Lesson 4: Proportional Relationships and Equations

Goals

- Generalise a process for finding missing values in a proportional relationship, and justify (orally) why this can be abstracted as $y = kx$, where k is the constant of proportionality.
- Generate an equation of the form $y = kx$ to represent a proportional relationship in a familiar context.
- Write the constant of proportionality to complete a row in the table of a proportional relationship where the value for the first quantity is 1.

Learning Targets

- I can write an equation of the form $y = kx$ to represent a proportional relationship described by a table or a story.
- I can write the constant of proportionality as an entry in a table.

Lesson Narrative

In this lesson, students build on their work with tables and represent proportional relationships using equations of the form $y = kx$. The activities revisit contexts from the previous two lessons, presenting values in tables and focusing on the idea that for each table, there is a number k so that all values in the table satisfy the equation $y = kx$.

Building On

- Add, subtract, multiply, and divide decimals to hundredths, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.

Addressing

- Recognise and represent proportional relationships between quantities.
- Represent proportional relationships by equations. For example, if total cost t is proportional to the number n of items purchased at a constant price p, the relationship between the total cost and the number of items can be expressed as $t = pn$.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Co-Craft Questions
- Three Reads

- Compare and Connect
- Discussion Supports
- Number Talk
- Think Pair Share

Student Learning Goals

Let's write equations describing proportional relationships.

4.1 Number Talk: Division

Warm Up: 5 minutes

This number talk encourages students to think about the numbers in division problems and how they can use the result of one division problem to find the answer to a similar problem with a different, but related, divisor. Four problems are given, however, given limited time it may not be possible to share every possible strategy. Consider gathering only two or three different strategies per problem. Each problem is chosen to elicit a slightly different reasoning, so, as students explain their strategies, ask how the factors impacted their product.

In the final question, ask students to choose a value for x for which they could easily find the quotient. If students do not use what they know based on the answer to the previous question, ask them if they could use what they know about that equation to reason about the last expression.

Instructional Routines

- Discussion Supports
- Number Talk

Launch

Display one problem at a time. Tell students to give a signal when they have an answer and a strategy. After each problem, give students 1 minute of quiet think time followed by a whole-class discussion. Pause after discussing the third question and tell students they will be using patterns they noticed in the previous problems to choose their own divisor for the last problem. Keep all problems displayed throughout the talk.

Representation: Internalise Comprehension. To support working memory, provide students with sticky notes or mini whiteboards.
Supports accessibility for: Memory; Organisation

Student Task Statement

Find each quotient mentally.

$645 \div 100$

$645 \div 50$

$48.6 \div 30$

$48.6 \div x$

Student Response

- $645 \div 100 = 6.45$
- $645 \div 50 = 12.9$
- $48.6 \div 30 = 1.62$
- Answers vary. Students could double the divisor in the previous problem for a value of 60 for x and divide the previous quotient by 2 to arrive at a new quotient of 0.81. Students could also half the divisor in the previous problem for a value of 15 for x and multiply the previous quotient by 2 to arrive at a new quotient of 3.24.

Activity Synthesis

Ask students to share their strategies for each problem. Record and display their explanations for all to see. To involve more students in the conversation, consider asking:

- “Who can restate ___’s reasoning in a different way?”
- “Did anyone solve the problem the same way but would explain it differently?”
- “Did anyone solve the problem in a different way?”
- “Does anyone want to add on to _____’s strategy?”
- “Do you agree or disagree? Why?”

For the fourth question, ask students to share their divisor choice and reasoning behind the choice. Record these divisors for all to see and ask the rest of the class to find the quotient based on the divisor choice. Ask students to refer back to patterns and regularity they noticed in the first three problems that influenced their decisions.

Speaking: Discussion Supports.: Display sentence frames to support students when they explain their strategy. For example, "First, I _____ because . . ." or "I noticed _____ so I . . . ." Some students may benefit from the opportunity to rehearse what they will say with a partner before they share with the whole class.
Design Principle(s): Optimise output (for explanation)

4.2 Feeding a Crowd, Revisited

10 minutes

This activity revisits a context seen previously. Students solved problems like this in KS2 without formulating them in terms of ratios or rates (“If 1 cup of rice serves 3 people, how many people can you serve with 12 cups of rice?”). In this activity, they ultimately find an equation for the proportional relationship.

As students find missing values in the table, they should see that they can always multiply the number of food items by the constant of proportionality. When students see this pattern and represent the number of people served by x cups of rice (or s spring rolls) as $3x$ (or $\frac{1}{2}s$), they are expressing regularity in repeated reasoning.

Only one row in each table is complete. Based on their experience in the previous lesson, students are more likely to multiply the entries in the left-hand column by 3 (or $\frac{1}{2}$) than to use scale factors, at least for the first and third rows. If they do, they are more likely to see how to complete the last row in each table.

Some students might use unit rates: If 1 cup of rice can serve 3 people, then x cups of rice can serve $3x$ people. So, students have different ways to generate $3x$ as the expression that represents the number of people served by x cups of rice: completing the table for numerical values and continuing the pattern to the last row; or finding the unit rate and using it in the case of x cups.

Monitor students for different approaches as they are working.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Compare and Connect

Launch

Tell students that this activity revisits a context they worked on in an earlier lesson.

Anticipated Misconceptions

If students have trouble encapsulating the relationship with an expression, encourage them to draw diagrams or to verbalise the relationship in words.

Student Task Statement

1. A recipe says that 2 cups of dry rice will serve 6 people. Complete the table as you answer the questions. Be prepared to explain your reasoning.

    a. How many people will 1 cup of rice serve?

b. How many people will 3 cups of rice serve? 12 cups? 43 cups?

c. How many people will x cups of rice serve?

cups of dry rice	number of people
1	
2	6
3	
12	
43	
x	

2. A recipe says that 6 spring rolls will serve 3 people. Complete the table as you answer the questions. Be prepared to explain your reasoning.

a. How many people will 1 spring roll serve?

b. How many people will 10 spring rolls serve? 16 spring rolls? 25 spring rolls?

c. How many people will n spring rolls serve?

number of spring rolls	number of people
1	
6	3
10	
16	
25	
n	

3. How was completing this table different from the previous table? How was it the same?

Student Response

a. 3 people since $1 \times 3 = 3$

b. 9 people. 36 people. 129 people.

c. $3x$ people.

cups of dry rice	number of people

1	3
2	6
3	9
12	36
43	129
x	$3x$

a. $\frac{1}{2}$. Since $1 \times \frac{1}{2} = \frac{1}{2}$

b. 5 people; 8 people; 12.5 people.

c. $\frac{1}{2}n$

d. Answers vary.

number of spring rolls	number of people
1	$\frac{1}{2}$
6	3
10	5
16	8
25	12.5
n	$\frac{1}{2}n$

Activity Synthesis

Select students who have used the following approaches in the given order:

- Recognise that to move from the first column to the second, you multiply by 3.
- Say that 3 is the constant of proportionality.
- Recognise that the equation $2 \times ? = 6$ can be used to find the constant of proportionality algebraically.
- Say that the 3 can be interpreted as the number of people per 1 cup of rice.

If no student sees that these insights are connected to prior work, explicitly connect them with the lessons of the previous two days. At the end of the discussion, suggest to students that we let y represent the number of people who can be served by x cups of rice. Ask students to write an equation that gives the relationship of x and y (this builds on earlier work in KS3, but it may be rusty). Be sure to write the equation where all students can see it and help students interpret its meaning in the context: “To find y, the number of people served, we can multiply the number of cups of rice, x, by 3.”

Representation: Internalise Comprehension. Use colour and annotations to illustrate student thinking. As students describe their approaches, use colour and annotations to scribe their thinking on a display.
Supports accessibility for: Visual-spatial processing; Conceptual processing Conversing: Compare and Connect. As students explain their different approaches to completing the tables, ask “What is similar, what is different?” about their methods. Draw students’ attention to the different ways the constant of proportionality was represented across the different strategies. For instance, when discussing the first question, ask “Where do you see the 3 in each approach?” These exchanges strengthen students’ mathematical language use and reasoning based on constant of proportionality.
Design Principle(s): Cultivate conversation; Maximise meta-awareness

4.3 Denver to Chicago

10 minutes

This activity revisits a context seen previously. This time, students represent the proportional relationship between distance and time with an equation. Students once again make use of structure and use repeated reasoning, but there is also a focus on moving back and forth between the abstract representation and the context.

As part of this activity, students calculate distance and speed. Students should know from earlier in KS3 that speed is the quotient of distance travelled by amount of time elapsed, so they can divide 915 by 1.5 to get the speed. Students that do not begin the problem in that way can be directed back to the similar task in previous lessons to make connections and correct themselves. Once students have the speed, which is constant throughout this problem, they identify this as the constant of proportionality and use it to find the missing values.

Monitor for students who solve the problem in different ways.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Co-Craft Questions

Launch

Tell students that this activity revisits a context from an earlier lesson.

Representation: Internalise Comprehension. Activate or supply background knowledge. Provide students with access to calculators to ensure inclusive participation in the activity.
Supports accessibility for: Memory; Conceptual processing Writing: Co-Craft Questions. To help students consider the relationships between distance, time, and speed within this

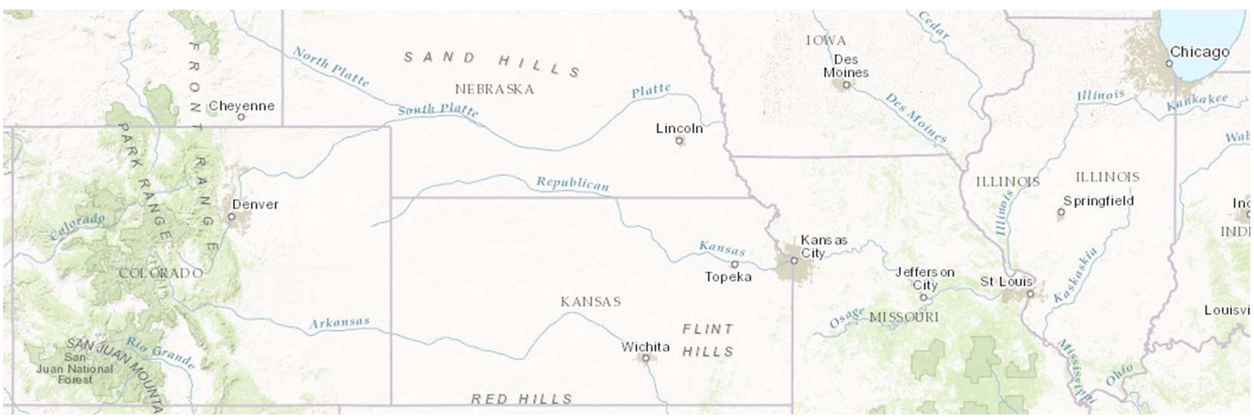
context, present to students the initial prompt and map image of the first question without the table. Ask students to write down possible mathematical questions that they can ask about this situation. Have pairs compare their questions and share out with the whole class. Listen for phrases such as "miles per hour," "constant of proportionality," "distance travelled," "unit of time," etc. This helps students produce the language of mathematical questions and talk about the relationships between the quantities in this task (e.g., distance, speed, and time) prior to being asked to analyse another's reasoning.
Design Principle(s): Maximise meta-awareness, Maximise output

Anticipated Misconceptions

Students who are having trouble understanding the task can draw a segment between Denver and Chicago and label it with the distance and the time. From there, they can draw a double number line diagram.

Student Task Statement

A plane flew at a constant speed between Denver and Chicago. It took the plane 1.5 hours to fly 915 miles.



1. Complete the table.

time (hours)	distance (miles)	speed (miles per hour)
1		
1.5	915	
2		
2.5		
t		

2. How far does the plane fly in one hour?

3. How far would the plane fly in t hours at this speed?

4. If d represents the distance that the plane flies at this speed for t hours, write an equation that relates t and d.

5. How far would the plane fly in 3 hours at this speed? In 3.5 hours? Explain or show your reasoning.

Student Response

1.

time (hours)	distance (miles)	speed (miles per hour)
1	610	610
1.5	915	610
2	1 220	610
2.5	1 525	610
t	$610t$	610

2. 610 miles since $915 \div 1.5 = 610$ for the speed and at a speed of 610 miles per hour, it would travel 610 miles after 1 hour.

3. $610t$ miles

4. Equation: $d = 610t$ or equivalent

5. 1 830 miles; 2 135 miles; I multiplied each number by 610.

Are You Ready for More?

A rocky planet orbits Proxima Centauri, a star that is about 1.3 parsecs from Earth. This planet is the closest planet outside of our solar system.

1. How long does it take light from Proxima Centauri to reach Earth? (A parsec is about 3.26 light years. A light year is the distance light travels in one year.)

2. There are two twins. One twin leaves on a spaceship to explore the planet near Proxima Centauri travelling at 90% of the speed of light, while the other twin stays home on Earth. How much does the twin on Earth age while the other twin travels to Proxima Centauri? (Do you think the answer would be the same for the other twin? Consider researching "The Twin Paradox" to learn more.)

Student Response

1. $1.3 \times 3.26 \approx 4.24$ or about 4.24 years

2. $4.24 \div 0.9 \approx 4.7$ or about 4.7 years

Activity Synthesis

Select students to discuss and share their solutions. Ask them to identify difficulties which might include: getting started, noticing the pattern, dividing with decimals, completing the values in the table, creating the equation. This problem increases the level of difficulty by having so much missing information, and by using decimals in the table. It is important to identify if there are parts that are confusing for students to move them forward.

As part of the discussion, write the equation for all to see, and ask students to describe in words how to interpret its meaning in the context of the situation (To find d, the distance travelled by the plane in miles, multiply the hours of travel, t, by the plane's speed in miles per hour, 610.).

4.4 Revisiting Bread Dough

Optional: 10 minutes

This activity gives students more practice writing an equation that represents the proportional relationship examined in a previous lesson: the amount of flour and honey in a recipe. Students can then use their equation to answer additional questions about the situation.

Instructional Routines

- Three Reads
- Think Pair Share

Launch

Tell students that this activity revisits a context from an earlier lesson. Give students quiet work time followed by partner discussion.

Reading: Three Reads. Use this routine to support reading comprehension of the baking situation, without solving it for students. In the first read, students read the prompt with the goal of comprehending the situation (e.g., there is always the same ratio of honey to flour; different days have different numbers of batches). Delay asking students to complete the table. In the second read, ask students to analyse the prompts to understand the mathematical structure by naming quantities. Listen for, and amplify, the two important quantities that vary in relation to each other in this situation: f is the number of cups of flour; h is the number of tablespoons of honey. After the third read, ask students to brainstorm possible strategies to determine the amount of flour needed if you know how much honey will be used.
Design Principle(s): Support sense-making

Student Task Statement

A bakery uses 8 tablespoons of honey for every 10 cups of flour to make bread dough. Some days they bake bigger batches and some days they bake smaller batches, but they always use the same ratio of honey to flour.

1. Complete the table.
2. If f is the cups of flour needed for h tablespoons of honey, write an equation that relates f and h.
3. How much flour is needed for 15 tablespoons of honey? 17 tablespoons? Explain or show your reasoning.

honey (tbsp)	flour (cup)
1	
8	10
16	
20	
h	

Student Response

1.

honey (tbsp)	flour (cup)
1	$\frac{5}{4}$
8	10
16	20
20	25
h	$\frac{5}{4}h$

2. $f = \frac{5}{4}h$ or equivalent.

3. $18\frac{3}{4}$ cups. $21\frac{1}{4}$ cups. I multiplied each number by $\frac{5}{4}$.

Activity Synthesis

Ask students to compare answers with their partner and discuss their reasoning until they reach an agreement.

Then, invite students to share how they used their equation from question 2 to answer question 3 with the whole class.

Lesson Synthesis

Briefly revisit the three activities, demonstrating the use of new and old terms. For example:

- We examined a proportional relationship between cups of rice and people served. What was the *constant of proportionality* in this task? What did the constant of proportionality represent? What *equation* did we write for this situation?
- We examined a proportional relationship where we knew that a plane was flying at a constant speed. What was the constant of proportionality for this relationship? What does the constant of proportionality represent in terms of the context? What equation did we determine would represent this situation?

As a way to help students synthesise their learning, consider asking them to work with a partner and create a mind map of the features they have noticed these proportional relationships have in common. You might collect these to check for understanding. A class version can be created to be referenced, revised, or augmented during the unit.

4.5 It's Snowing in Syracuse

Cool Down: 5 minutes

Student Task Statement

Snow is falling steadily in Syracuse, New York. After 2 hours, 4 inches of snow has fallen.

1. If it continues to snow at the same rate, how many inches of snow would you expect after 6.5 hours? If you get stuck, you can use the table to help.
2. Write an equation that gives the amount of snow that has fallen after x hours at this rate.
3. How many inches of snow will fall in 24 hours if it continues to snow at this rate?

time (hours)	snow (inches)
	1
1	
2	4
6.5	
x	

Student Response

1. 13 inches. Two inches fell in 1 hour, and 6.5 is $1 \times (6.5)$, and $2 \times (6.5) = 13$.

2. $y = 2x$ where x is the number of hours that have passed and y is the depth of the accumulated snow.

3. 48 inches. $24 \times 2 = 48$.

Student Lesson Summary

The table shows the amount of red paint and blue paint needed to make a certain shade of purple paint, called Venusian Sunset.

Note that "parts" can be *any* unit for volume. If we mix 3 cups of red with 12 cups of blue, you will get the same shade as if we mix 3 teaspoons of red with 12 teaspoons of blue.

red paint (parts)	blue paint (parts)
3	12
1	4
7	28
$\frac{1}{4}$	1
r	$4r$

The last row in the table says that if we know the amount of red paint needed, r, we can always multiply it by 4 to find the amount of blue paint needed, b, to mix with it to make Venusian Sunset. We can say this more succinctly with the equation $b = 4r$. So the amount of blue paint is proportional to the amount of red paint and the constant of proportionality is 4.

We can also look at this relationship the other way around.

If we know the amount of blue paint needed, b, we can always multiply it by $\frac{1}{4}$ to find the amount of red paint needed, r, to mix with it to make Venusian Sunset. So $r = \frac{1}{4}b$. The amount of red paint is proportional to the amount of blue paint and the constant of proportionality $\frac{1}{4}$.

blue paint (parts)	red paint (parts)
12	3
4	1
28	7
1	$\frac{1}{4}$

b	$\frac{1}{4}b$

In general, when y is proportional to x, we can always multiply x by the same number k—the constant of proportionality—to get y. We can write this much more succinctly with the equation $y = kx$.

Lesson 4 Practice Problems

1. **Problem 1 Statement**

   A certain ceiling is made up of tiles. Every square metre of ceiling requires 10.75 tiles. Fill in the table with the missing values.

square metres of ceiling	number of tiles
1	
10	
	100
a	

   Solution

square metres of ceiling	number of tiles
1	10.75
10	107.5
9.3	100
a	$10.75 \times a$

2. **Problem 2 Statement**

   On a flight from New York to London, an airplane travels at a constant speed. An equation relating the distance travelled in miles, d, to the number of hours flying, t, is $t = \frac{1}{500}d$. How long will it take the airplane to travel 800 miles?

   Solution

   1.6 hours since $\frac{1}{500} \times 800 = 1.6$

3. **Problem 3 Statement**

   Each table represents a proportional relationship. For each, find the constant of proportionality, and write an equation that represents the relationship.

s	P
2	8
3	12
5	20
10	40

Constant of proportionality:

Equation: $P =$

d	C
2	6.28
3	9.42
5	15.7
10	31.4

Constant of proportionality:

Equation: $C =$

Solution

a. Constant of proportionality: 4. Equation: $P = 4s$

b. Constant of proportionality: 3.14 Equation: $C = 3.14d$

4. **Problem 4 Statement**

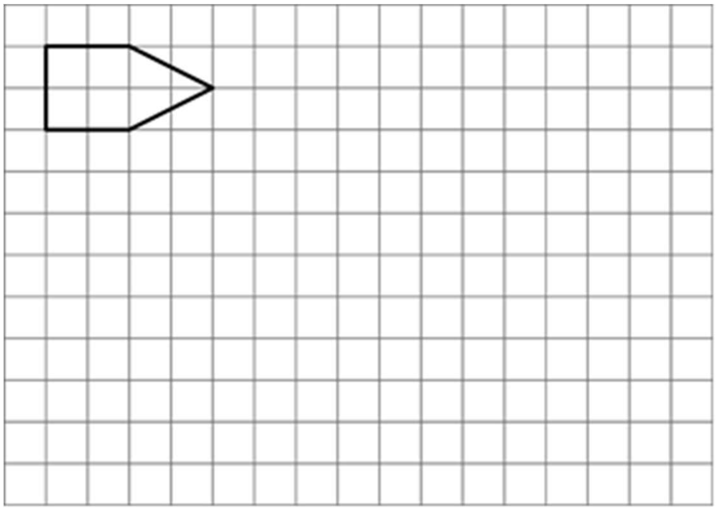
A map of Colorado says that the scale is 1 inch to 20 miles or 1 to 1 267 200. Are these two ways of reporting the scale the same? Explain your reasoning.

Solution

Yes. Sample response: There are 12 inches in a foot and 5 280 feet in 1 mile, so that's 63 360 inches in a mile and 1 267 200inches in 20 miles.

5. **Problem 5 Statement**

Here is a polygon on a grid.



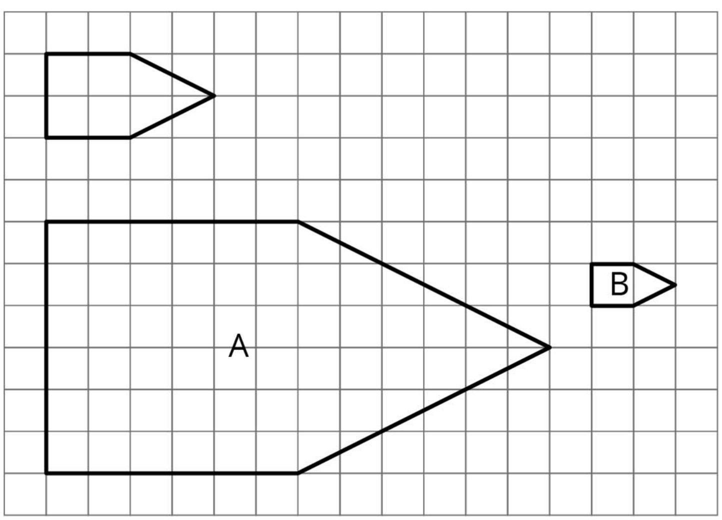
a. Draw a scaled copy of the polygon using a scale factor 3. Label the copy A.

b. Draw a scaled copy of the polygon with a scale factor $\frac{1}{2}$. Label it B.

c. Is Polygon A a scaled copy of Polygon B? If so, what is the scale factor that takes B to A?

Solution

a.



b. Yes, A is a scaled copy of B with a scale factor of 6.