for 6 hours preparing for the art fair. She spends 3 hours on a sculpture and then paints 27 picture frames.

Here are three equations:

- $3x + 6 = 27$
- $6x + 3 = 27$
- $27x + 3 = 6$

a. Decide which equation represents each story. What does x represent in each equation?

b. Find the solution to each equation. Explain or show your reasoning.

c. What does each solution tell you about its situation?

Solution

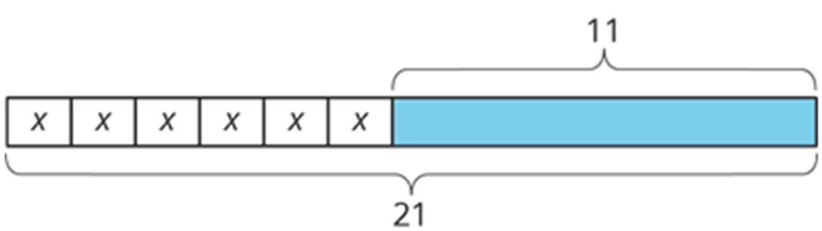
a. Tickets to the show: $6x + 3 = 27$, x represents the cost of a ticket. Diego's juice: $3x + 6 = 27$, x represents the number of ounces of juice he gave each friend. The art fair: $27x + 3 = 6$, x represents the number of hours spent on each picture frame.

b. $6x + 3 = 27$ $x = 4$.
$3x + 6 = 27$ $x = 7$.
$27x + 3 = 6$ $x = \frac{1}{9}$.
Explanations vary.

c. Each ticket to the show cost £4. Diego gave each friend 7 ounces of juice. Jada spent $\frac{1}{9}$ of an hour painting each picture frame.

4. **Problem 4 Statement**

Here is a diagram and its corresponding equation. Find the solution to the equation and explain your reasoning.



$6x + 11 = 21$

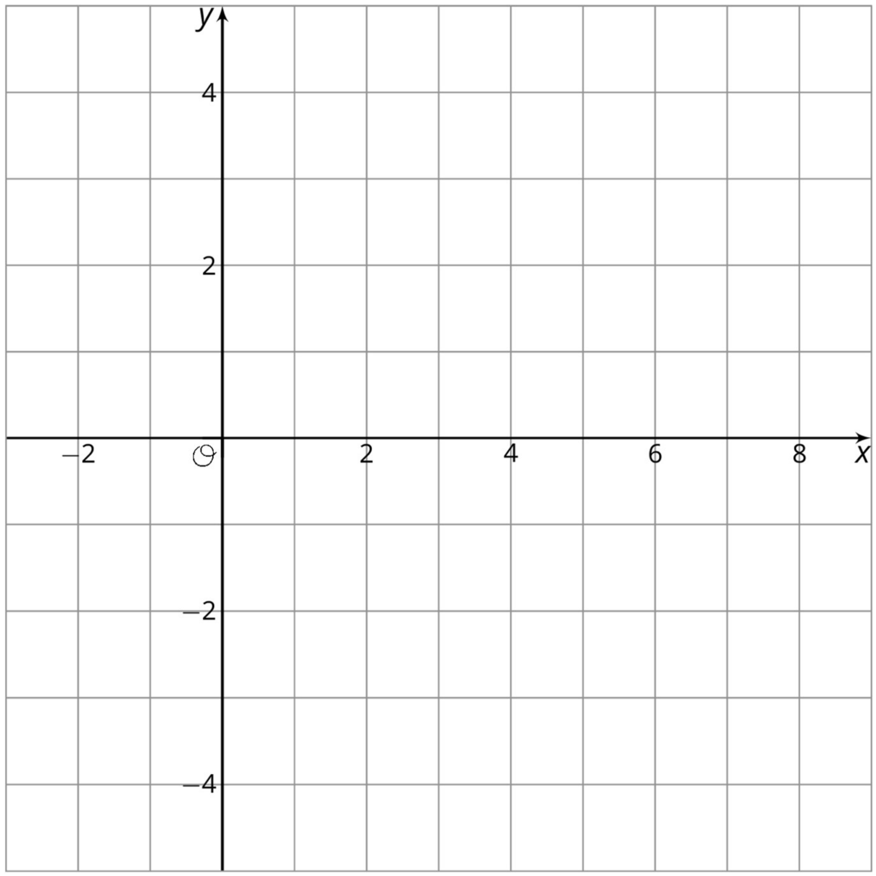
Solution

$x = \frac{10}{6}$ (or equivalent). Explanations vary.

5. **Problem 5 Statement**

a. Plot these points on the coordinate grid:

$A = (3,2), B = (7.5,2), C = (7.5,-2.5), D = (3,-2)$

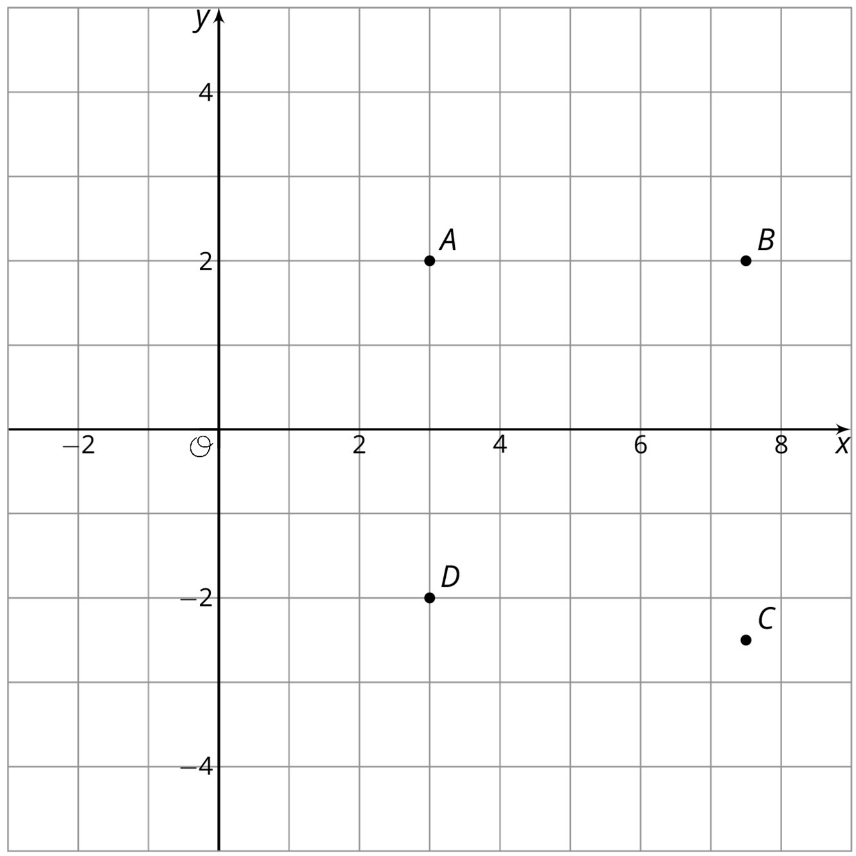


b. What is the vertical difference between D and A?

c. Write an expression that represents the vertical distance between B and C.

Solution

a.



b. The vertical difference between D and A is -4 units.

c. An expression for the vertical distance between B and C is $|2 - (-2.5)|$.