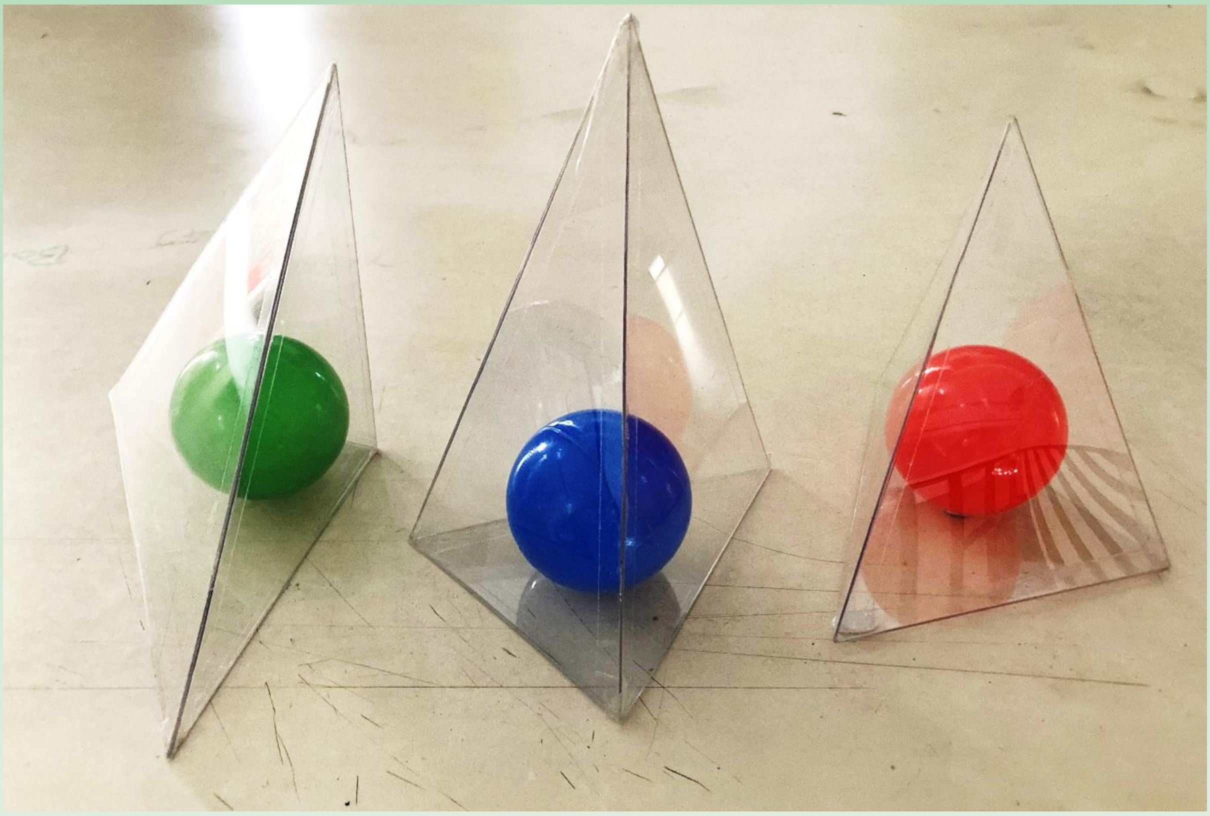
Mathematics Project Competition for Secondary Schools 2023/24



Title: Incenters and Inspheres of Tetrahedrons

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<table>
<tr><td colspan="5">Mathematics Project Competition (2023/24)
數學專題習作比賽 (2023/24)
Information Sheet 資料頁</td></tr>
<tr><td>Category
參賽組別</td><td colspan="4">☑* A 組：初中習作 (Category A: Junior secondary project)
☐* B 組：中一小型習作 (Category B: S1 mini-project)</td></tr>
<tr><td>Title of Project
專題習作題目</td><td colspan="4">Incenters and Inspheres of Tetrahedrons</td></tr>
<tr><td>Name of School</td><td colspan="4">Wah Yan College, Kowloon</td></tr>
<tr><td>學校名稱</td><td colspan="4">華仁書院 (九龍)</td></tr>
<tr><td rowspan="7">Team members
隊員</td><td colspan="2">Name in English</td><td>中文姓名</td><td>Class 班別</td></tr>
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</table>

Content

1. Introduction

In Form 3, we have learnt the perpendicular bisector property and angle bisector property, and we learnt how to apply these properties to construct the circumcenter and incenter of triangles.

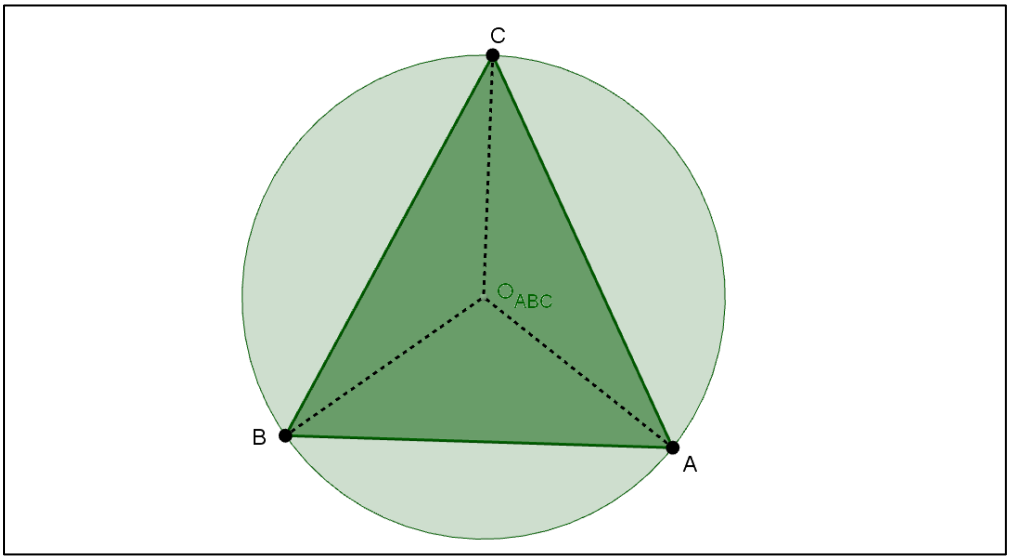
Meanwhile, we know that there is a question in HKDSE 2023 relating to 3D trigonometry, involving tetrahedrons and the circumcenter of triangles.

It inspires us to extend our knowledge on centers of triangles in 2D. How can we construct the circumcenter and incenter of tetrahedrons in 3D?

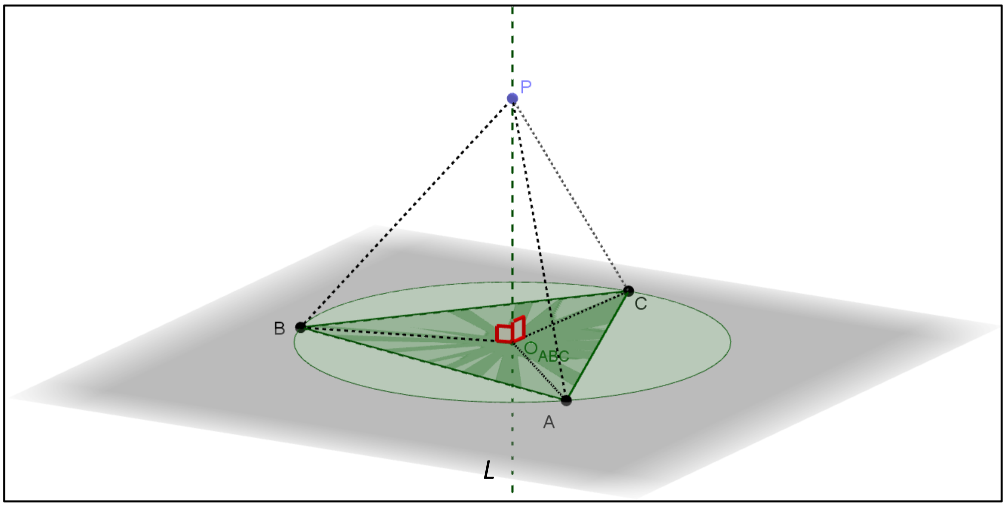
Finally, we find that it is quite easy to construct the circumcenter of tetrahedrons. But the story is totally different for the incenter of tetrahedrons. In this project, we would like to share our finding, and our journey of studying the incenter and insphere of tetrahedrons.

2. From 2-D to 3-D: Constructing the circumcenter of a tetrahedron by the circumcenters of its faces

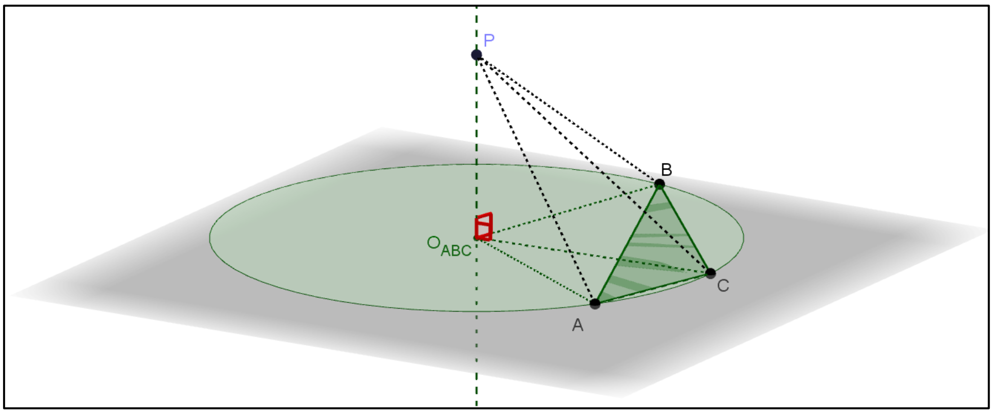
How can we make use of the circumcenters of triangles to construct the circumcenter of a tetrahedron? Recall that if O_{ABC} is the circumcenter of $\triangle ABC$, then we have $AO_{ABC} = BO_{ABC} = CO_{ABC}$.



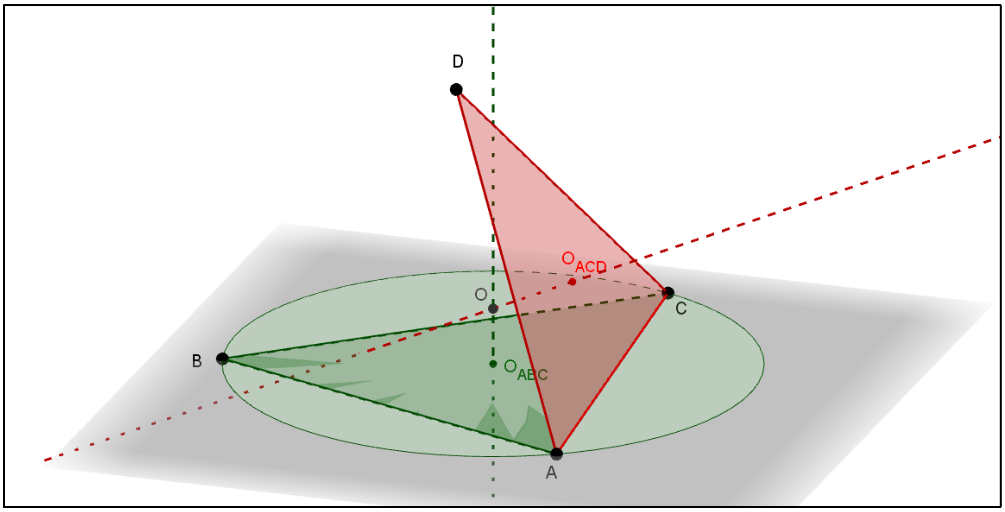
If L is a line perpendicular to $\triangle ABC$ (i.e. $L \perp AO_{ABC}$, $L \perp BO_{ABC}$ and $L \perp CO_{ABC}$) and pass through O_{ABC}, then for any point P lies on L we have PA = PB = PC (since $\triangle PAO_{ABC} \cong \triangle PBO_{ABC} \cong \triangle PCO_{ABC}$ (S.A.S.)).



Remark: Note that the above result is also valid no matter △ABC is acute, obtuse or right-angled.

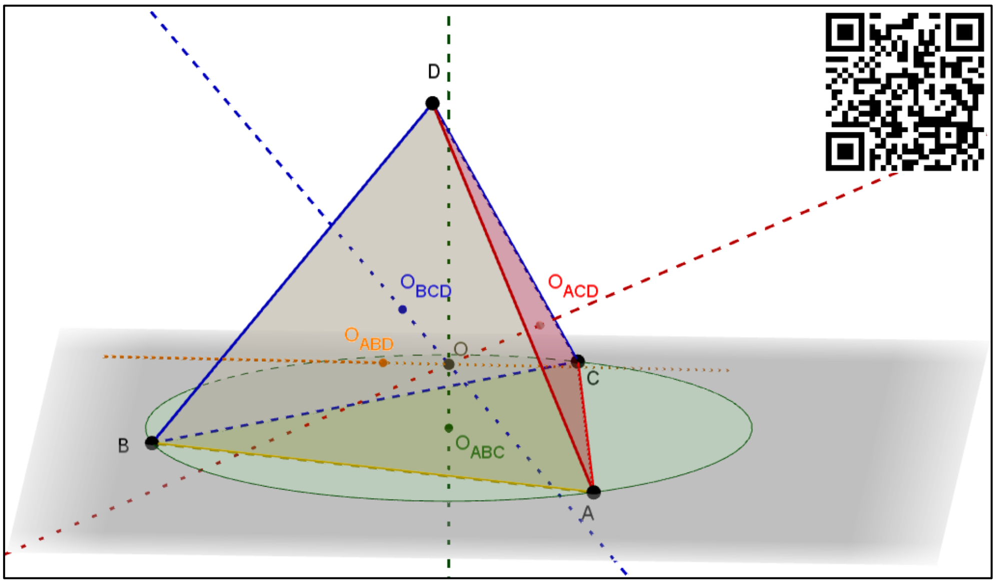


Therefore, by choosing any two faces of a tetrahedron ABCD, we can construct two lines which are perpendicular to the faces and passing through their circumcenters respectively. If the two lines intersect at O, then we have AO = BO = CO = DO.



In fact, for a tetrahedron ABCD with faces △ABC, △ABD, △ACD and △BCD, let's denote their circumcenters be O_{ABC}, O_{ABD}, O_{ACD} and O_{BCD} respectively, and construct the four perpendiculars which passes through the circumcenters and their corresponding faces.

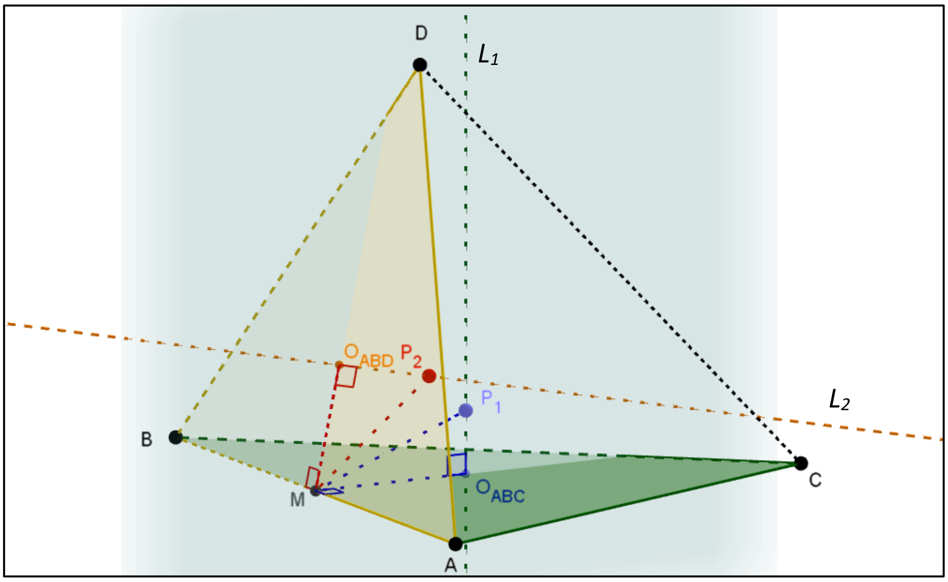
Then all the perpendiculars are concurrent. The intersection of the perpendiculars is the circumcenter of the tetrahedron.



(GeoGebra file available at https://www.geogebra.org/m/tgmxavre)

It is easy to derive that all the perpendiculars are concurrent. The proof is as follows:

For tetrahedron ABCD, consider △ABC and △ABD and let M be the mid-point of AB.



Denote the circumcenters of △ABC and △ABD by O_{ABC} and O_{ABD} respectively.

Construct a line L_1 which passes through O_{ABC} and perpendicular to △ABC, and a line L_2 which passes through O_{ABD} and perpendicular to △ABD.

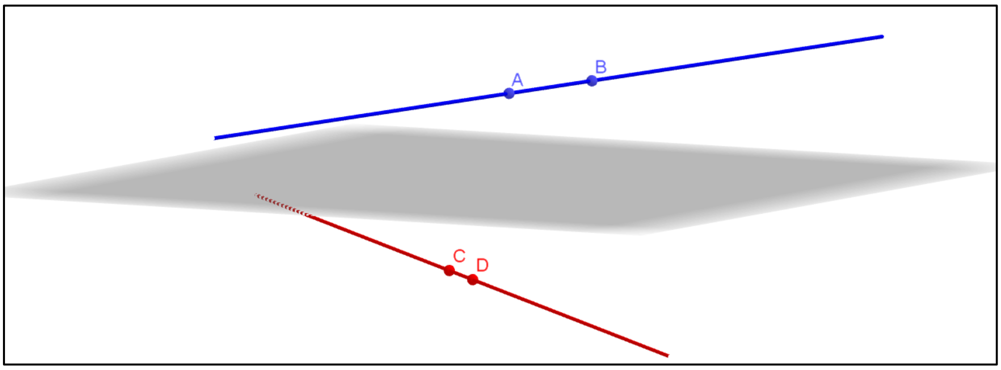
Let P_1 and P_2 be points on L_1 and L_2 respectively.

Then

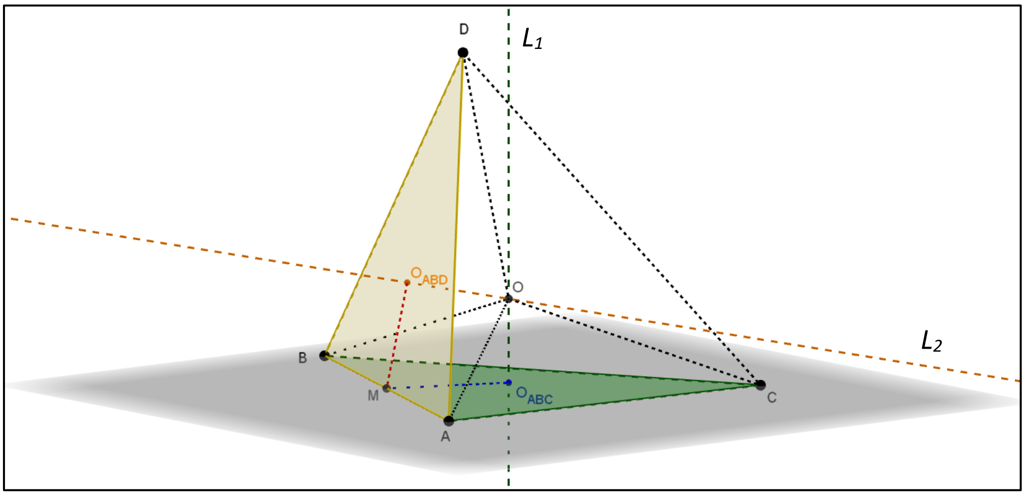
$L_1 \perp MO_{ABC}$	and	$L_2 \perp MO_{ABD}$	(by definition)
$AB \perp MO_{ABC}$	and	$AB \perp MO_{ABD}$	(circumcenter)
$AB \perp MP_1$	and	$AB \perp MP_2$	(theorem of three perpendiculars)

Therefore, all the points O_{ABC}, O_{ABD}, P_1 and P_2 lie on the plane which passes through M and perpendicular to △ABC and △ABD. In other words, L_1 and L_2 lie on the same plane. Since L_1 and L_2 are not parallel, they will intersect at a point.

(Note: It is necessary to show that L_1 and L_2 lie on the same plane as two non-parallel lines in 3-dimensional space may not have intersection (refer to the figure below).)



Now denote the intersection of L_1 and L_2 be O.



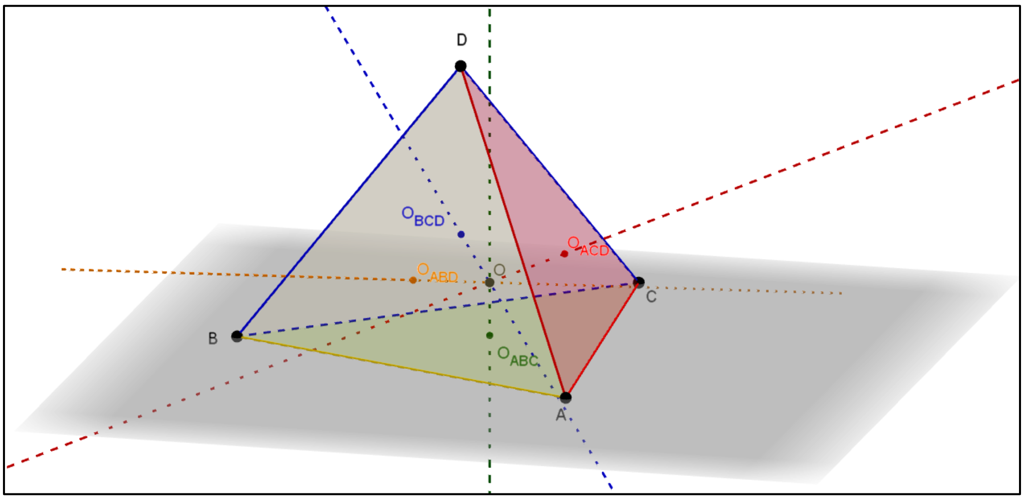
Then O lies on L_1 , deduces that OA = OB = OC.

Meanwhile, O lies on L_2 deduces OA = OB = OD.

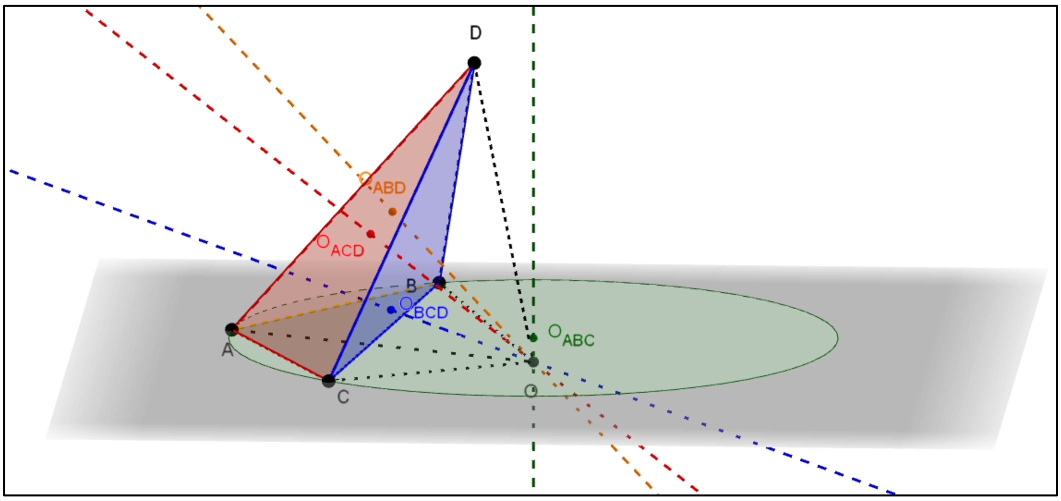
Therefore, OA = OB = OC = OD.

As OA = OC = OD, the line perpendicular to △ACD and passing through O_{ACD} (circumcenter of △ACD) will pass through O.

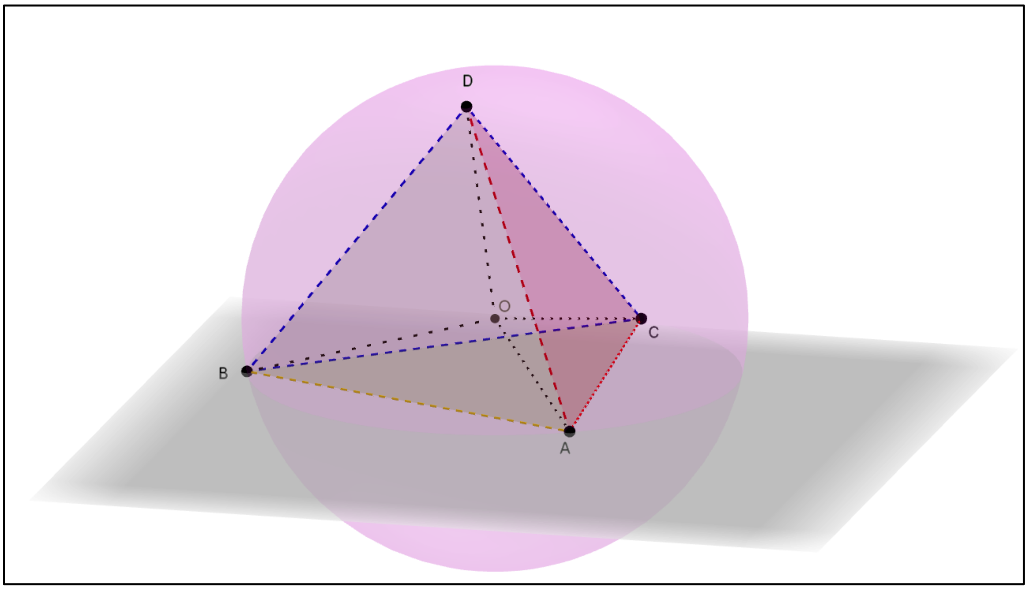
Similarly, the line perpendicular to △BCD and passing through O_{BCD} (circumcenter of △BCD) will pass through O also.



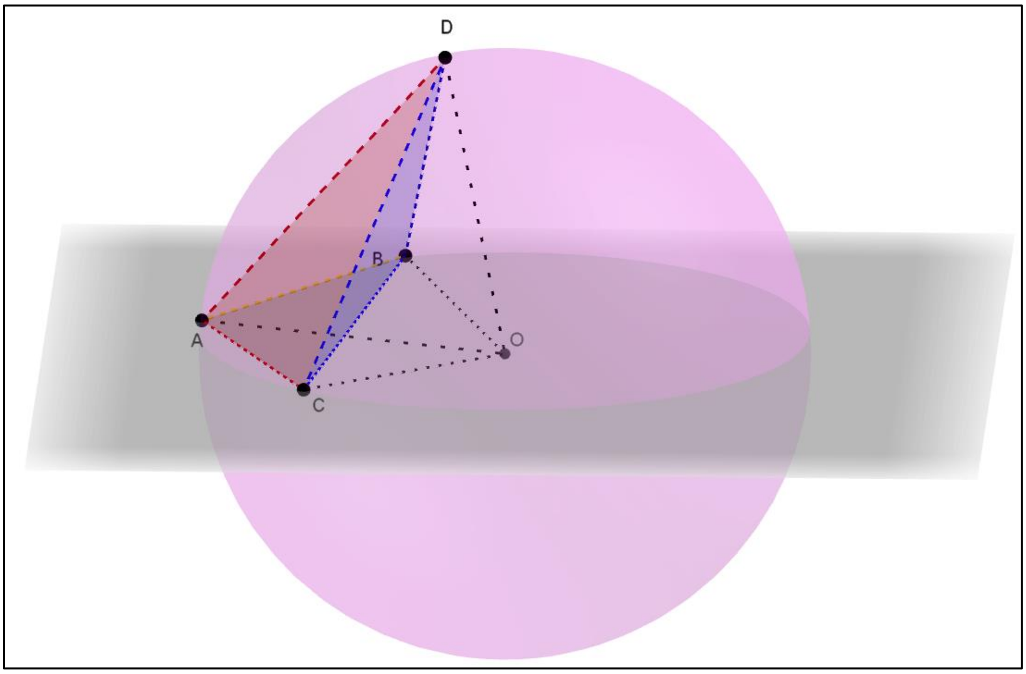
In conclusion, all the perpendiculars of the faces passing through the circumcenters of the corresponding faces are concurrent at point O, which is the circumcenter of the tetrahedron. It is true for all tetrahedrons with different shapes.



As OA = OB = OC = OD, we can find a sphere with center O and passing through all the vertices of the tetrahedron ABCD. It is the circumsphere of the tetrahedron ABCD.



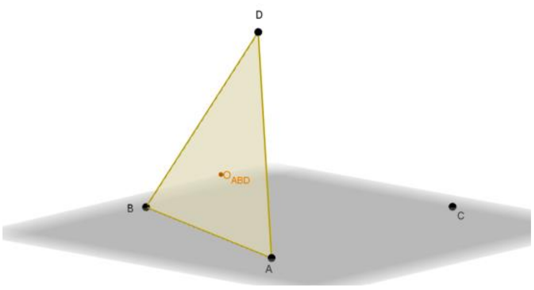
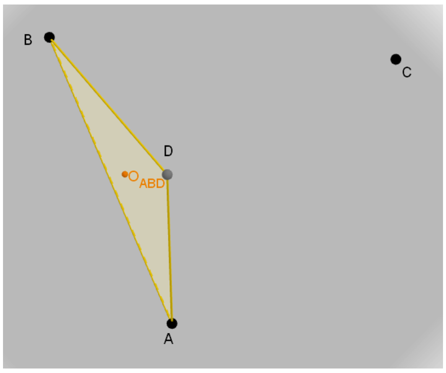
Note that the circumcenter of tetrahedron is not always inside a tetrahedron. For example, the following tetrahedron has its circumcenter lies outside the tetrahedron:



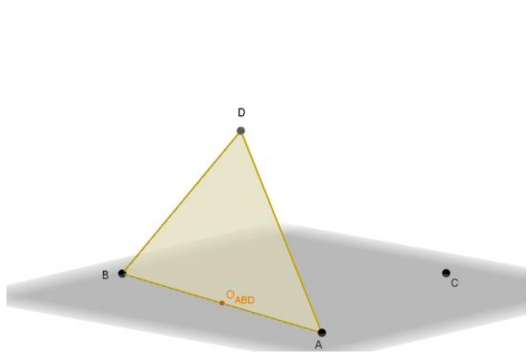
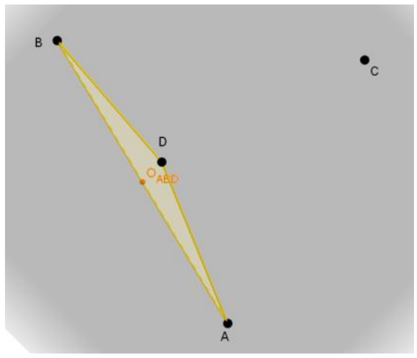
From the above example, it is suggested that the circumcenter of a tetrahedron will lie outside the tetrahedron if at least one of its faces is obtuse-angled triangle.

In GeoGebra, it is quite difficult to determine whether a triangle in 3-D is an obtuse-angled triangle by observing its shape. Nevertheless, we can determine that by the location of the circumcenter instead.

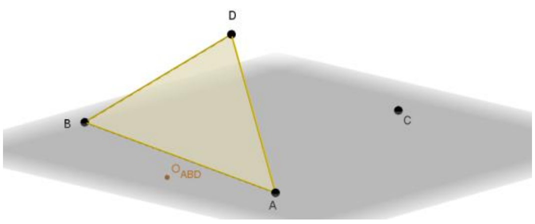
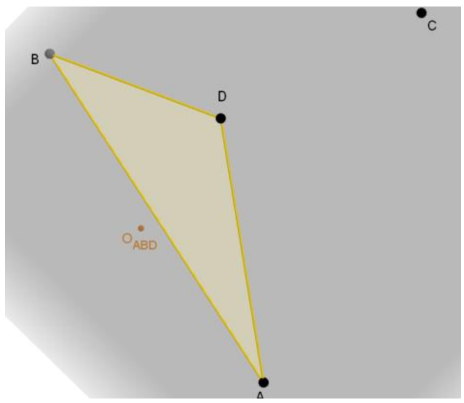
For example, △ABD below is an acute-angled triangle. Note that its circumcenter O_{ABD} lies inside △ABD:

3-D view of △ABD	Top view of △ABD
	

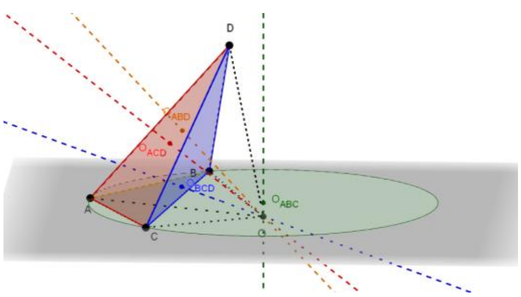
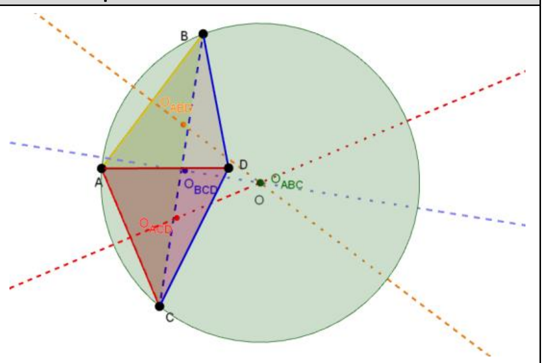
△ABD below is a right-angled triangle and its circumcenter O_{ABD} lies on AB:

3-D view of △ABD	Top view of △ABD
	

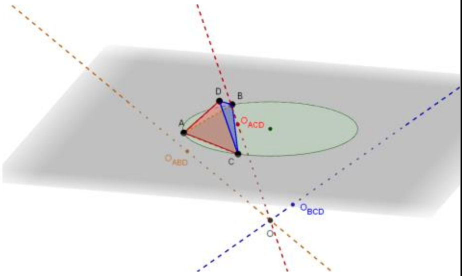
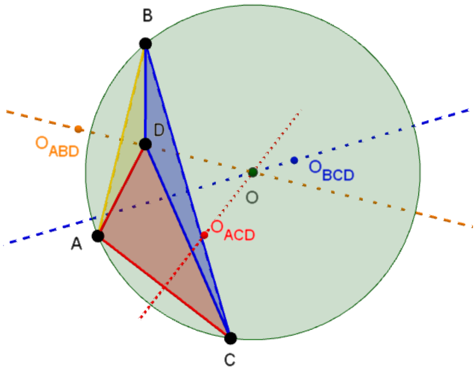
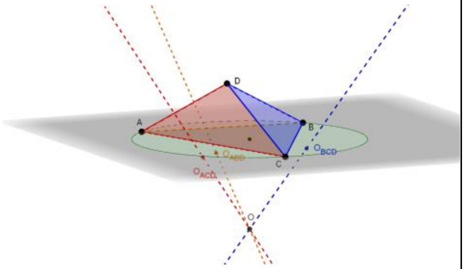
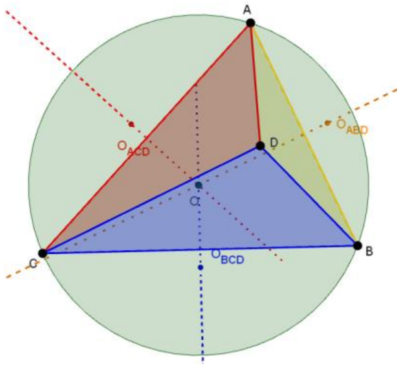
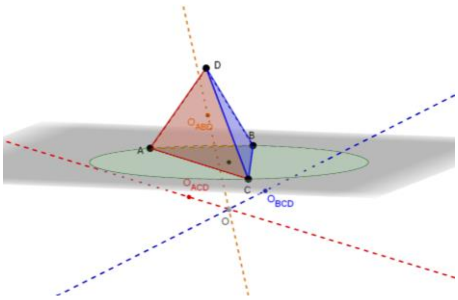
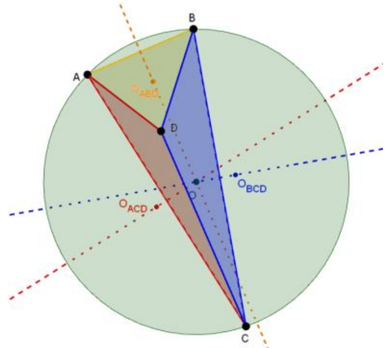
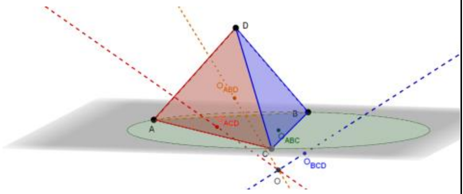
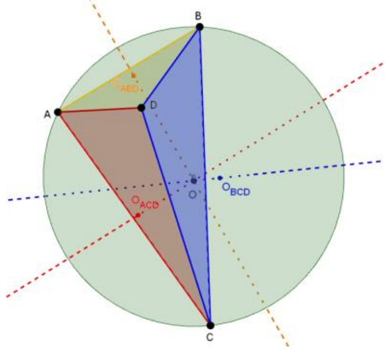
△ABD below is an obtuse-angled triangle and its circumcenter O_{ABD} lies outside △ABD:

3-D view of △ABD	Top view of △ABD
	

Then using the tetrahedron ABCD below as an example, only the circumcenter of △ABC (O_{ABC}) lies outside the triangle, whilst the circumcenters of △ABD, △ACD and △BCD (O_{ABD}, O_{ACD} and O_{BCD} respectively) lie inside the corresponding triangles. Therefore, we can determine that only 1 out of 4 of the faces of the tetrahedron ABCD is obtuse-angled triangle.

3-D view of tetrahedron ABCD	Top view of tetrahedron ABCD
	

In the following cases, all the circumcenter O of the tetrahedron ABCD lie outside it:

Case	3-D view of tetrahedron ABCD	Top view of tetrahedron ABCD
All faces are obtuse-angled triangles	 All O_{ABC}, O_{ABD}, O_{ACD} and O_{BCD} lie outside the triangles. O lies outside the tetrahedron.	
3 out of 4 faces (△ABD, △ACD and △BCD) are obtuse-angled triangles	 O_{ABD}, O_{ACD} and O_{BCD} lie outside the triangles. O lies outside the tetrahedron.	
2 out of 4 faces (△ACD and △BCD) are obtuse-angled triangles	 O_{ACD} and O_{BCD} lie outside the triangles. O lies outside the tetrahedron.	
Only 1 out of 4 faces (△BCD) is obtuse-angled triangle	 Only O_{BCD} lie outside the triangle. O lies outside the tetrahedron.	

By varying the shape of the tetrahedron ABCD, it shows that the circumcenter of a tetrahedron will lie outside the tetrahedron if at least one of its faces is obtuse-angled triangle:

The above cases show that the following statement is true:

"If at least one of the faces of a tetrahedron is an obtuse-angled triangle, then the circumcenter of the tetrahedron will lie outside it."

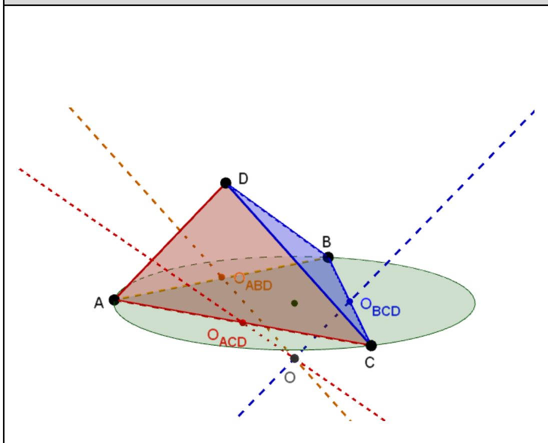
However, we find the converse of the statement is not true. That is:

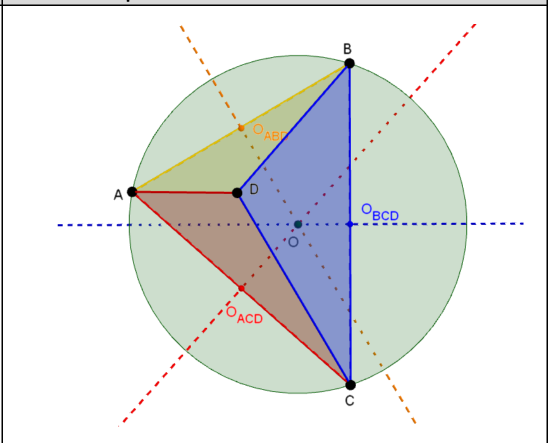
"If the circumcenter of a tetrahedron lies outside it, it does NOT imply that at least one of the faces of the tetrahedron is an obtuse-angled triangle."

For example,

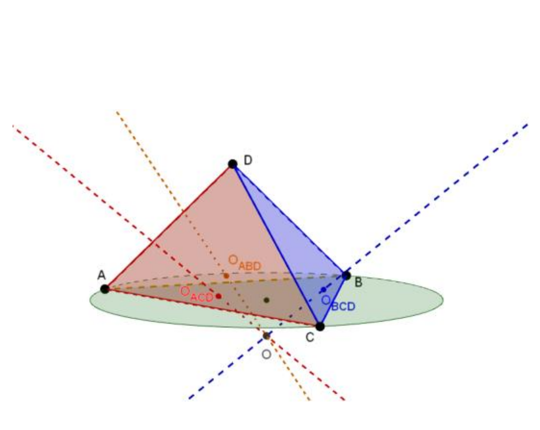
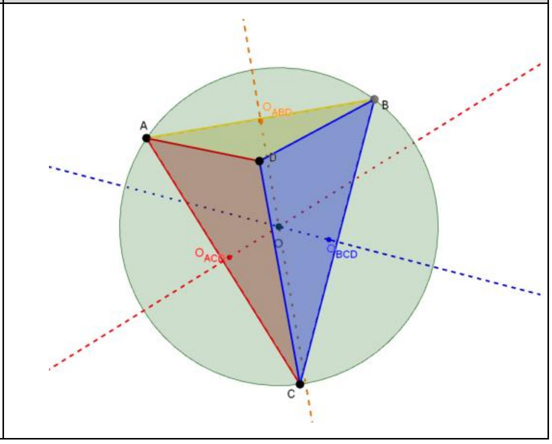
1. △ABD, △ACD and △BCD are right-angled triangles (their corresponding circumcenters O_{ABD}, O_{ACD} and O_{BCD} lie on the mid-point of AB, AC and BC respectively) and △ABC is an acute-angled triangle, but the circumcenter O of tetrahedron ABCD still lies outside it.

3-D view of tetrahedron ABCD	Top view of tetrahedron ABCD

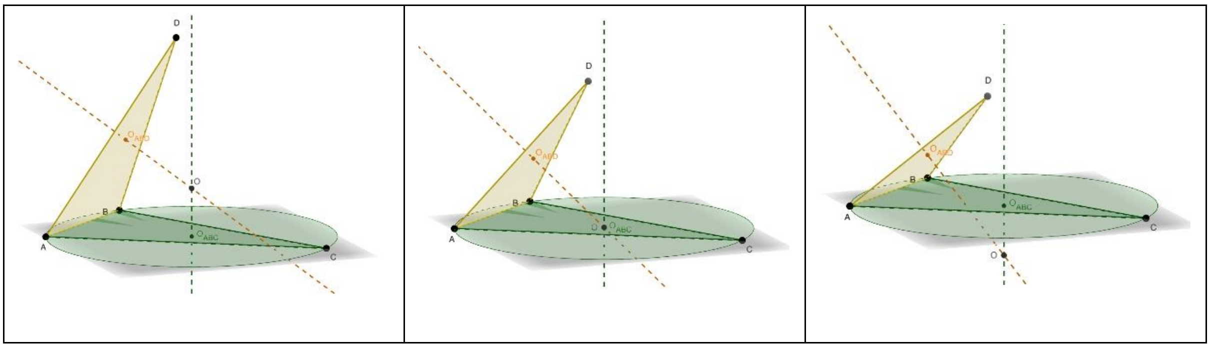




2. All △ABC, △ABD, △ACD and △BCD are acute-angled triangles, but the circumcenter O of tetrahedron ABCD still lies outside it.

3-D view of tetrahedron ABCD	Top view of tetrahedron ABCD
	

In fact, we notice that even △ABC, △ABD, △ACD and △BCD are acute-angled triangles, the circumcenter O of the tetrahedron ABCD will be shifted from inside to outside of the tetrahedron if the angle between faces is decreased.

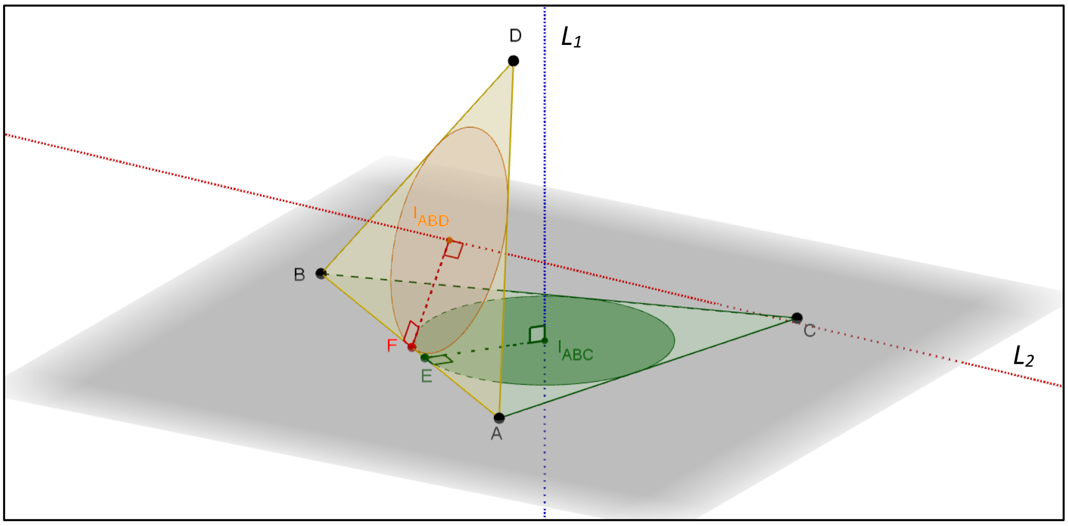


So, what is/are the condition(s) for the circumcenter of a tetrahedron lies outside it? This is not an easy question and probably we will conduct further exploration in the future.

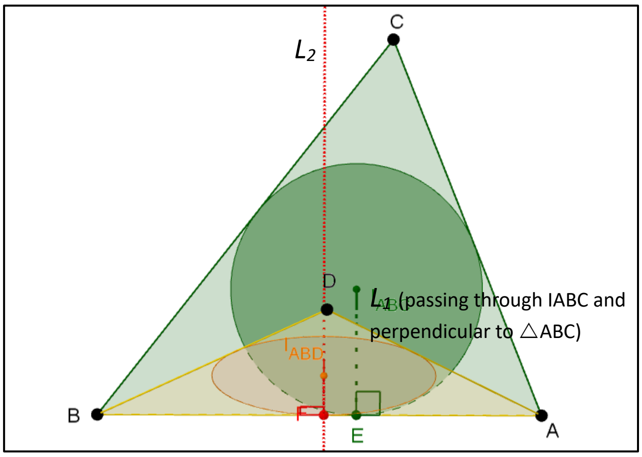
3. Can we locate the insphere of a tetrahedron by the incenter of its faces?

Different from the circumcenter, we find that it is almost impossible to use the incenters of the faces of a tetrahedron to locate the incenter of the tetrahedron! The reason is as follows:

- Denote the incenters of △ABC and △ABD be I_{ABC} and I_{ABD} respectively.
- Construct a line L_1 which passes through I_{ABC} and perpendicular to △ABC, and another line L_2 which passes through I_{ABD} and perpendicular to △ABD. Let E and F be the foot of perpendicular from I_{ABC} to AB and I_{ABD} to AB respectively.
- Unless I_{ABC} and I_{ABD} are also the circumcenters of △ABC and △ABD, or else E and F will be two distinct points on AB. It makes L_1 and L_2 lie on two different planes. In other words, L_1 and L_2 have no intersection.

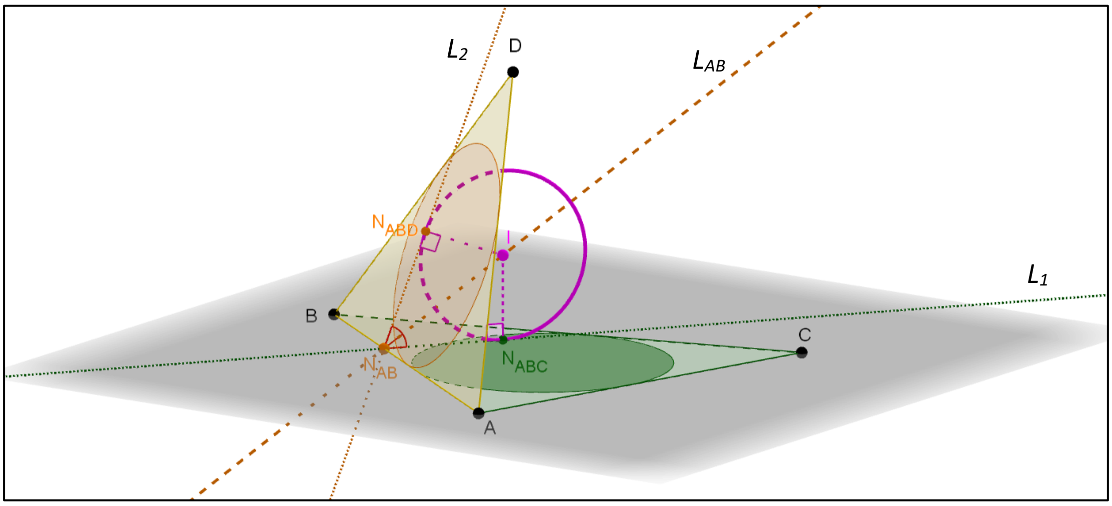


From the top view, it is easier to observe that L_1 and L_2 have no intersection.



In fact, the incenter of a tetrahedron should lie on angle bisectors of its faces. For example, by considering two faces △ABC and △ABD of a tetrahedron ABCD:

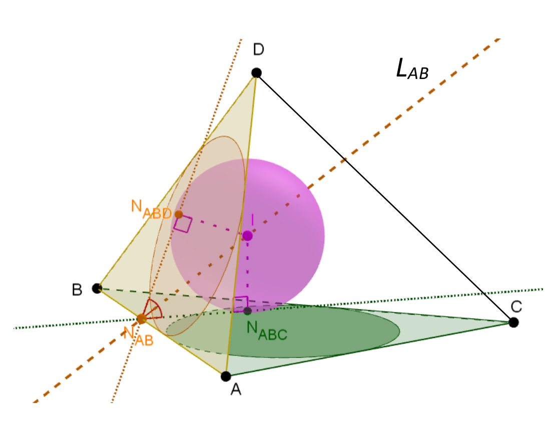
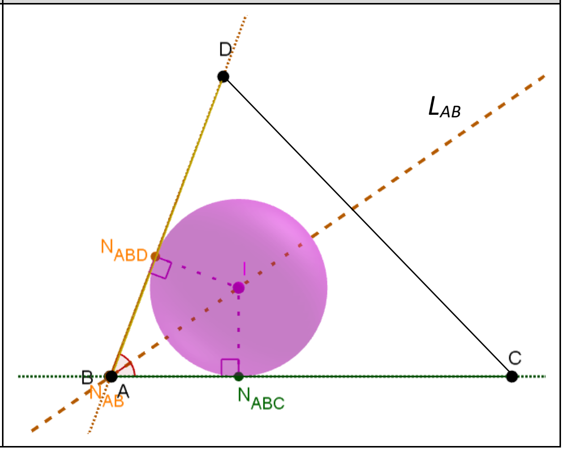
- Let N_{AB} be a point lying on AB.
- Denote L_1 be a line lying on the plane consisting of △ABC and passing through N_{AB} such that $L_1 \perp AB$.
- Denote L_2 be a line lying on the plane consisting of △ABD and passing through N_{AB} such that $L_2 \perp AB$.
- Denote L_{AB} be the angle bisector of the angle between L_1 and L_2. Note that L_{AB} is also the angle bisector of the angle between △ABC and △ABD.
- Let I be a point lies on L_{AB}.



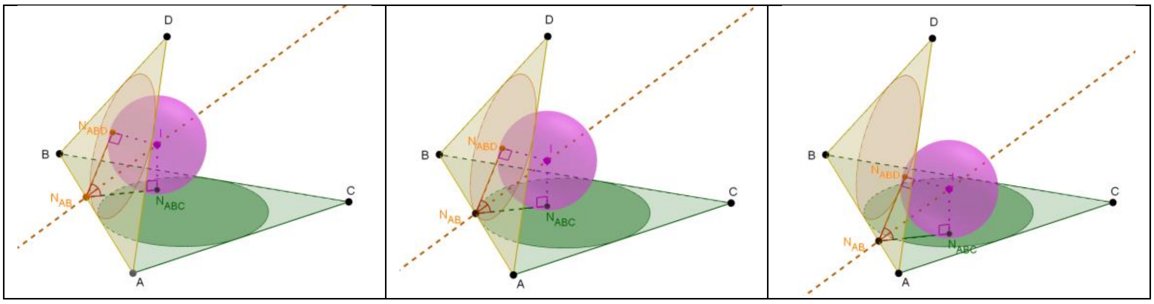
Since $L_1 \perp AB$, $L_2 \perp AB$ and L_{AB} be the angle bisector of the angle between L_1 and L_2, all L_1, L_2 and L_{AB} lie on the same plane.

From angle bisector property, it shows that the perpendicular distance from I to L_1 is equal to the perpendicular distance from I to L_2 (that is $IN_{ABC} = IN_{ABD}$ in the above figure).

The above result also implies that the perpendicular distance from I to △ABC is equal to the perpendicular distance from I to △ABD. In short, if we want to construct a sphere touching both △ABC and △ABD, the center of the sphere must lie on the angle bisector of the angle between △ABC and △ABD.

3-D view of tetrahedron ABCD	Side view of tetrahedron ABCD
	

However, the location of angle bisector of the angle between △ABC and △ABD can be varied as follows:



Therefore, locating incenter of a tetrahedron by using angle bisectors of the angles between the faces of the tetrahedron is very challenging. As the location of the angle bisectors can be varied, how can we locate the angle bisectors in order to find their intersection (which is incenter of the tetrahedron) successfully?

To facilitate our study, we construct a 3-D figure in GeoGebra which consists of:

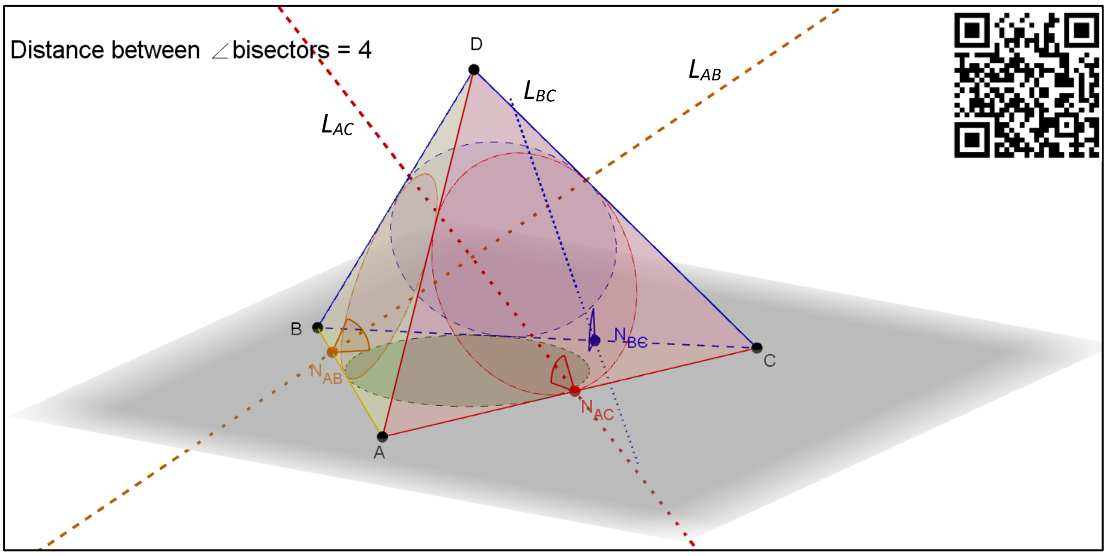
- a tetrahedron ABCD
- points N_{AB}, N_{AC} and N_{BC} which lie on AB, AC and BC respectively
- the inscribed circles of △ABC, △ABD, △ACD and △BCD
- angle bisector L_{AB} which passes through N_{AB} and bisects the angle between △ABD and △ABC.
- angle bisector L_{AC} which passes through N_{AC} and bisects the angle between △ACD and △ABC.
- angle bisector L_{BC} which passes through N_{BC} and bisects the angle between △BCD and △ABC.

In order to indicate whether the angle bisectors are concurrent, we calculate the distance between angles bisectors L_{AB}, L_{AC} and L_{BC} by using the inbuilt function in GeoGebra as follows:

Distance between angle bisectors

$$= \frac{1}{2}\,[\text{Distance}(L_{AB}, L_{AC}) + \text{Distance}(L_{AC}, L_{BC}) + \text{Distance}(L_{BC}, L_{AB})]$$

If the above distance is zero, then the angles bisectors L_{AB}, L_{AC} and L_{BC} will be concurrent.

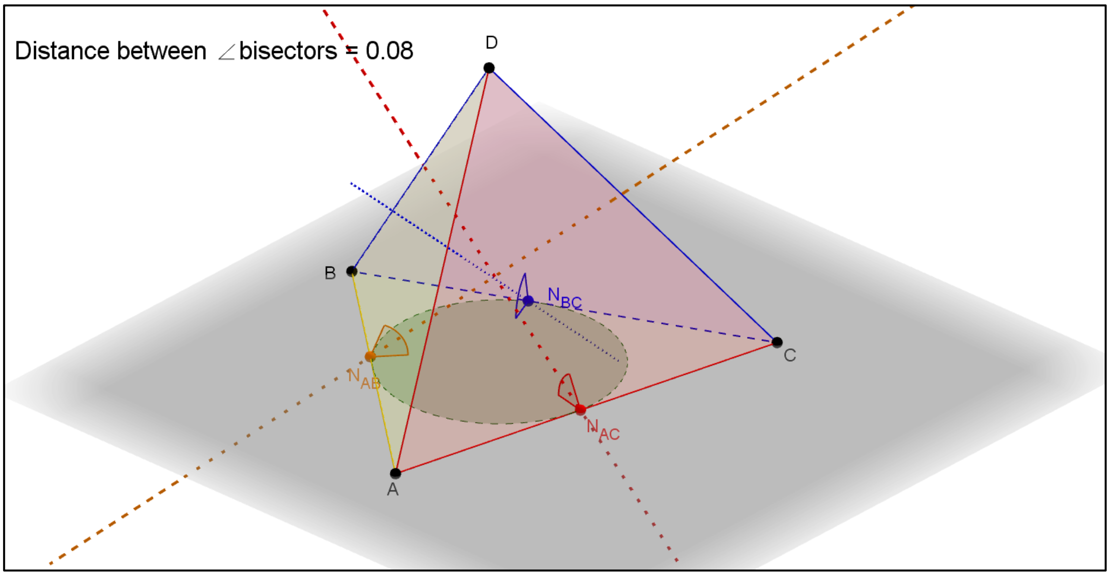


(GeoGebra file available at https://www.geogebra.org/m/szae8xx7)

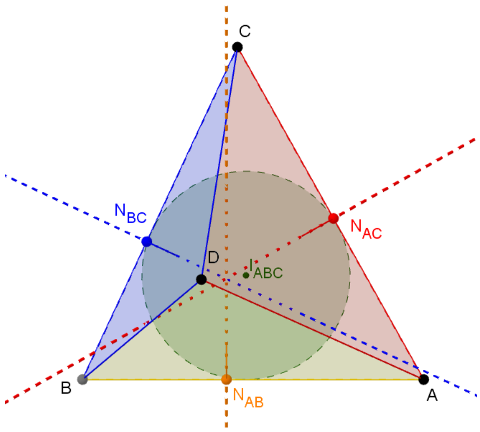
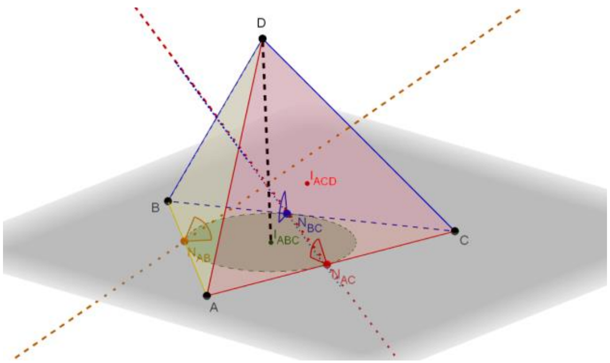
Then we try to change the position of N_{AB}, N_{AC} and N_{BC} to minimize the distance between ∠ bisectors to be zero.

<u>First attempt</u>:

We have minimized the distance between ∠ bisectors to be 0.08 in a tetrahedron with all faces are acute-angled triangles.

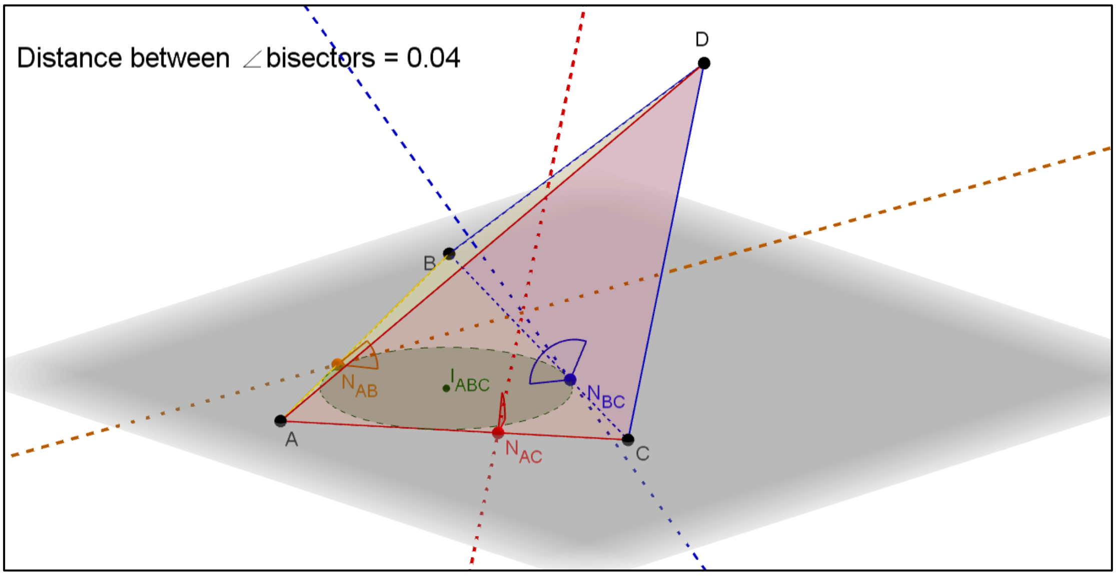


Then we have the following observation:

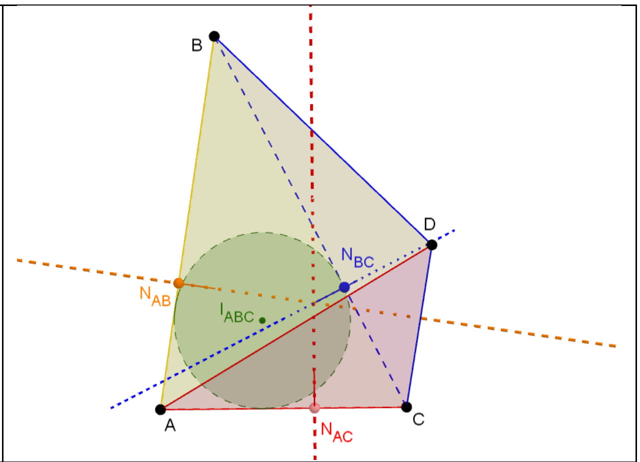
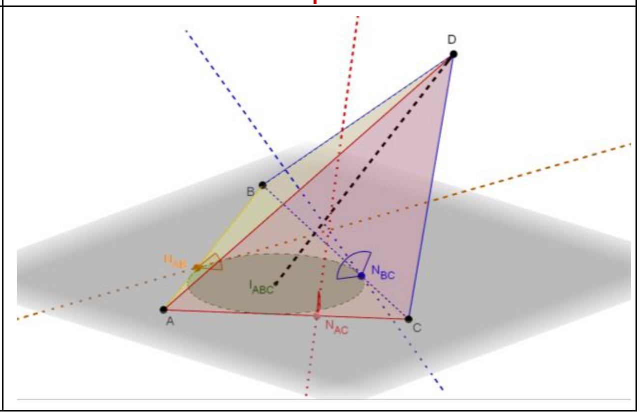
1. From the top view, we see that the projection of the intersection of the three angle bisectors L_{AB}, L_{AC} and L_{BC} does not lie on the incenter I_{ABC} of the base △ABC. 2. N_{AB}, N_{AC} and N_{BC} are close to the points of contact of the inscribed circle of △ABC and AB, AC and BC respectively.	
3. The intersection of the three angle bisectors L_{AB}, L_{AC} and L_{BC} is close to the line segment joining the point D and I_{ABC}.	

Second attempt:

We have minimized the distance between ∠ bisectors to be 0.04 in a tetrahedron with two faces (△ABD and △ACD) are obtuse-angled triangles.



Then we have the following observation:

1. Again, the projection of the intersection of the three angle bisectors L_{AB}, L_{AC} and L_{BC} does not lie on the incenter I_{ABC} of the base △ABC. 2. Different from the first attempt, N_{AC} are far away from the point of contact of the inscribed circle of △ABC and AC.	
3. The intersection of the three angle bisectors L_{AB}, L_{AC} and L_{BC} is far away from the line segment joining the point D and I_{ABC}.	

To conclude, it seems that the incenter of a tetrahedron is not related to the incenters of its faces, and not related to the inscribed circles of its faces. Furthermore, it seems that the incenter of a tetrahedron cannot be located through angle bisectors, which is different from the circumcenter of a tetrahedron.

So, is it possible to locate the incenter of a tetrahedron by construction? In the next part, we will try to locate it by using 3-D trigonometry.

Remark:

From the internet, we find that the incenter of a tetrahedron is the intersection of the dihedral angle bisectors of the tetrahedron. Here is the dihedral angle bisector theorem:

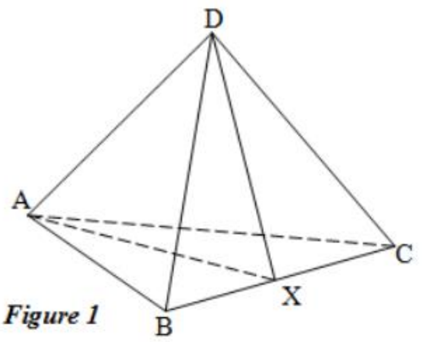


Figure 1

Theorem 1 (Dihedral angle bisector theorem)
Let $ABCD$ be a tetrahedron (see figure 1). Point X is on $\overline{BC}$ and plane AXD is the dihedral angle bisector of dihedral angle $\angle(ADB, ADC)$. Then

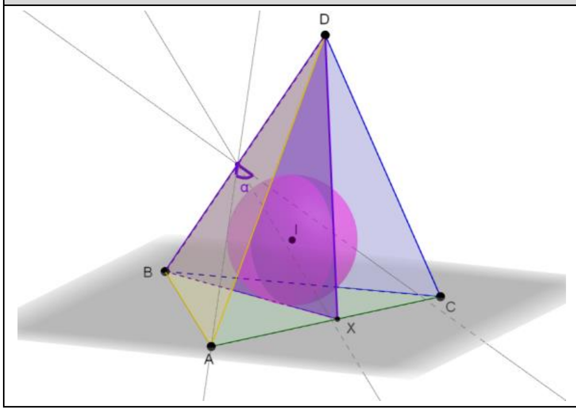
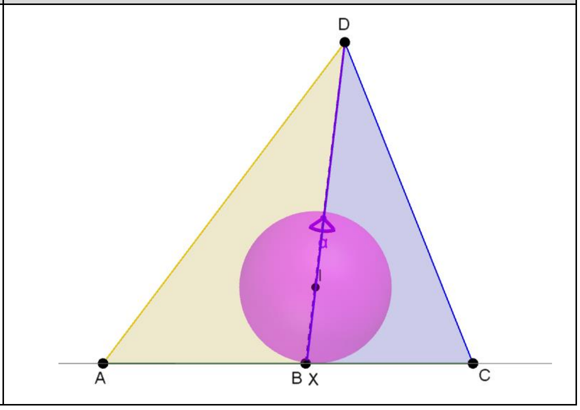
$$\frac{BX}{XC} = \frac{S_{ABD}}{S_{ACD}}.$$

The proof is not long. The ratio $\frac{V_{ABXD}}{V_{ACXD}}$ is equal to $\frac{S_{BXD}}{S_{CXD}}$ because both tetrahedrons have a common height from A to (CBD). The ratio of areas can also be simplified further to $\frac{BX}{XC}$

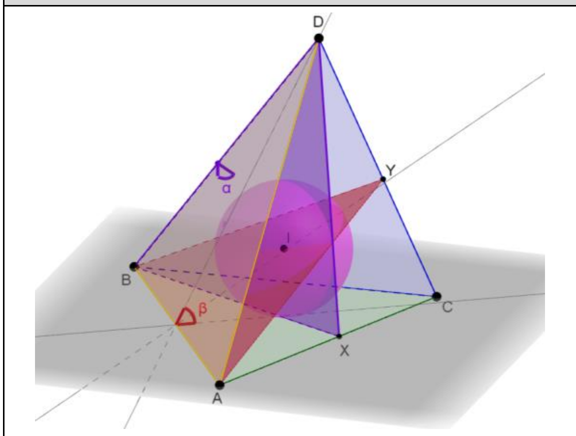
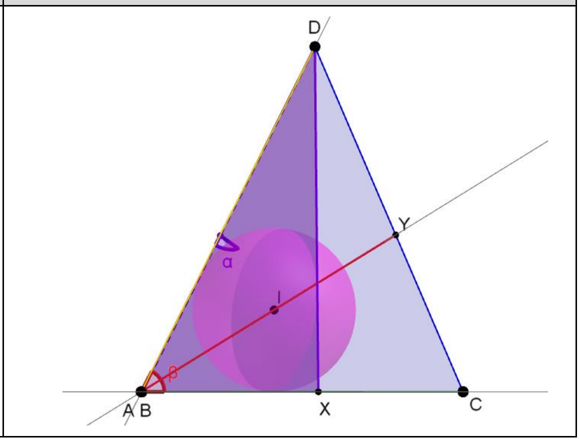
(Extracted from https://math.stackexchange.com/questions/627464/generalization-of-angle-bisector-to-tetrahedron)

In the above figure, △AXD is the dihedral angle bisector of the angles between △ABD and △ACD. To locate the incenter of a tetrahedron, we need to construct three dihedral angle bisectors of the tetrahedron. Here is an example illustrated by GeoGebra:

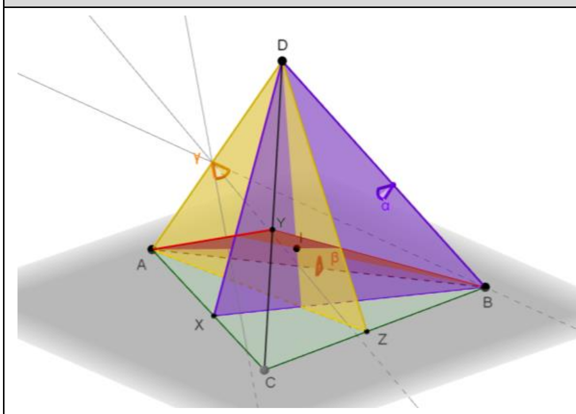
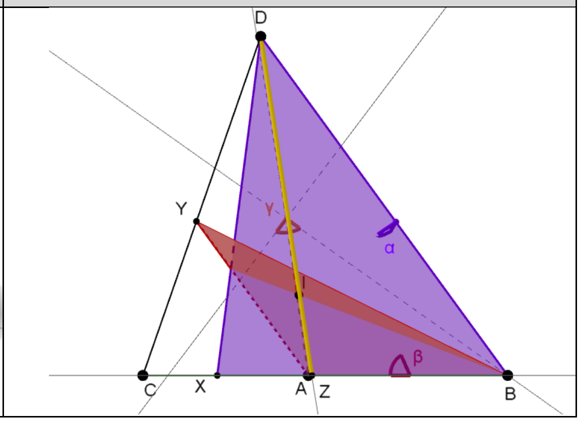
1. α is the angle between △ABD and △BCD, and △BXD bisects α.
   In other words, △BXD is the dihedral angle bisector of △ABD and △BCD.

<table>
<tr><th>3-D view of tetrahedron ABCD</th><th>Side view of tetrahedron ABCD</th></tr>
<tr><td>
</td><td>
</td></tr>
</table>

2. β is the angle between △ABD and △ABC, and △AYB bisects β.
   In other words, △AYB is the dihedral angle bisector of △ABD and △ABC.

<table>
<tr><th>3-D view of tetrahedron ABCD</th><th>Side view of tetrahedron ABCD</th></tr>
<tr><td>
</td><td>
</td></tr>
</table>

3. γ is the angle between △ABD and △ACD, and △AZD bisects γ.
   In other words, △AZD is the dihedral angle bisector of △ABD and △ACD.

<table>
<tr><th>3-D view of tetrahedron ABCD</th><th>Side view of tetrahedron ABCD</th></tr>
<tr><td>
</td><td>
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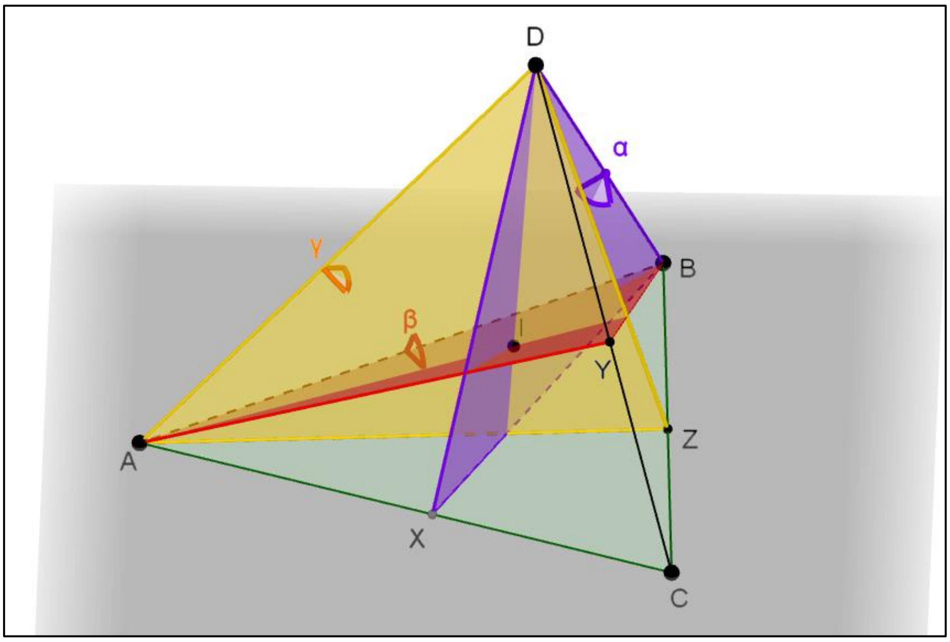
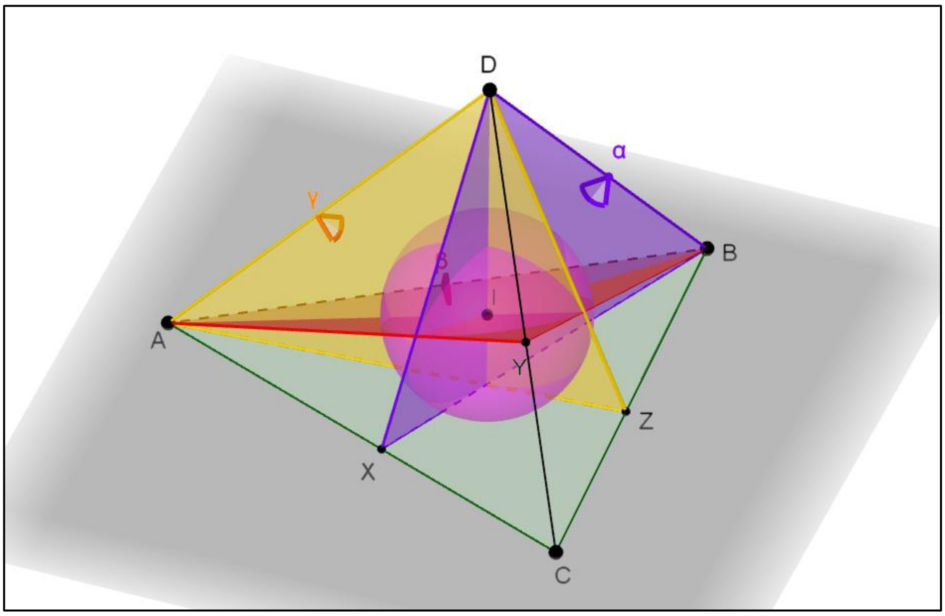
</table>

4. Then the intersection of the dihedral angle bisectors △BXD, △AYB and △AZD is the incenter of the tetrahedron ABCD.



5. Note that the dihedral angle bisectors △BXD, △AYB and △AZD can also cut the insphere into two equal halves.

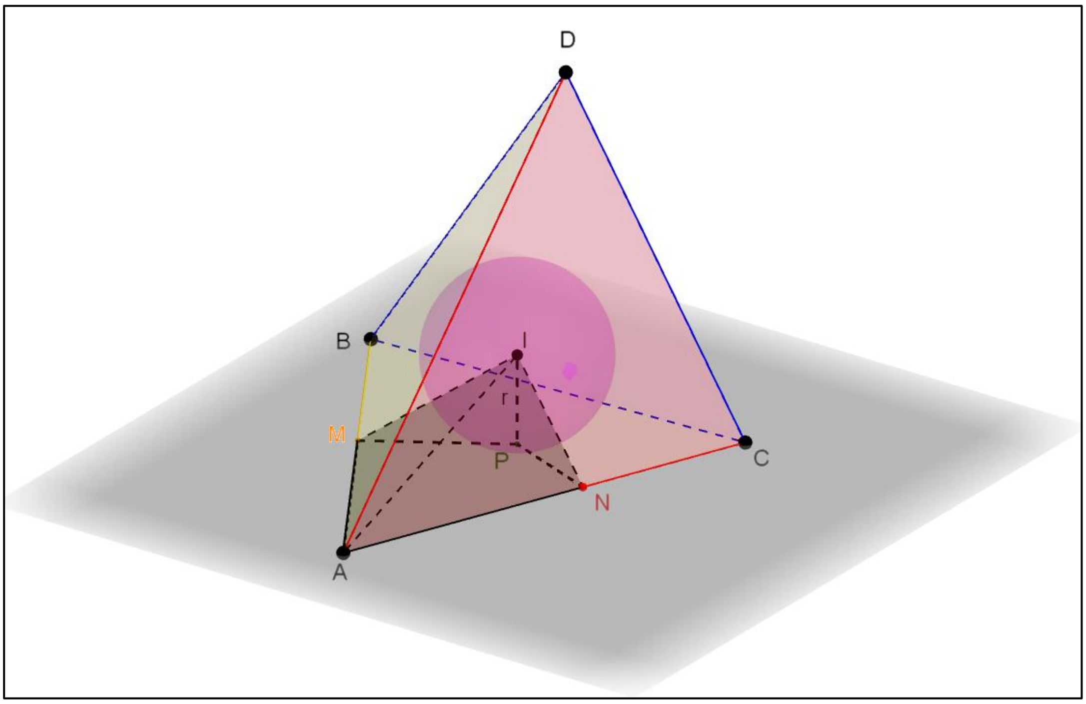


In practice, it is difficult to construct planes and find their intersection. Therefore, we will keep trying to locate the incenter of a tetrahedron by using straight lines in the rest of our project.

4. Locating the incenter of a tetrahedron by 3-D trigonometry

Suppose all the lengths of edges of tetrahedron ABCD (i.e. AB, AC, AD, BC, BD and CD) are known, then we suggest a way to locate the incenter I of the tetrahedron ABCD as follows:

1. Find AM and AN, where M and N lie on AB and AC respectively such that $IM \perp AM$ and $IN \perp AN$.
2. Find $\angle IMP$ and $\angle INP$, where P is the projection of I on $\triangle ABC$.

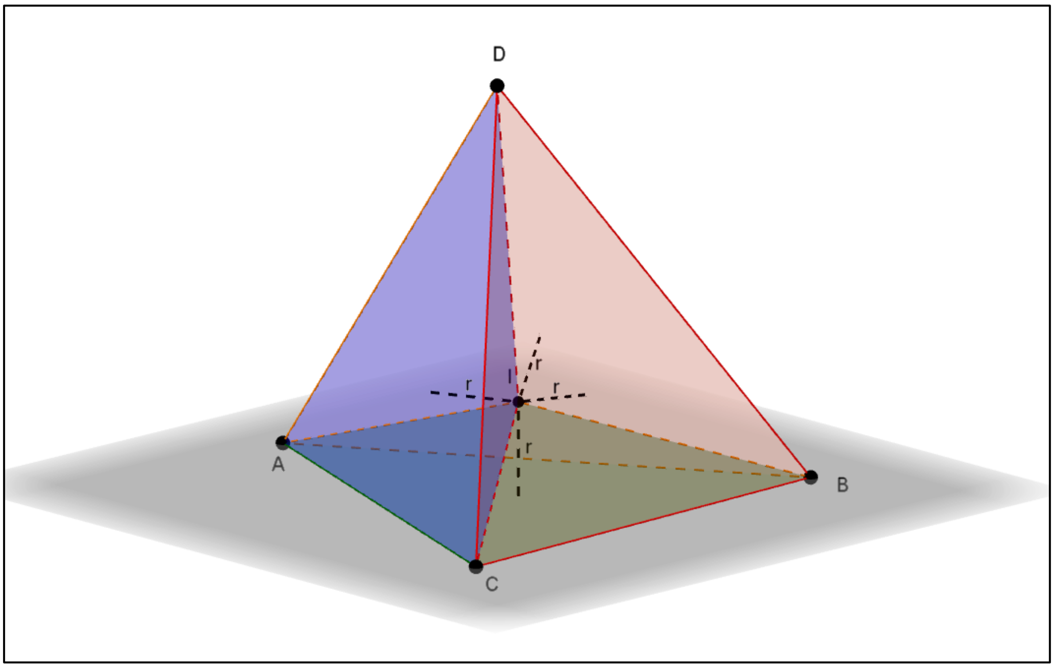


Then we can use AM, AN, $\angle IMP$ and $\angle INP$ to locate the incenter I of the tetrahedron ABCD.

Suppose all the lengths of edges of tetrahedron ABCD (i.e. AB, AC, AD, BC, BD and CD) are known, then a way to solve $\angle IMP$, $\angle INP$, AM and AN, hence locate the incenter I of the tetrahedron ABCD is as follows:

1. Calculate the inradius r of the tetrahedron ABCD. It can be done by using the fact:

   Volume of ABCD
   = Volume of ABCI + Volume of ABDI + Volume of ACDI + Volume of BCDI
   $= \frac{r}{3}$ (Area of △ABC + Area of △ABD + Area of △ACD + Area of △BCD)



   The areas of △ABC, △ABD, △ACD and △BCD can be found by using Heron's formula, for example:

   Area of △ABC = $\sqrt{s(s-AB)(s-AC)(s-BC)}$, where $s = \frac{AB+AC+BC}{2}$

   Although the volume of tetrahedron can be found by using $\frac{1}{3}Ah$, but in real scenario the height h of a tetrahedron is not easy to be found. Therefore, it is suggested to use the following formula to calculate the volume instead:

   Volume of tetrahedron ABCD = $\frac{\sqrt{4AB^2AC^2AD^2 - c^2AB^2 - b^2AC^2 - a^2AD^2 + abc}}{12}$

   where $a = AB^2 + AC^2 - BC^2$, $b = AB^2 + AD^2 - BD^2$ and $c = AC^2 + AD^2 - CD^2$.

   Then we can calculate the inradius r.

2. We need to find $\angle IMP$ and $\angle INP$. It can be done by using the cosine formula and dihedral angle formula:

Let α be the angle between $\triangle ABD$ and $\triangle ABC$,
β be the angle between $\triangle ACD$ and $\triangle ABC$.

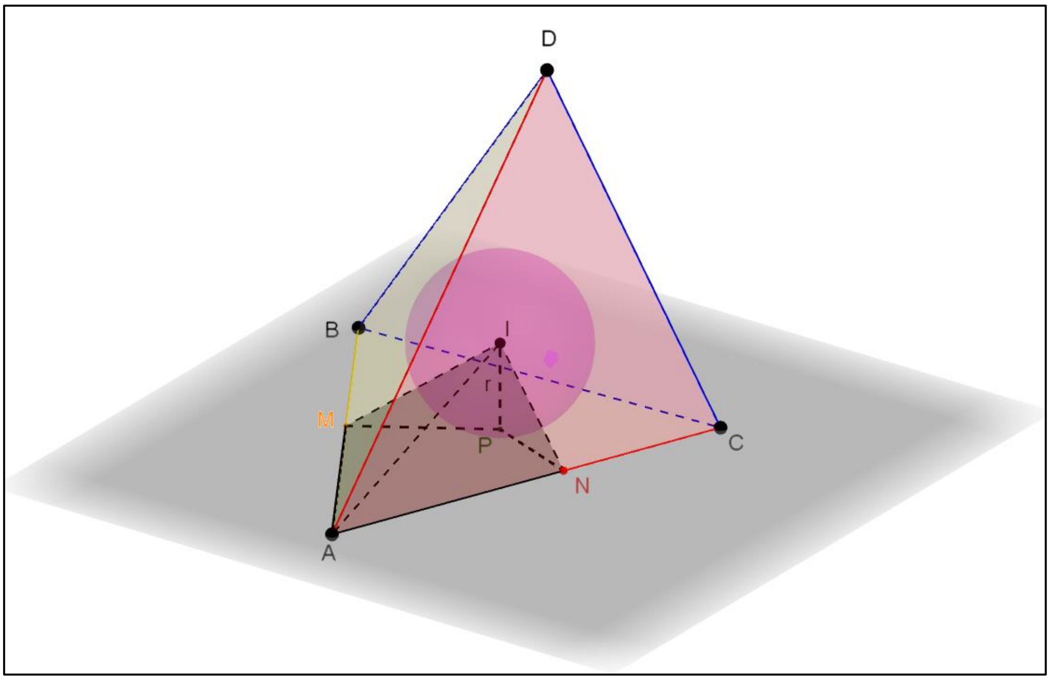
Then $\angle IMP = \frac{\alpha}{2}$ and $\angle INP = \frac{\beta}{2}$.

Using dihedral angle formula, we have

$$\cos\alpha = \frac{\cos\angle CAD - \cos\angle BAD \cos\angle BAC}{\sin\angle BAD \sin\angle BAC} \quad \text{and} \quad \cos\beta = \frac{\cos\angle BAD - \cos\angle CAD \cos\angle BAC}{\sin\angle CAD \sin\angle BAC}$$

while $\angle CAD$, $\angle BAD$ and $\angle BAC$ can be found by using cosine formula, i.e.

$$\cos\angle CAD = \frac{AC^2+AD^2-CD^2}{2(AC)(AD)}, \quad \cos\angle BAD = \frac{AB^2+AD^2-BD^2}{2(AB)(AD)}, \quad \cos\angle BAC = \frac{AB^2+AC^2-BC^2}{2(AB)(AC)}$$



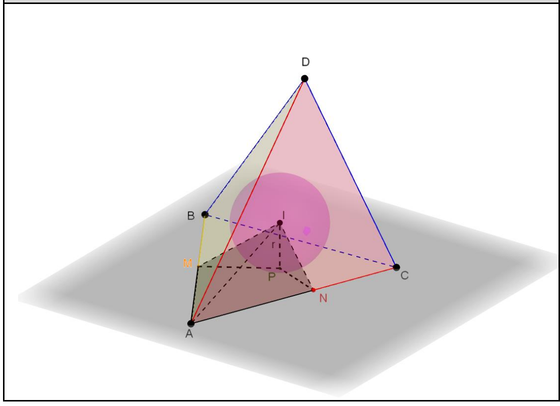
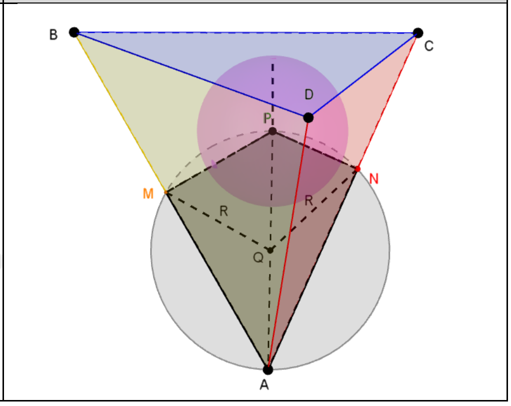
3. After solving the inradius r, $\angle IMP$ and $\angle INP$, we can find MP and NP by

$$MP = \frac{r}{\tan\angle IMP} \quad \text{and} \quad NP = \frac{r}{\tan\angle INP}$$

4. Consider the quadrilateral AMPN.

Since IM $\perp$ AM, IP $\perp$ PM and IN $\perp$ AN, IP $\perp$ PN,
By the theorem of three perpendiculars, we have PM $\perp$ AM and PN $\perp$ AN.

As $\angle$AMP + $\angle$ANP = 180°, the opposite angles of AMPN are supplementary and AMPN is a cyclic quadrilateral.

3-D view of tetrahedron ABCD	Top view of tetrahedron ABCD
	

As $\angle$AMP = 90°, by the converse of the $\angle$ in semi-circle, AP is a diameter of the circle passing through A, M, P and N.

Let R be the radius of the circle AMPN. By using the extended law of sines,

$$AP = 2R = \frac{MN}{\sin\angle MAN} = \frac{MN}{\sin\angle BAC}$$

and MN can be calculated by using the cosine formula:

$$\begin{aligned} MN^2 &= MP^2 + NP^2 - 2(MP)(NP)\cos\angle MPN \\ &= MP^2 + NP^2 - 2(MP)(NP)\cos(180^\circ - \angle MAN) \\ &= MP^2 + NP^2 + 2(MP)(NP)\cos\angle BAC \end{aligned}$$

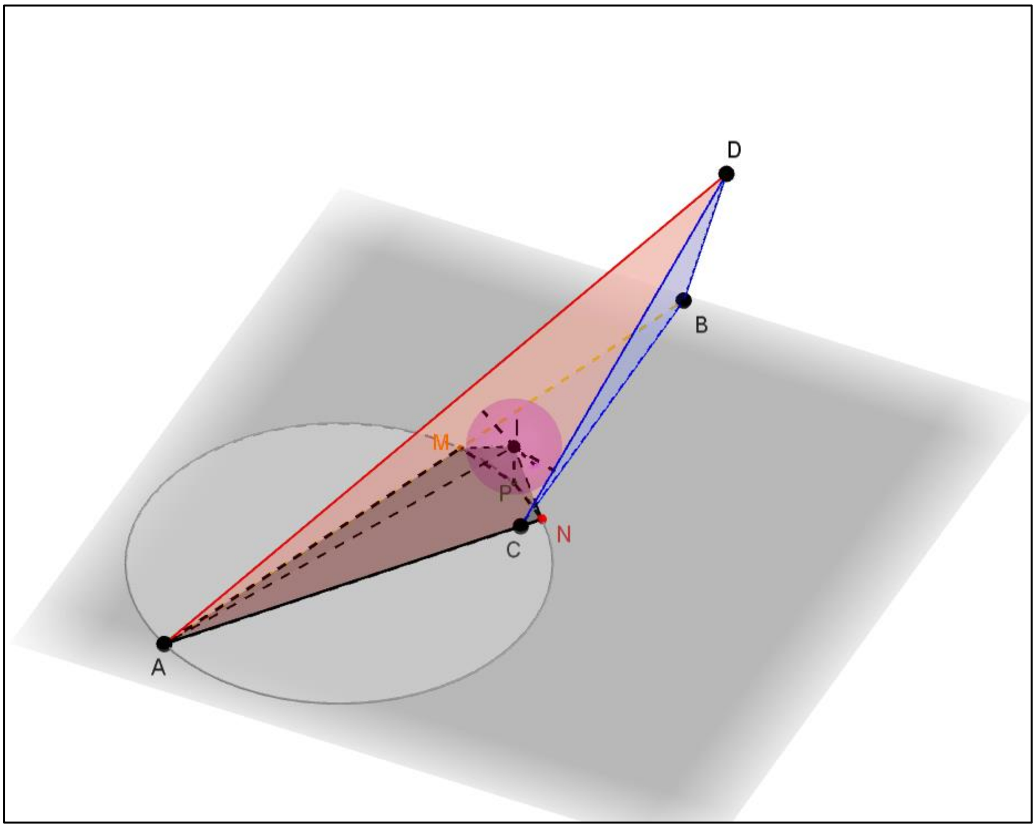
[Note that cos $\angle$BAC has been found in step 2]

Finally, AM and AN can be calculated by using Pythagoras' Theorem:

$AM^2 = AP^2 - MP^2$, $AN^2 = AP^2 - NP^2$

As $\angle$IMP, $\angle$INP, AM and AN have been found, we can use this information to locate the incenter of the tetrahedron ABCD.

Note that for some tetrahedrons with some obtuse-angled triangles as its faces, the points M and N may lie outside the line segment AB and AC. The figure below shows an example where N lies on the AC produced. Nevertheless, the works above is still valid for this type of tetrahedrons.

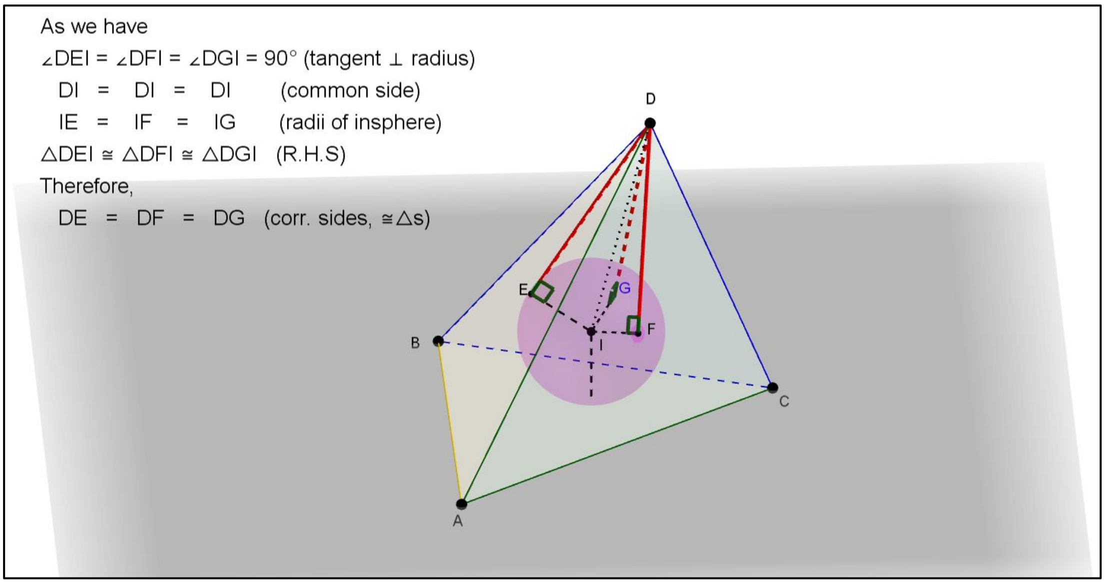


Although the above method can be used to locate the incenter of a tetrahedron, it makes use of angles ∠IMP and ∠INP which let this method less practical in real scenario. Therefore, we try to find another method to locate the incenter of a tetrahedron by using lengths only.

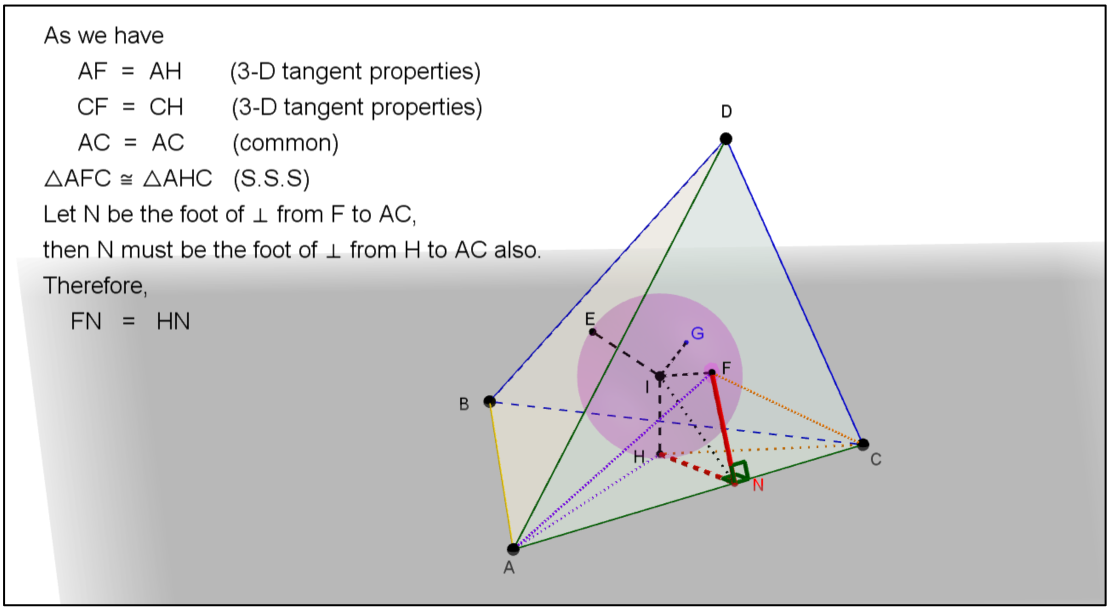
Now we consider using the points of contact between the insphere and the faces of a tetrahedron to locate the incenter of the tetrahedron.

Let E, F, G and H be the points of contact of the insphere of the tetrahedron ABCD and △ABD, △ACD, △BCD and △ABC respectively. Then we have the following theorems (we temporary denote them as 3-D tangent properties):

3-D tangent properties (first version):



3-D tangent properties (second version):



(Remark: From the result on P.6, the segments NF, NH and NI lie on the same plane.)

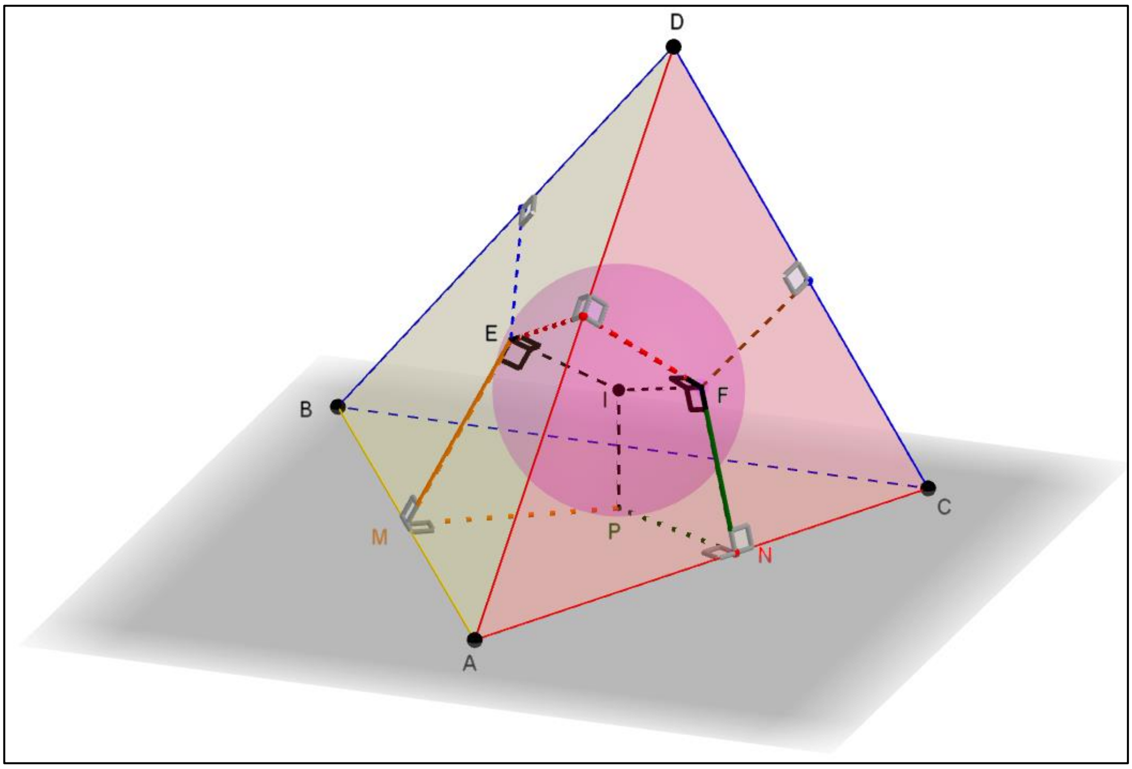
From the above results, we have an alternative method to locate the points of contacts between the insphere and the faces of a tetrahedron:

Refer to the figure below. Recall that we can calculate MP, NP, AM and AN.

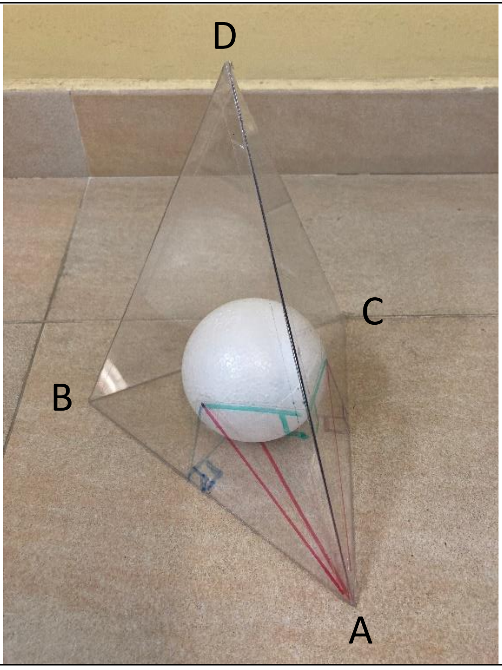
As ME = MP and EM ⊥ AM, we can determine the location of the point E on △ABD by using the lengths AM and MP.

Similarly, as NF = MP and FN ⊥ AN, we can determine the location of the point F on △ACD by using the lengths AN and NP.

Then by constructing a line passing through E and perpendicular to △ABD, and another line passing through F and perpendicular to △ACD, we can locate the intersection of the two perpendiculars, and it is the incenter of the tetrahedron ABCD.



Example:

We have a model of tetrahedron, and we are going to locate its incenter. The measurement of the tetrahedron are as follows: AB = 23.2 cm AC = 23.2 cm AD = 29.7 cm BC = 22.1 cm BD = 26.6 cm CD = 27.1 cm	

Then

$s_{ABD} = \frac{AB+BD+AD}{2} = \frac{23.2+26.6+29.7}{2} = 39.75$ cm

Area of △ABD = $\sqrt{s_{ABD}(s_{ABD} - AB)(s_{ABD} - BD)(s_{ABD} - AD)}$

= $\sqrt{39.75(39.75 - 23.2)(39.75 - 26.6)(39.75 - 29.7)}$

= 295 cm^2 (corr. To 3 sig. fig.)

$s_{BCD} = \frac{BC+CD+BD}{2} = \frac{22.1+27.1+26.6}{2} = 37.9$ cm

Area of △BCD = $\sqrt{s_{BCD}(s_{BCD} - BC)(s_{BCD} - CD)(s_{BCD} - BD)}$

= $\sqrt{37.9(37.9 - 22.1)(37.9 - 27.1)(37.9 - 26.6)}$

= 270 cm^2 (corr. To 3 sig. fig.)

$s_{ACD} = \frac{AC+CD+AD}{2} = \frac{23.2+27.1+29.7}{2} = 40$ cm

Area of △ACD = $\sqrt{s_{ACD}(s_{ACD} - AC)(s_{ACD} - CD)(s_{ACD} - AD)}$

= $\sqrt{40(40 - 23.2)(40 - 27.1)(40 - 29.7)}$

= 299 cm^2 (corr. To 3 sig. fig.)

$s_{ABC} = \frac{AB+BC+AC}{2} = \frac{23.2+22.1+23.2}{2} = 34.25$ cm

Area of △ABC = $\sqrt{s_{ABC}(s_{ABC} - AB)(s_{ABC} - BC)(s_{ABC} - AC)}$

= $\sqrt{34.25(34.25 - 23.2)(34.25 - 22.1)(34.25 - 23.2)}$

= 225 cm^2 (corr. To 3 sig. fig.)

$$\alpha = AB^2 + AC^2 - BC^2 = (23.2)^2 + (23.2)^2 - (22.1)^2 = 588.07$$

$$\beta = AB^2 + AD^2 - BD^2 = (23.2)^2 + (29.7)^2 - (26.6)^2 = 712.77$$

$$\gamma = AC^2 + AD^2 - CD^2 = (23.2)^2 + (29.7)^2 - (27.1)^2 = 685.92$$

Volume of the tetrahedron ABCD

$$= \sqrt{\frac{4AB^2AC^2AD^2 - \gamma^2AB^2 - \beta^2AC^2 - \alpha^2AD^2 + \alpha\beta\gamma}{12}}$$

$$= \sqrt{\frac{4(23.2)^2(23.2)^2(29.7) - (685.92)^2(23.2)^2 - (712.77)^2(23.2)^2 - (588.07)^2(29.7)^2 + (588.07)(712.77)(685.92)}{12}}$$

= 1821.840521 cm^3 (corr. to 6 d.p.)

$$\therefore \quad \frac{r}{3}(295 + 270 + 299 + 225) = 1821.840521$$

r = 5.016921 (corr. to 6 d.p.)

$\angle$CAD = $\cos^{-1}\frac{AC^2+AD^2-CD^2}{2(AC)(AD)} = \cos^{-1}\frac{(23.2)^2+(29.7)^2-(27.1)^2}{2(23.2)(29.7)} = 60.149674°$ (corr. to. 6 d.p.)

$\angle$BAD = $\cos^{-1}\frac{AB^2+AD^2-BD^2}{2(AB)(AD)} = \cos^{-1}\frac{(23.2)^2+(29.7)^2-(26.6)^2}{2(23.2)(29.7)} = 58.854067°$ (corr. to. 6 d.p.)

$\angle$BAC = $\cos^{-1}\frac{AB^2+AC^2-BC^2}{2(AB)(AC)} = \cos^{-1}\frac{(23.2)^2+(23.2)^2-(22.1)^2}{2(23.2)(23.2)} = 56.887155°$ (corr. to. 6 d.p.)

Denote θ be the angle between △ABD and △ABC,

ϕ be the angle between △ACD and △ABC.

$\theta = \cos^{-1}\left(\frac{\cos\angle CAD - \cos\angle BAD \cos\angle BAC}{\sin\angle BAD \sin\angle BAC}\right) = 72.531798°$ (corr. to. 6 d.p.)

$\phi = \cos^{-1}\left(\frac{\cos\angle BAD - \cos\angle CAD \cos\angle BAC}{\sin\angle CAD \sin\angle BAC}\right) = 70.264682°$ (corr. to. 6 d.p.)

$\angle IMP = \frac{\theta}{2} = 36.265899°$ (corr. to. 6 d.p.)

$\angle INP = \frac{\phi}{2} = 35.132341°$ (corr. to. 6 d.p.)

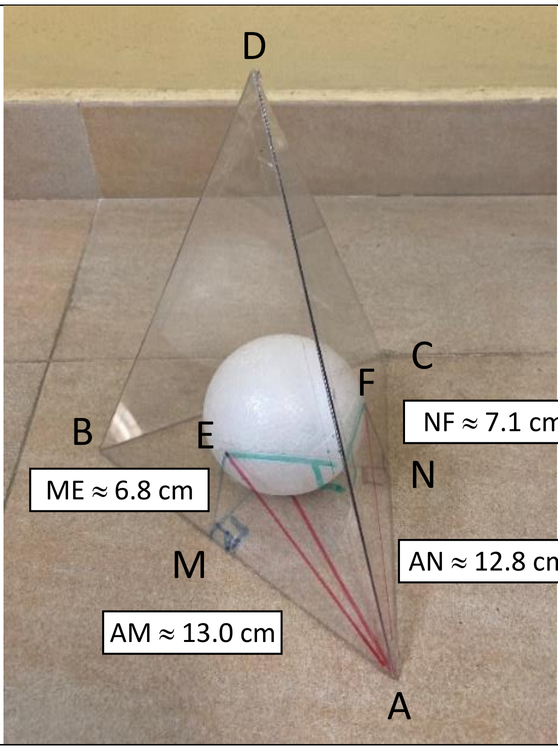
$MP = \frac{r}{\tan\angle IMP} = 6.838237$ cm (corr. to. 6 d.p.)

$NP = \frac{r}{\tan\angle INP} = 7.129799$ cm (corr. to. 6 d.p.)

$MN = \sqrt{MP^2 + NP^2 + 2(MP)(NP)\cos\angle BAC} = 12.282692$ cm (corr. to. 6 d.p.)

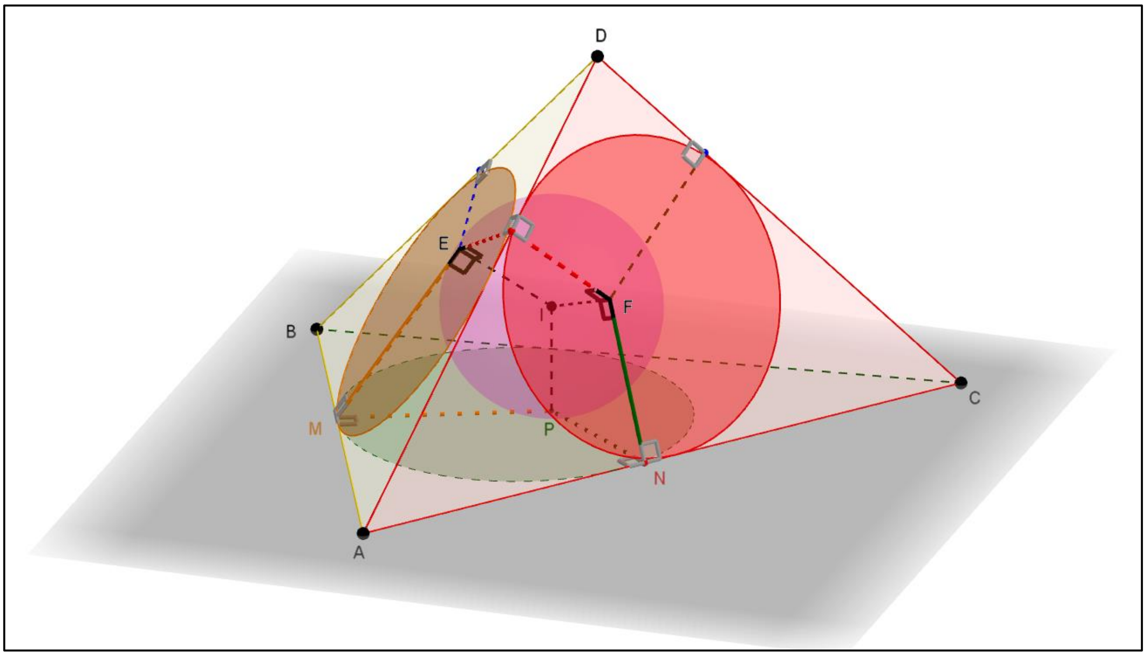
$AP = \frac{MN}{\sin\angle BAC} = 14.664215$ cm (corr. to. 6 d.p.)

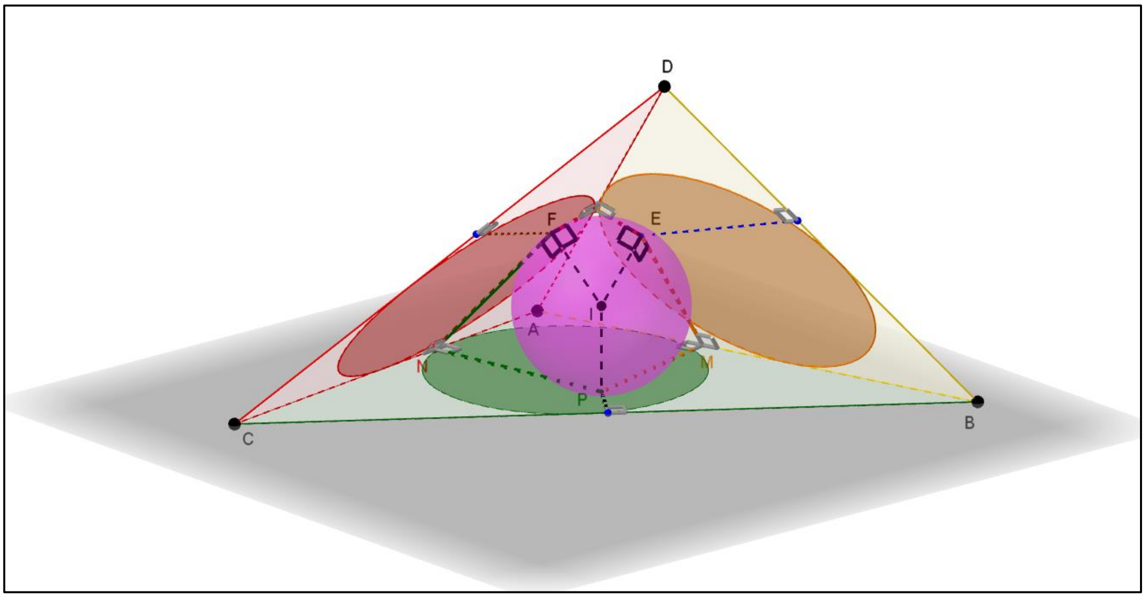
Therefore,

$AM = \sqrt{AP^2 - MP^2} \approx 13.0$ cm $AN = \sqrt{AP^2 - MP^2} \approx 12.8$ cm ME = MP ≈ 6.8 cm NF = NP ≈ 7.1 cm	

Remark:

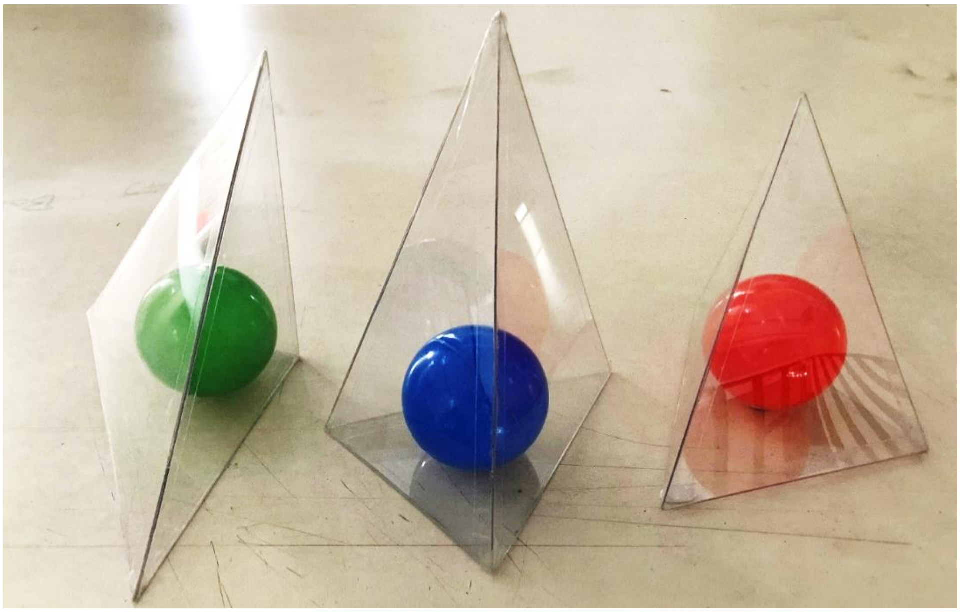
It is easy to observe that the points of contact of the insphere and the faces of the tetrahedron (points E, F and P in the figure) are the projection of the incenter I on the faces of the tetrahedron (△ABD, △ACD and △ABC respectively). In general, there are no trivial relationships between I and the incenters of any faces of the tetrahedron.





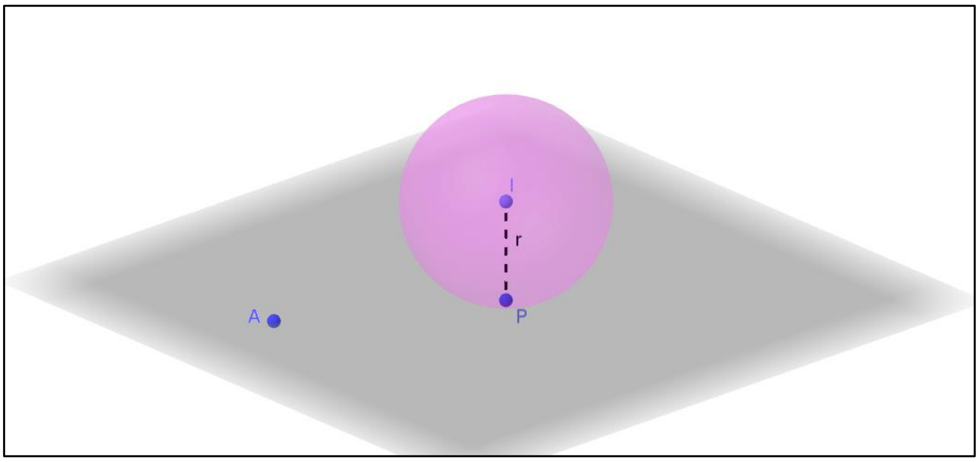
5. Constructing tetrahedrons with given inspheres

In this project, we have made some tetrahedrons with inspheres by using transparent plastic boards and plastic balls. How can we make different types tetrahedrons with inscribed plastic balls with given size?



To complete this task, we make use of GeoGebra to simulate the construction.

- First of all, a sphere with center I and given radius r is constructed. It touches the horizontal ground at point P,
- Then, a point A is created on the horizontal ground.

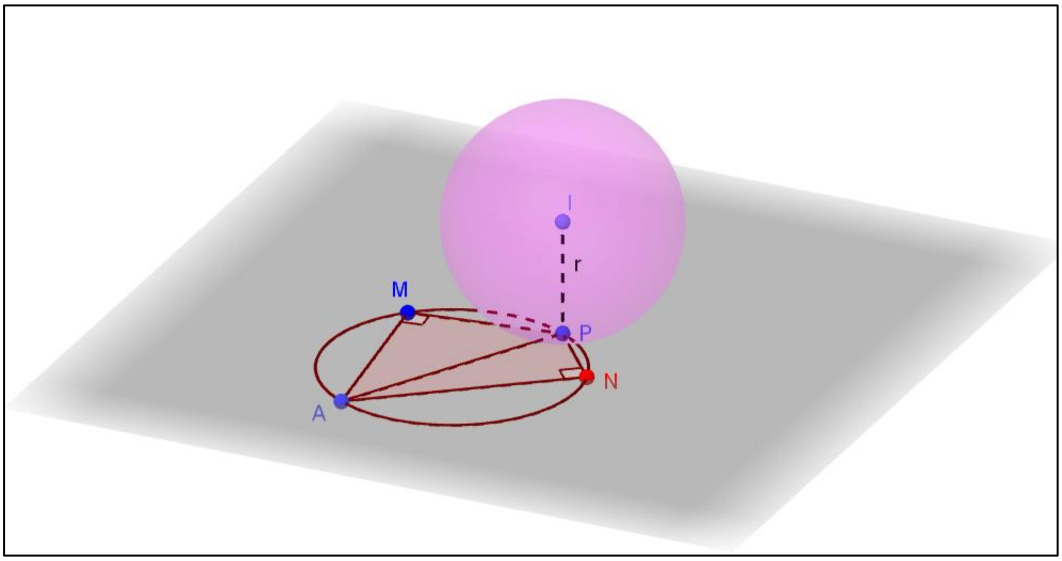


Then a tetrahedron with the insphere is being constructed as follows:

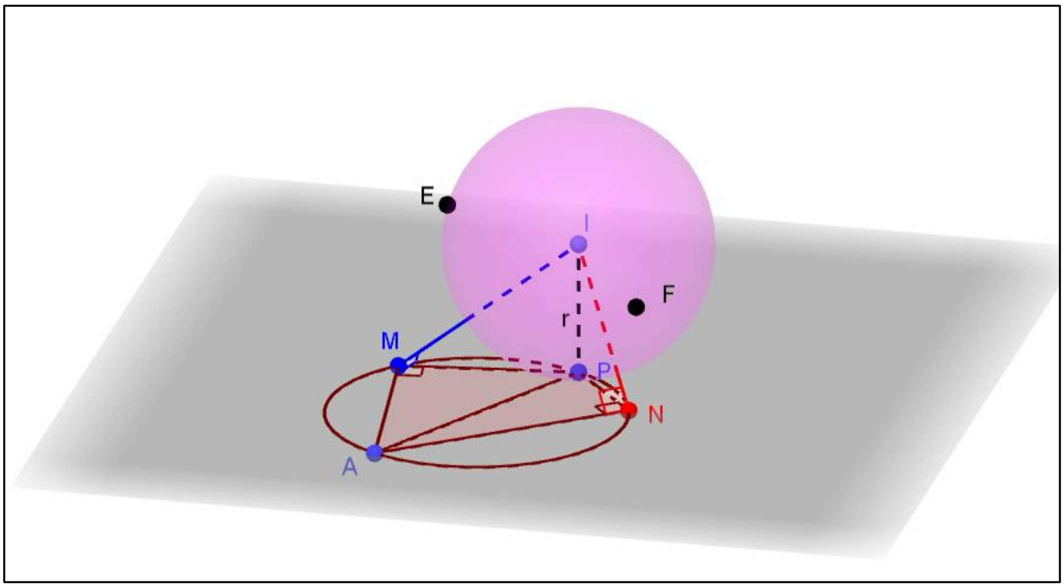
The following construction is illustrated in the GeoGebra file:

https://www.geogebra.org/m/zfkxk3de

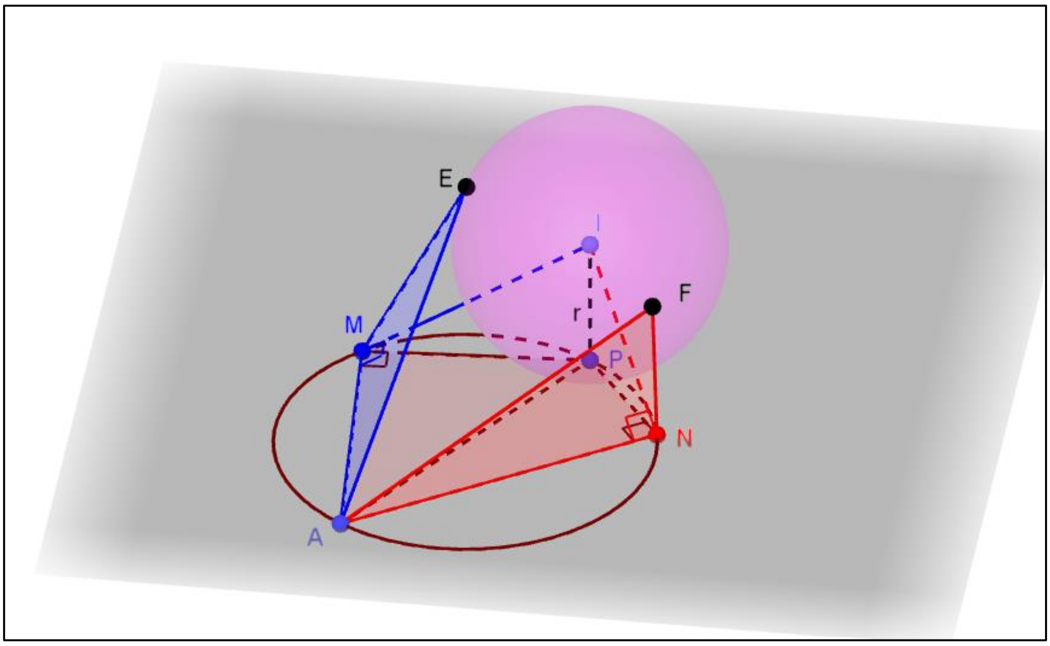
1. Construct a circle with diameter AP on the horizontal ground. Then create two points M and N where they lie on different arcs of the circle separated by AP.
   Note that M and N are freely to move along the arcs. It determines the size of ∠MAN, which is an interior angle of the base of the tetrahedron.



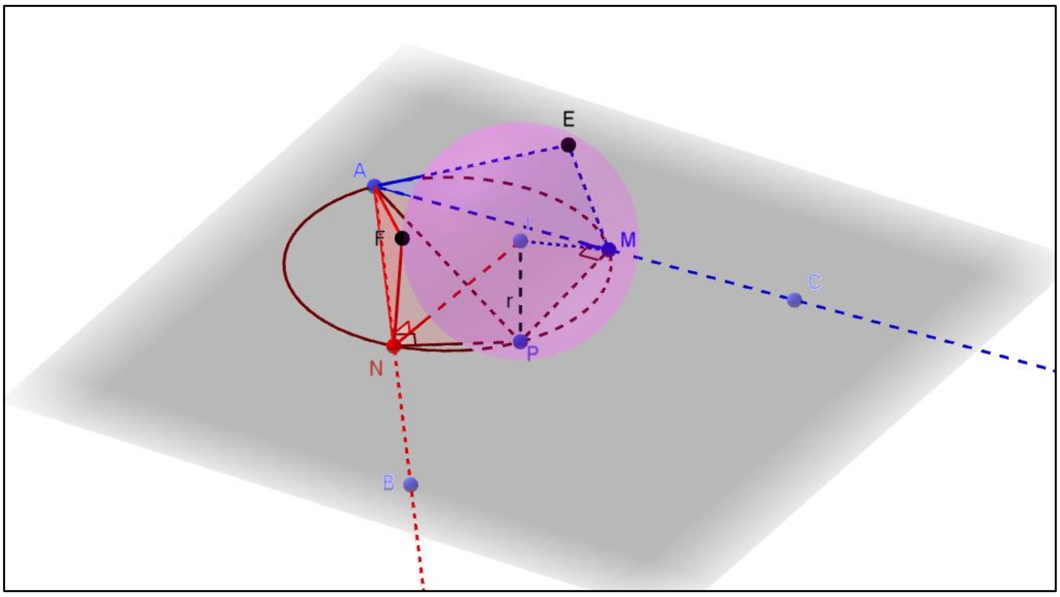
2. Join IM. Then reflect the point P along IM to create the point E. Note that △MPI and △MEI lie on the same plane and △MPI ≅ △MEI. From the previous result, it is deduced that point E lies on the sphere.
   Similarly, join IN and reflect the point P along IN to create the point F. F also lies on the sphere.



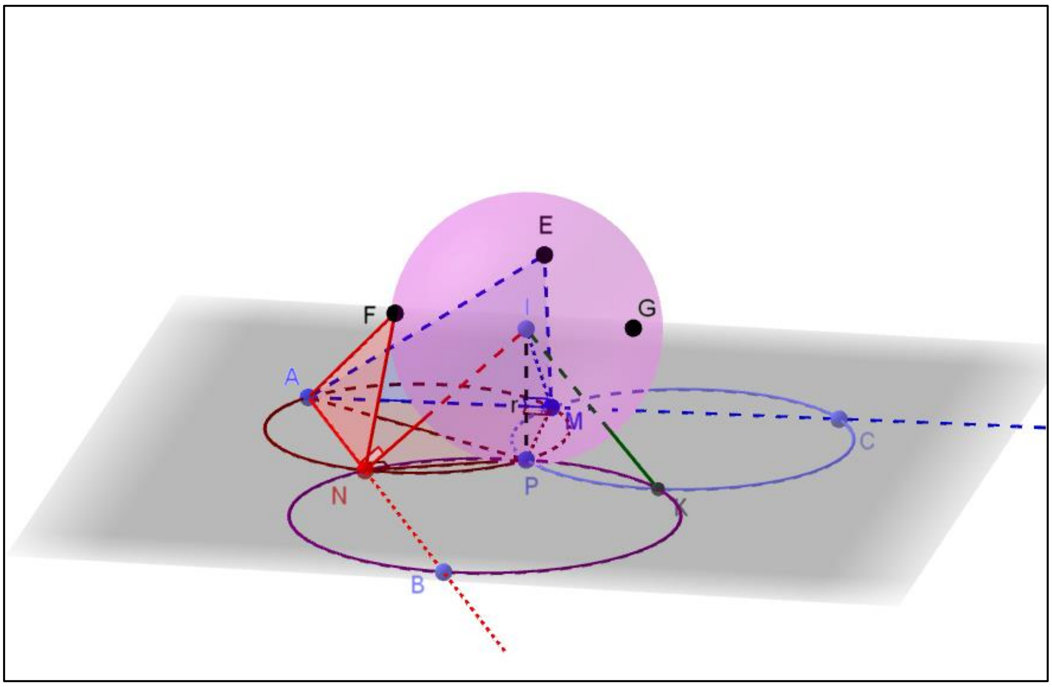
3. Construct △AME and △ANF.



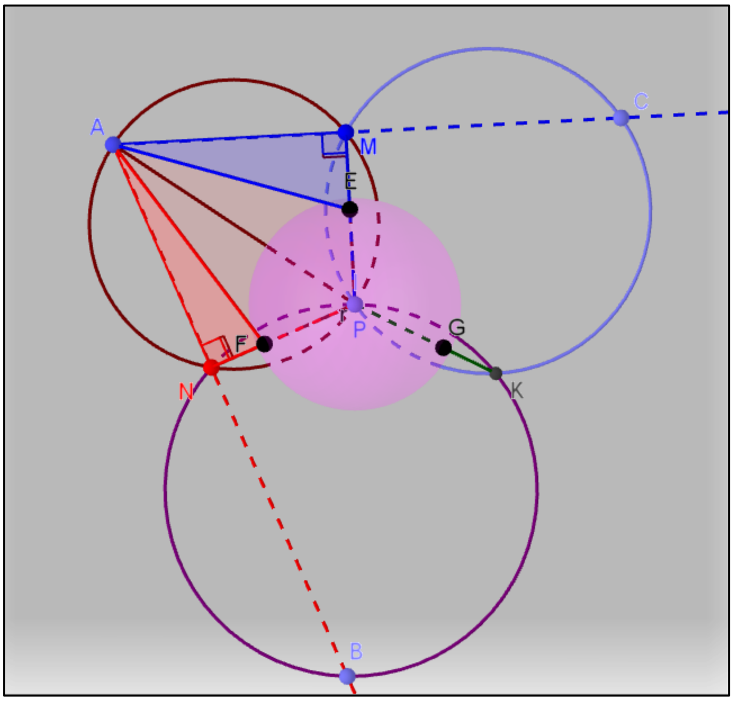
4. Produce AN and AM. Create two points B and C which lie on AN produced (or segment AN) and AM produce (or segment AM) respectively.
   Note that B and C are freely to move along the lines. It determines the shape of △ABC, which is the base of the tetrahedron.



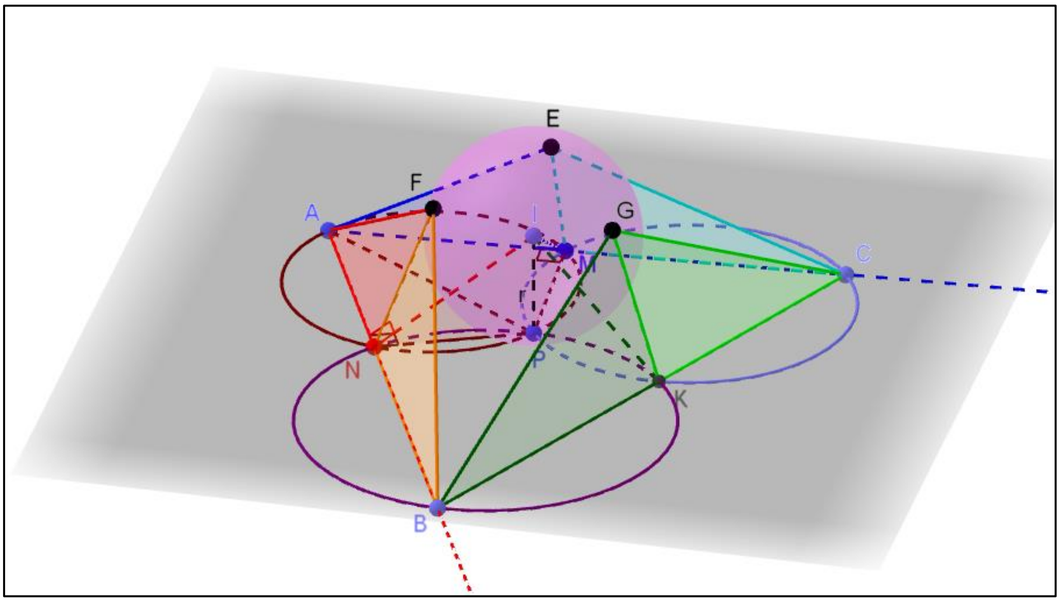
5. Construct two circles on the horizontal ground with BP and CP as diameters respectively. The two circles will intersect at P and another point. Denote this point be K.
Join IK, then reflect the point P along IK to create the point G. Note that G lies on the sphere.



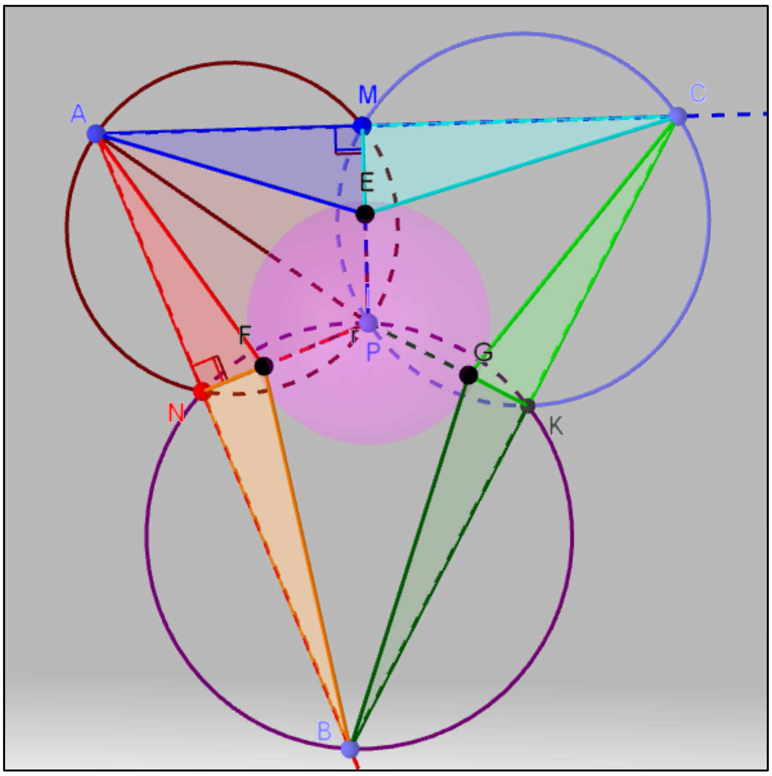
From the top view, it is observed that all the constructed circles pass through the point P.



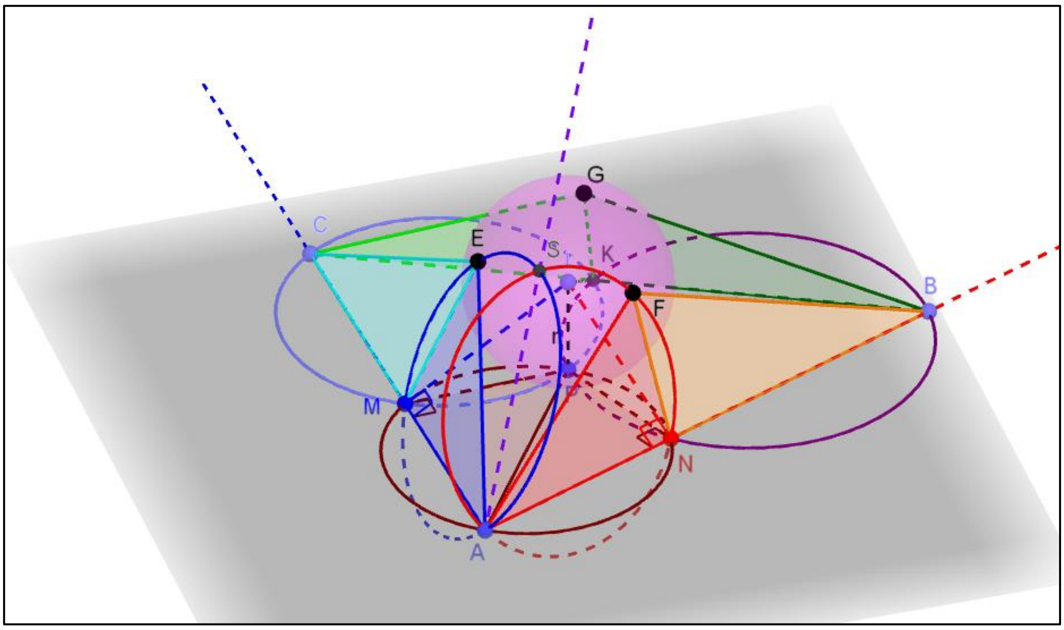
6. Construct △FNB, △GBK, △GCK and △ECM.
   Then three larger triangles △ABF, △BCG and △ACE are formed. Their bases AB, BC and AC are attaching the base of the tetrahedron △ABC, whilst their vertices F, G and E are touching the sphere.



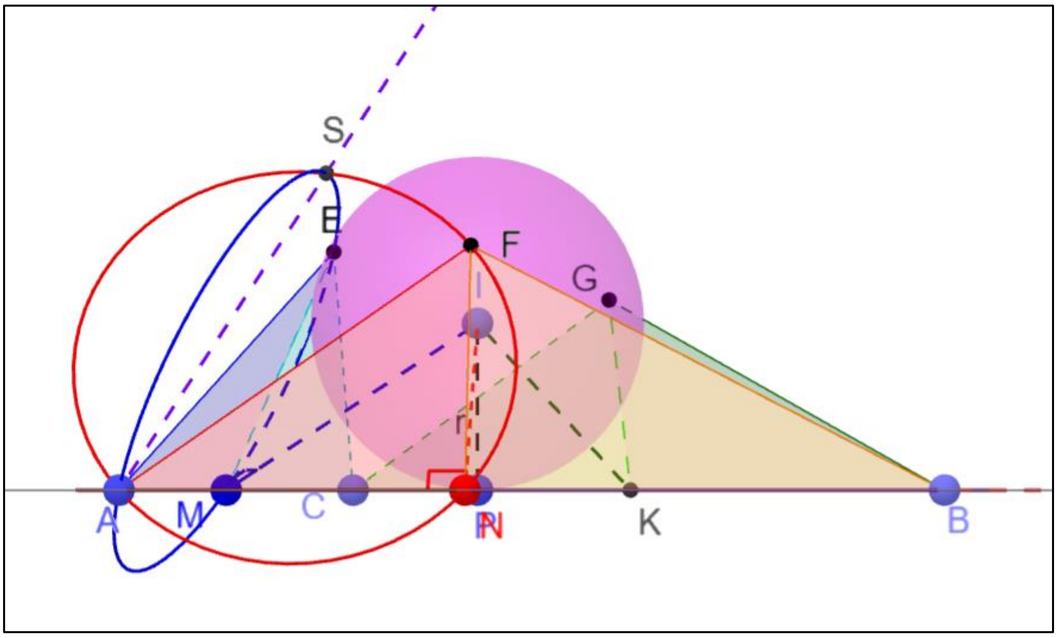
From the top view, it is observed that the sphere is "partially" inscribed with △ABF, △BCG and △ACE.



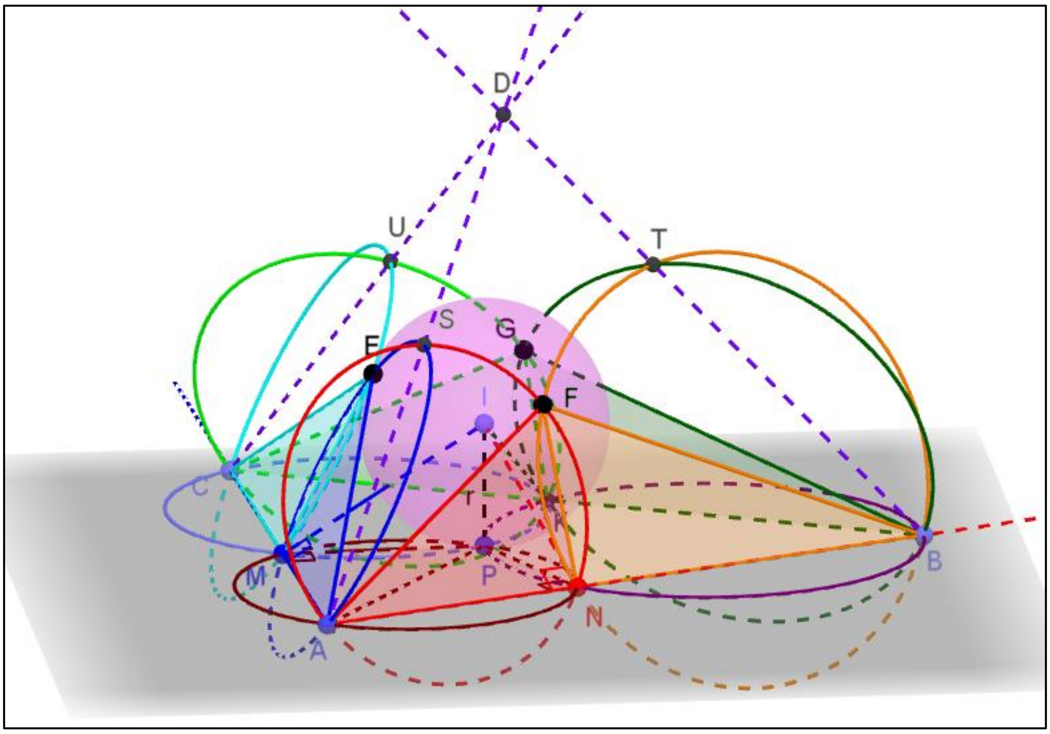
7. Construct a circle with diameter AE on the plane consisting of △AEM.
   Construct another circle with diameter AF on the plane consisting of △AFN.
   The two circles will intersect at A and another point. Denote this point be S.



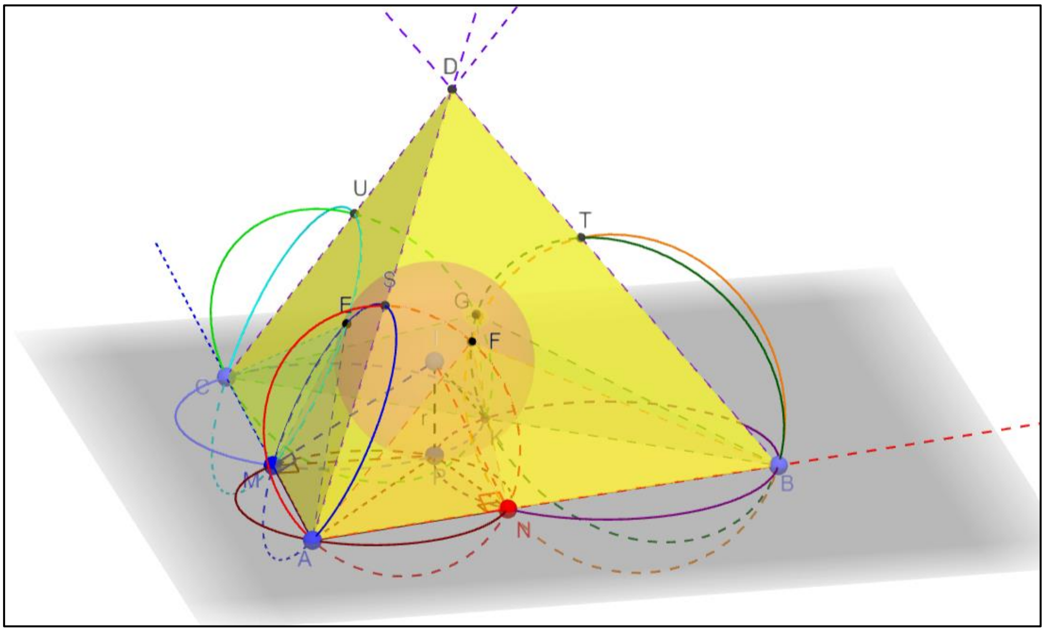
The following is the side view. Note that S does not lie on the sphere.



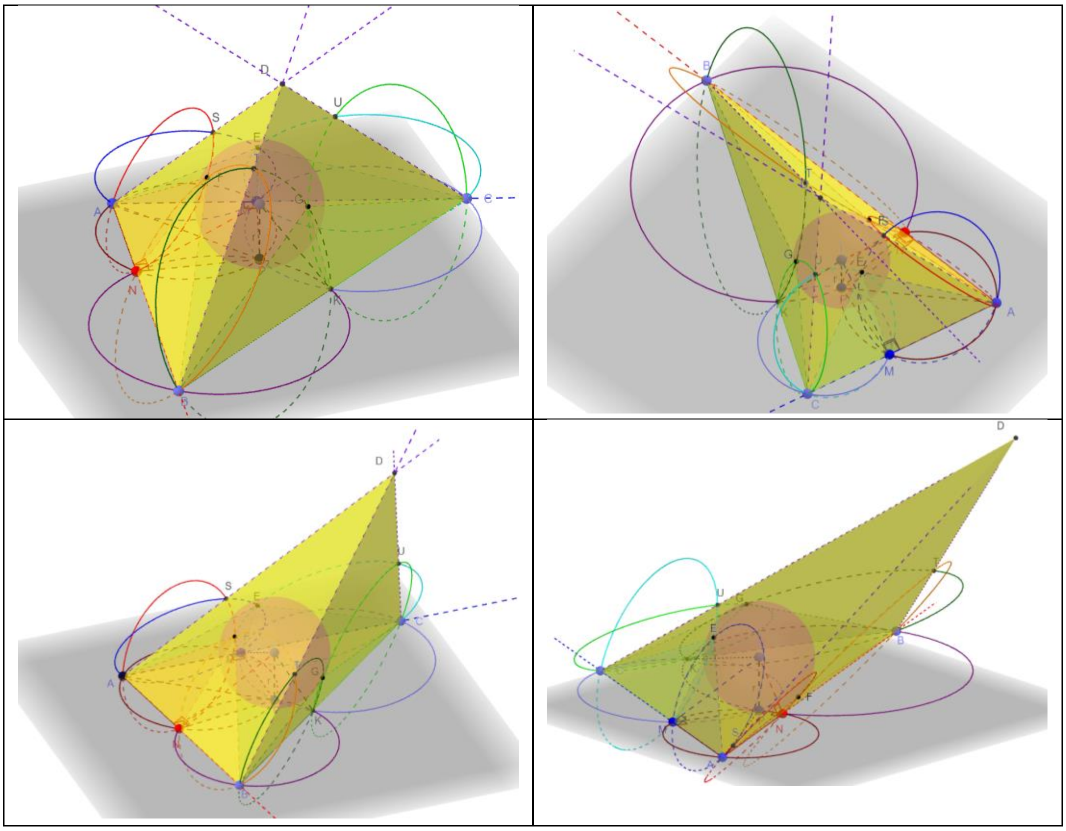
8. Construct four more circles with diameter BF, BG, CG and CE, on the planes consisting of △BFN, △BGK, △CGK and △CEM respectively.
   Denote T and U be the intersections first two circles and the last two circles respectively. Produce AS, BT and CU to meet at the point D.



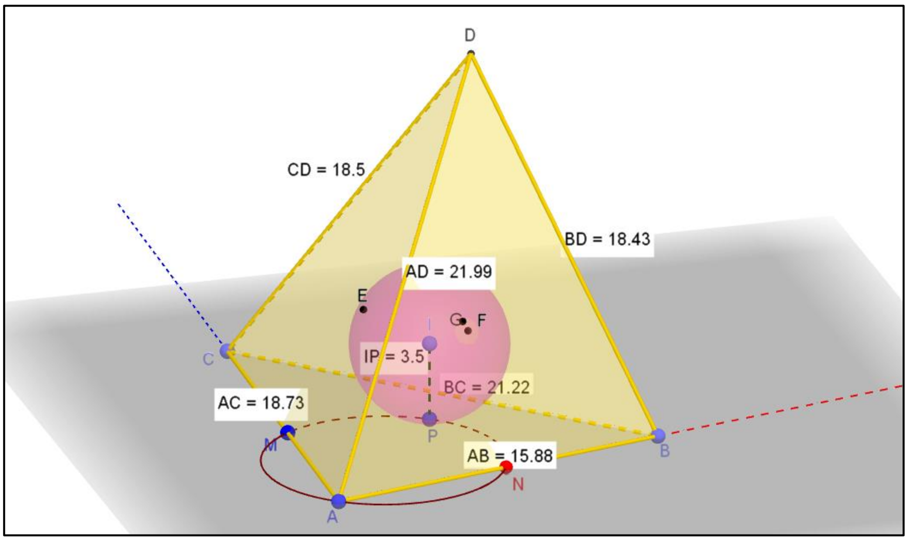
9. D is the apex of the tetrahedron ABCD with the base △ABC. The given sphere is inscribed in the tetrahedron.



10. By changing the position of the points A, N, M, B and C, we can form different tetrahedrons with the given insphere.



As a result, we can form different types of tetrahedrons with the given insphere. By showing the measures of the edges AB, AC, AD, BC, BD and CD, we can cut the faces △ABC, △ABD, △ACD and △BCD out of transparent plastic boards and use them to form the tetrahedron ABCD.



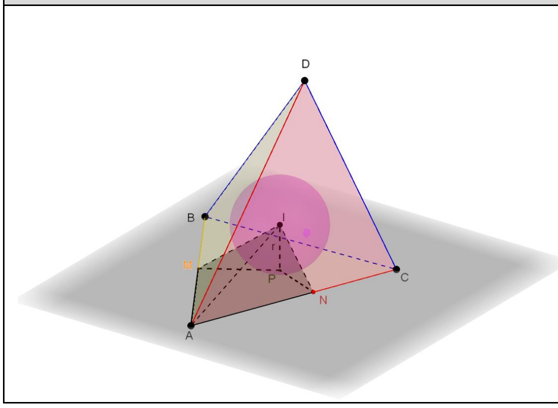
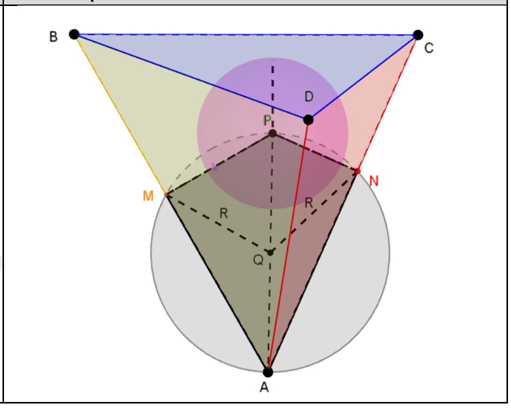
Remark: Why the above construction works?

Recall what we have shown in P.26:

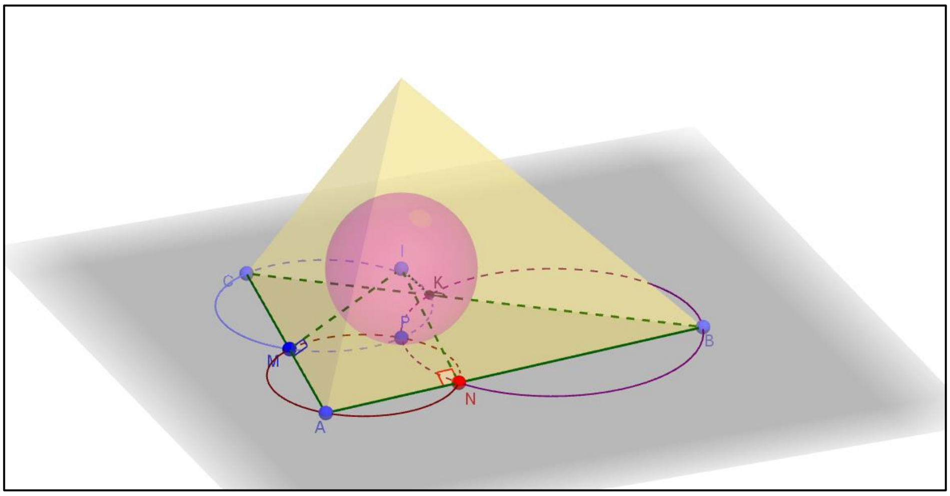
Denote I as the incenter of a tetrahedron ABCD and P be the project of I on △ABC. Let M and N be two points lying on AB and AC respectively such that IM ⊥ AM and IN ⊥ AN.

Since IM ⊥ AM, IP ⊥ PM and IN ⊥ AN, IP ⊥ PN,
By the theorem of three perpendiculars, we have PM ⊥ AM and PN ⊥ AN.

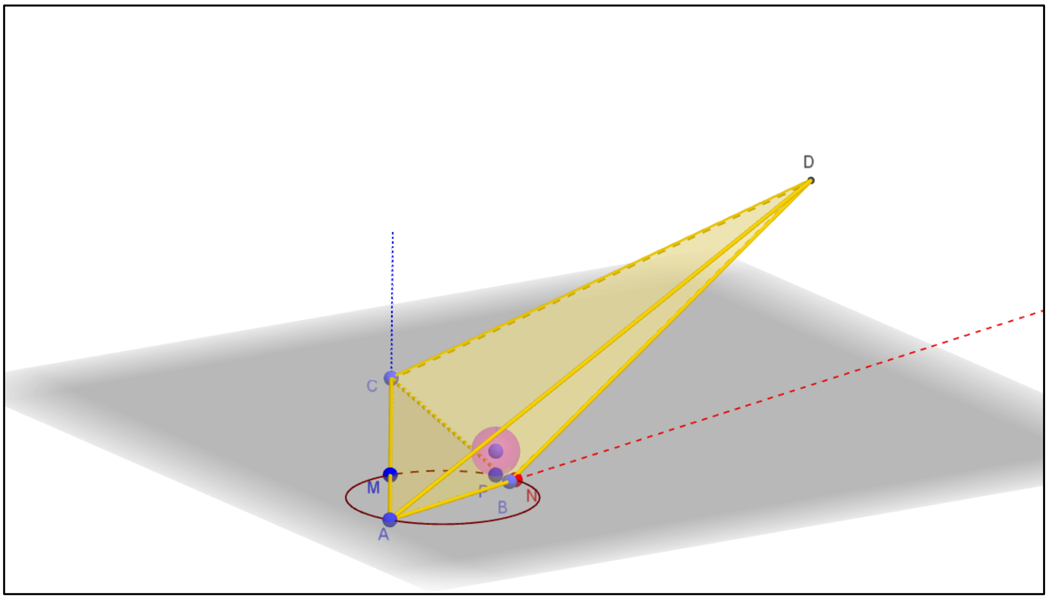
As ∠AMP + ∠ANP = 180°, the opposite angles of AMPN are supplementary and AMPN is a cyclic quadrilateral.

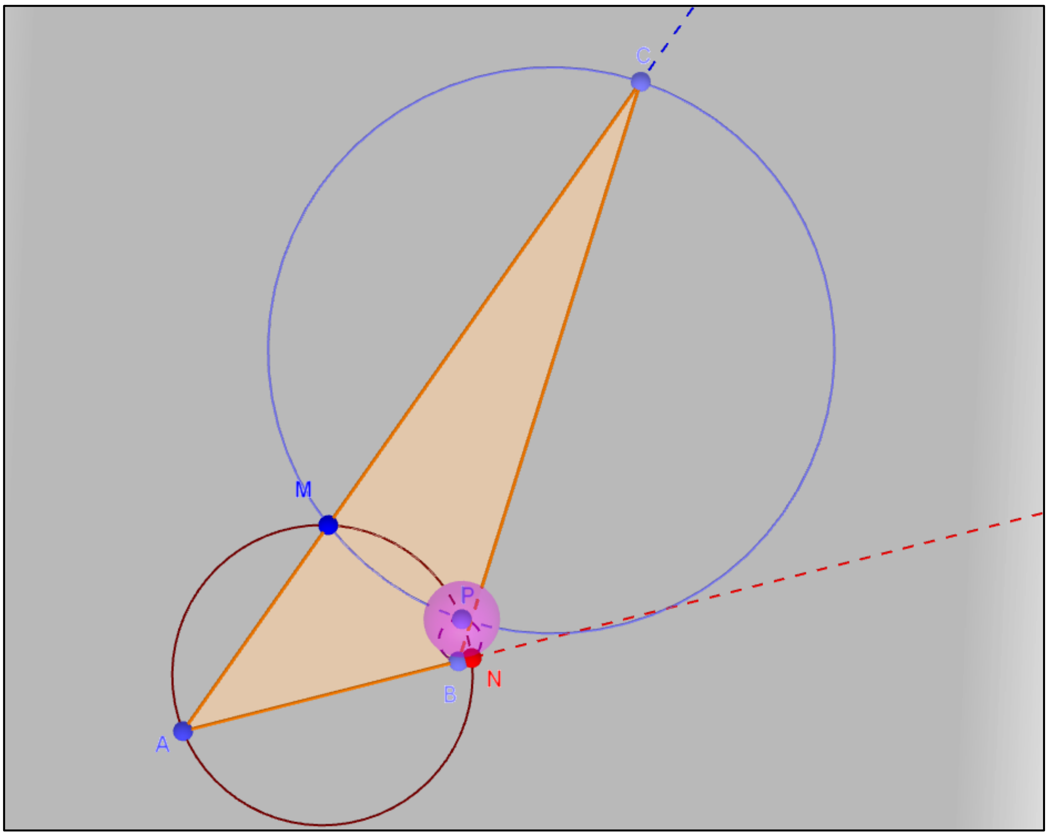
3-D view of tetrahedron ABCD	Top view of tetrahedron ABCD
	

Hence, if we also denote K as a point lying on BS such that IK ⊥ BC, then AMPN, BNPK and CKPM are all cyclic quadrilateral.



The above result is still valid if △ABC is a right-angled or obtuse-angled triangle, or N lies on AB produced. Meanwhile, it is observed that P must lies inside △ABC.



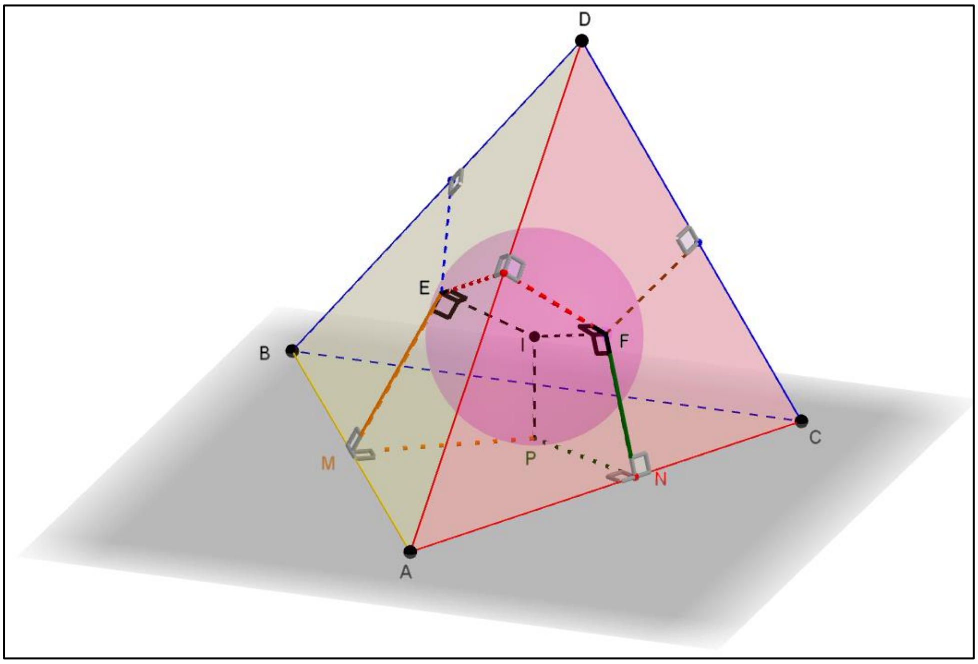


Therefore, by constructing the three circles with AP, BP and CP as diameters, we can preliminary locate the incenter of the tetrahedron ABCD with △ABC as its base.

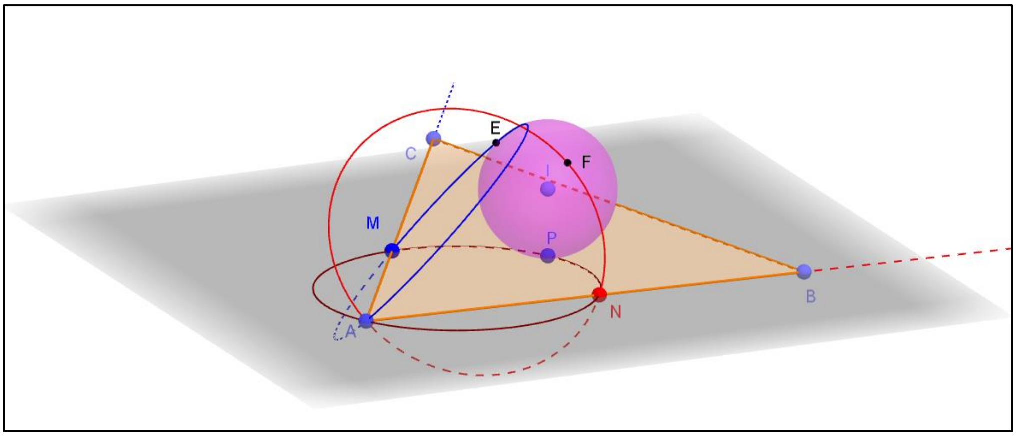
Then we proceed the construction from 2-D to 3-D.

Denote E and F be the projection of the incenter I on the faces △ABD and △ACD respectively.

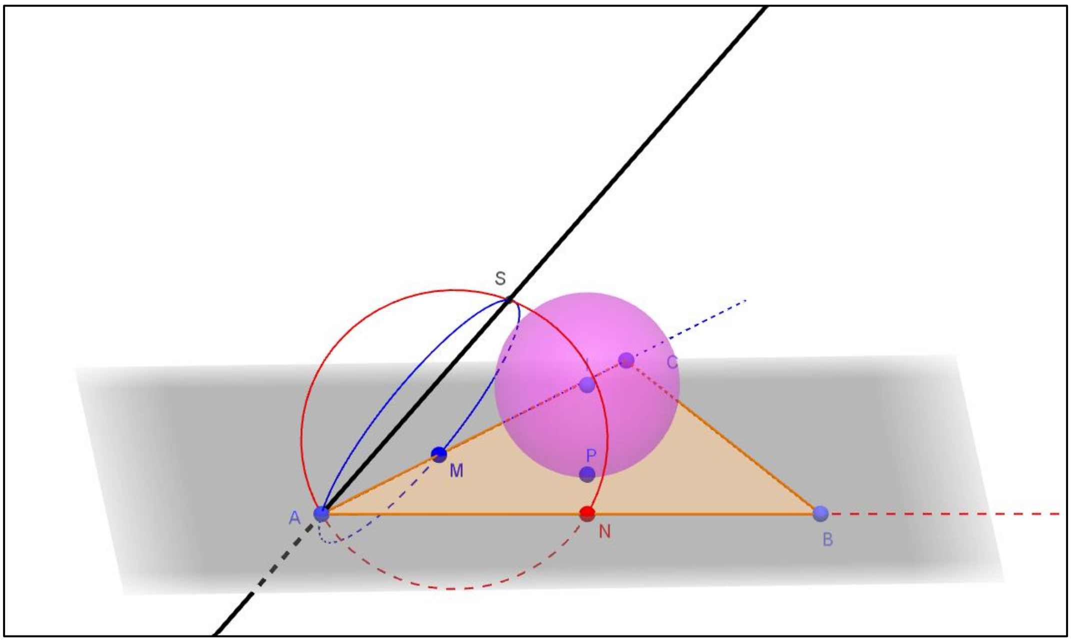
By applying the 3-D tangent properties s (second version) stated in P.28, we notice that △PMI and △EMI lies in the same plane and △PMI ≅ △EMI. Similarly, △PNI and △FNI lies in the same plane and △PNI ≅ △FNI.



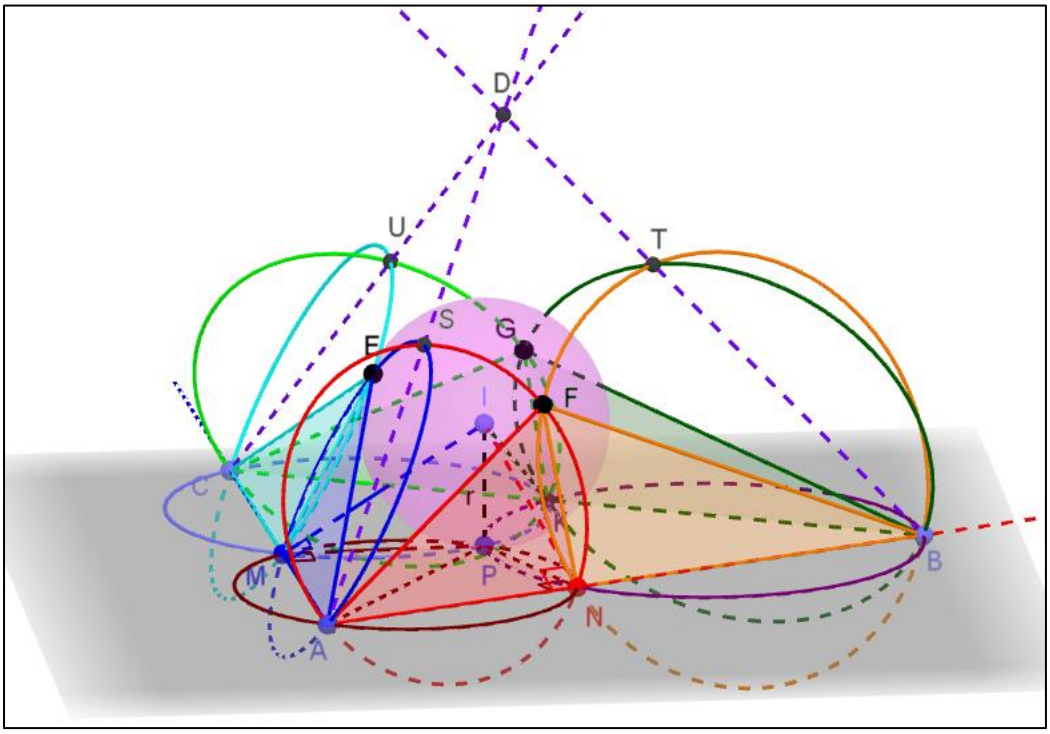
Therefore, E is the point of reflection of P along MI. Similarly, F is the point of reflection of P along NI. As a result, we can construct two circles with AE and AF as diameters (from 3-D tangent properties, AP = AE = AF). As a result, we can determine that the lateral faces of the tetrahedron ABCD should consist of △AME and △ANF.



The two circles with AE and AF as diameters will intersect at A and another point S. It is suggested the common edge of lateral faces with bases AB and AC will lie on this line.



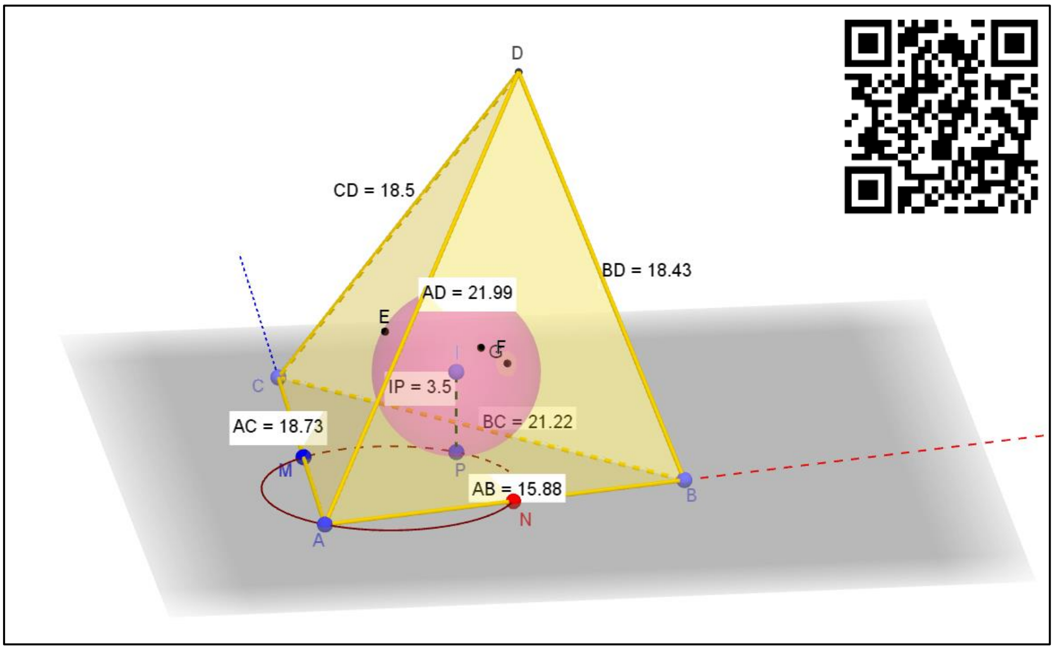
Therefore, we continue the construction and finally we can construct the apex D of the tetrahedron ABCD.



6. Making 3-D models of tetrahedrons with given insphere via GeoGebra Augmented Reality (AR)

As we know how to construct tetrahedrons with given insphere, we have developed a GeoGebra App which can help people to make real 3-D models of tetrahedron with given insphere:

GeoGebra App: https://www.geogebra.org/3d/ezaz4g8r

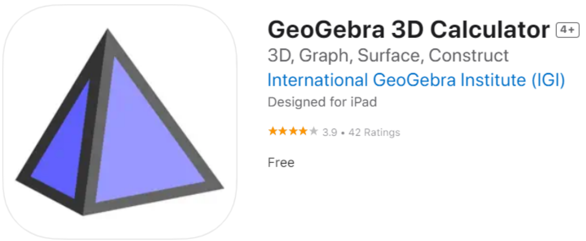


First of all, the users are required to drag on the point I to adjust the radius of the given insphere.

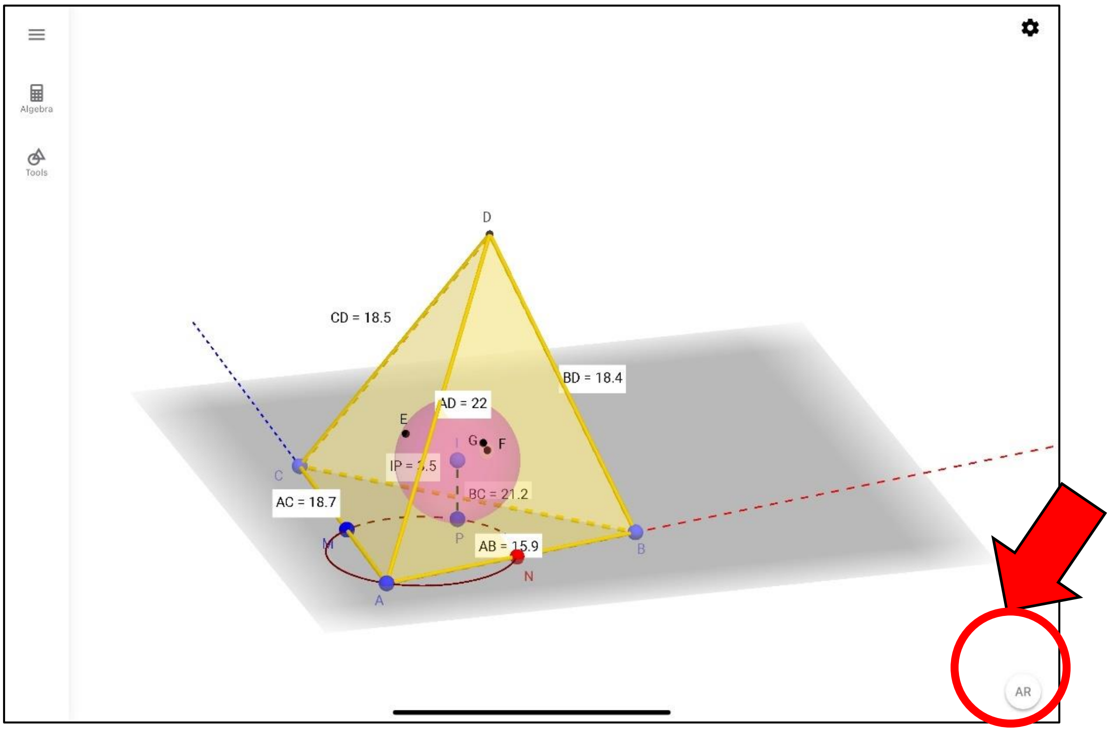
Then, by changing the position of the points A, M, N, B and C, the users can change the shape of the tetrahedron ABCD as they wish.

By referring the lengths of the edges AB, AC, AD, BC, BD and CD shown on the screen, the users can obtain the dimension of each face, which is sufficient for them to make the 3-D model of the tetrahedron.

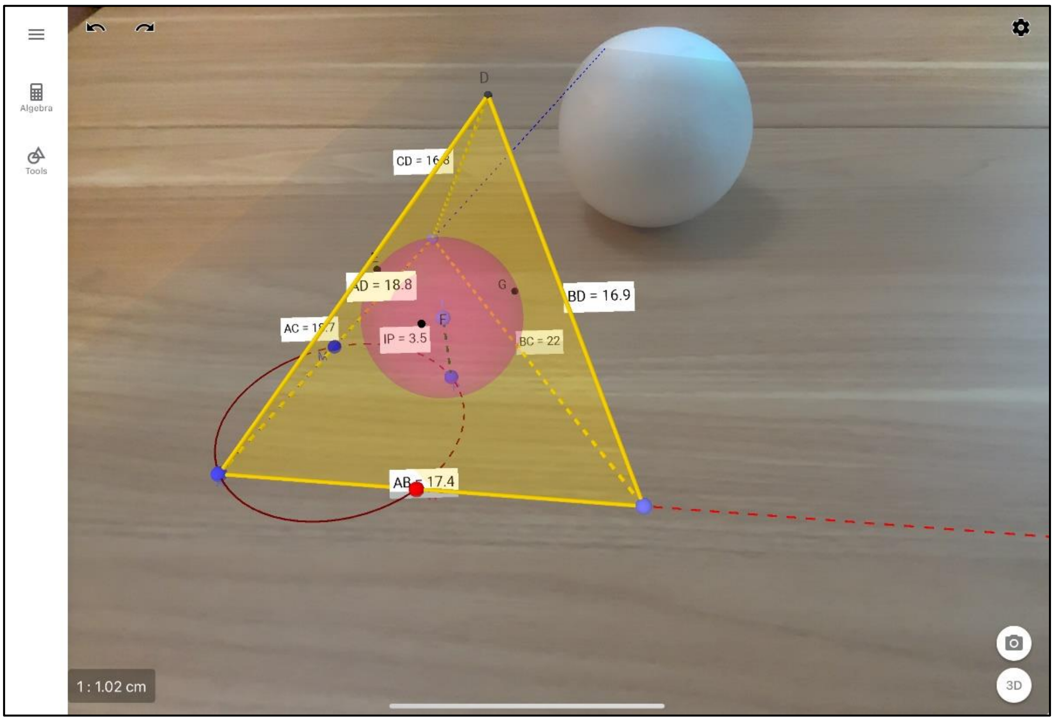
If the users want to know exactly how the size and shape of the tetrahedrons that can be formed, they can use the GeoGebra Augmented Reality (AR) with the App GeoGebra 3D Calculator:

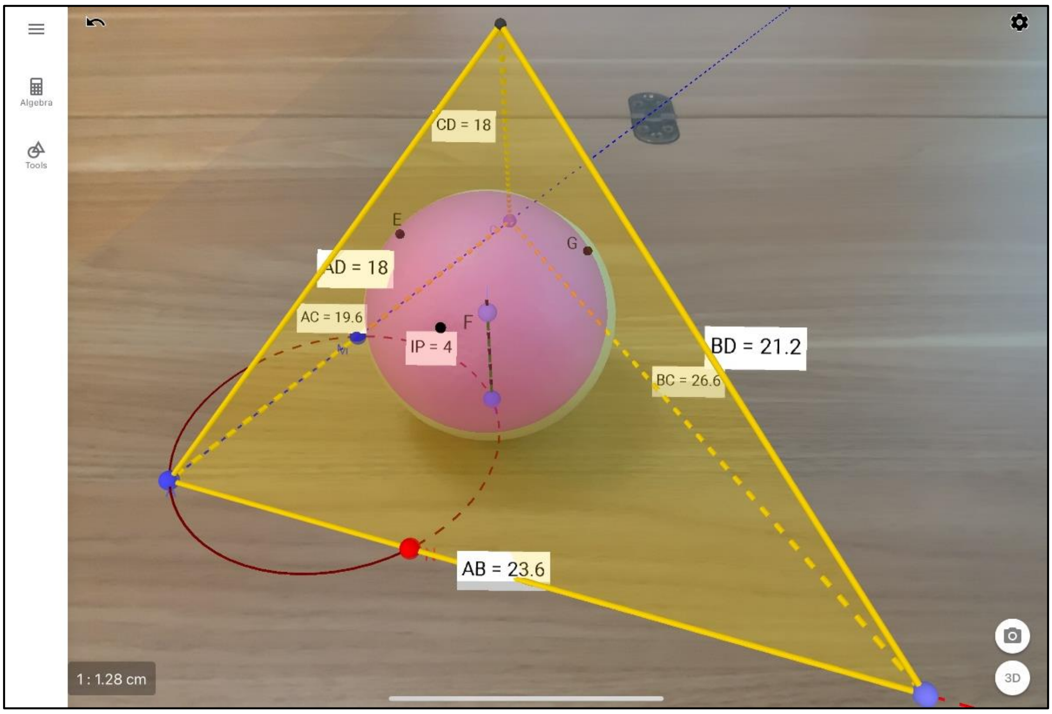
	iOS:	Android:

Once the GeoGebra 3D Calculator is installed, the users can use the built-in AR function by clicking the AR button on the screen:

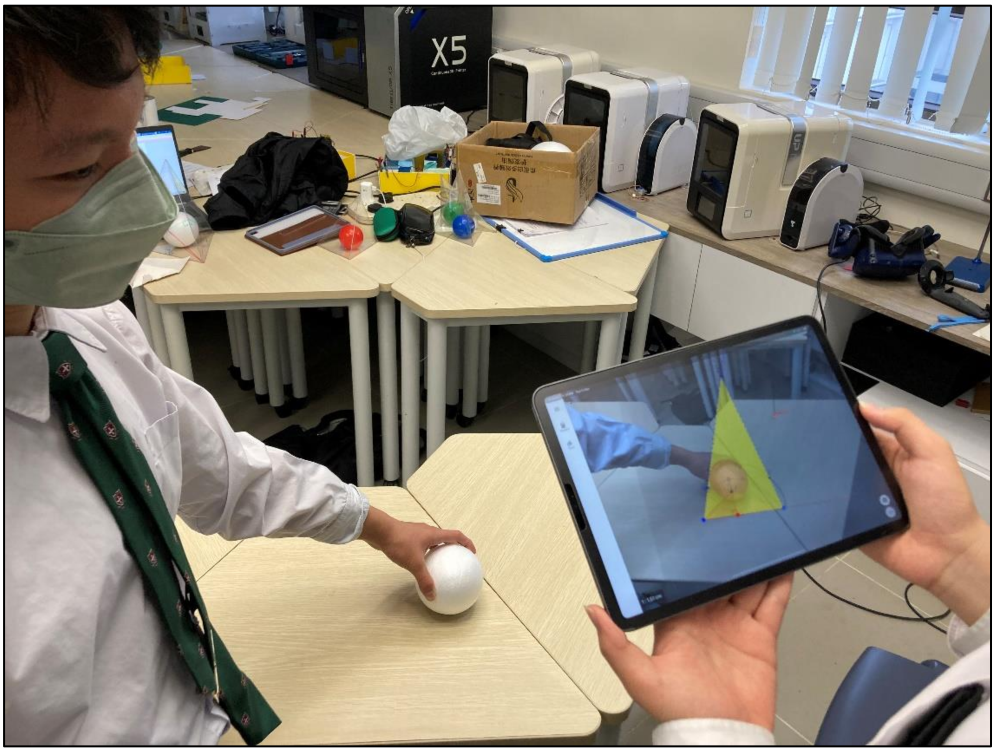


Once the AR function is activated, the user can know what the possible size and the possible shape of the tetrahedron that could be formed with the given sphere in real-time.





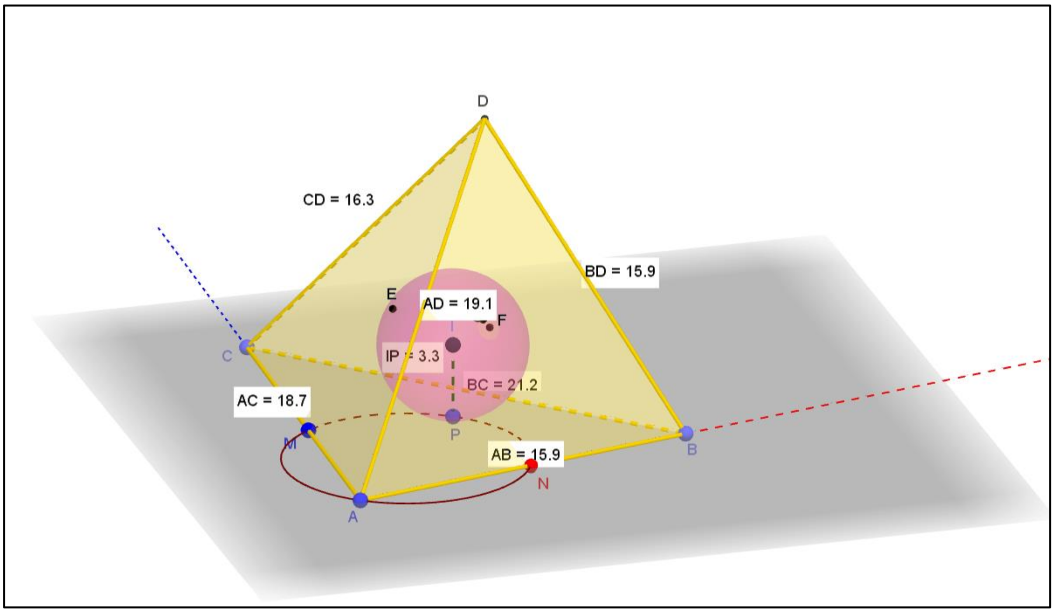
Through the App, it is very easy for us to make lots if different tetrahedrons with given insphere!



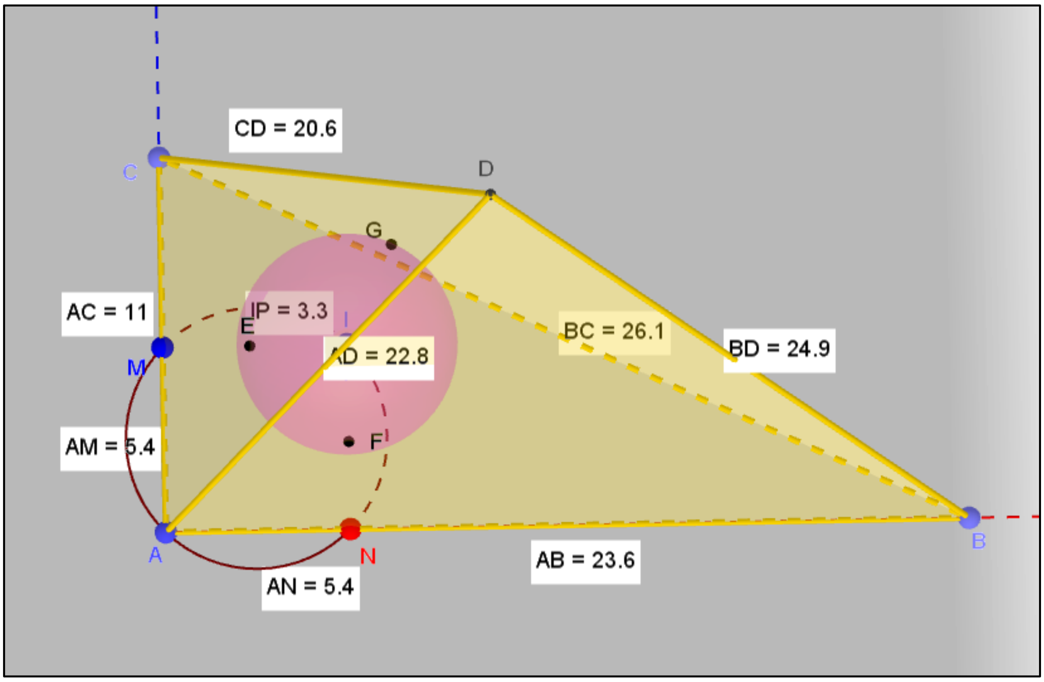
Here are the procedures that we made different tetrahedrons with given insphere:

(1) Making a 3-D model of trirectangular tetrahedron with given insphere

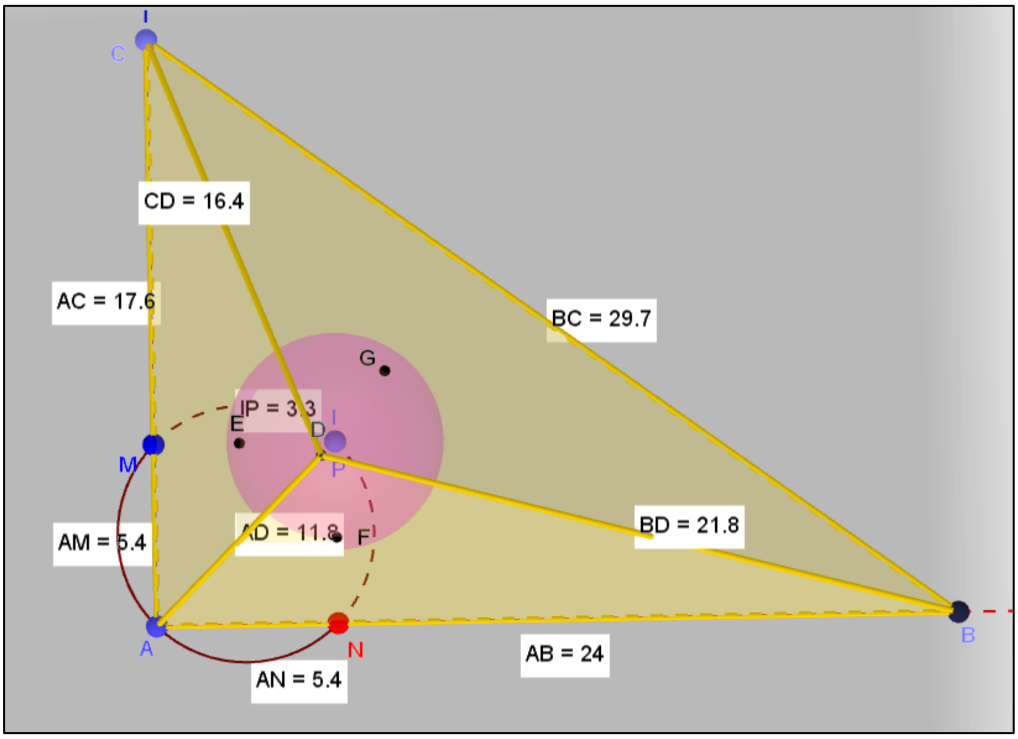
Step 1: We open the App and rotate the view of the tetrahedron to view it from the top.



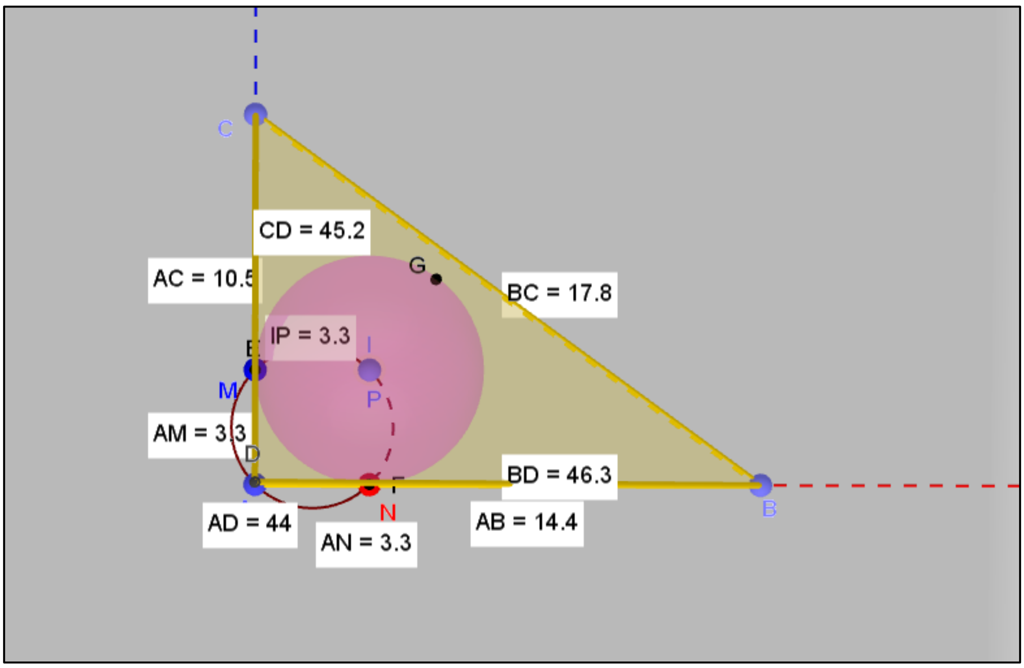
Step 2: Besides the lengths shown on the screen, we measure the lengths of AM and AN. Then we drag on the points A, M, N, B and C to make AM = AN and ∠BAC = 90°. Instead of showing the size of ∠BAC, we check whether $AB^2 + AC^2 = BC^2$ to determine if △ABC is a right-angled triangle.



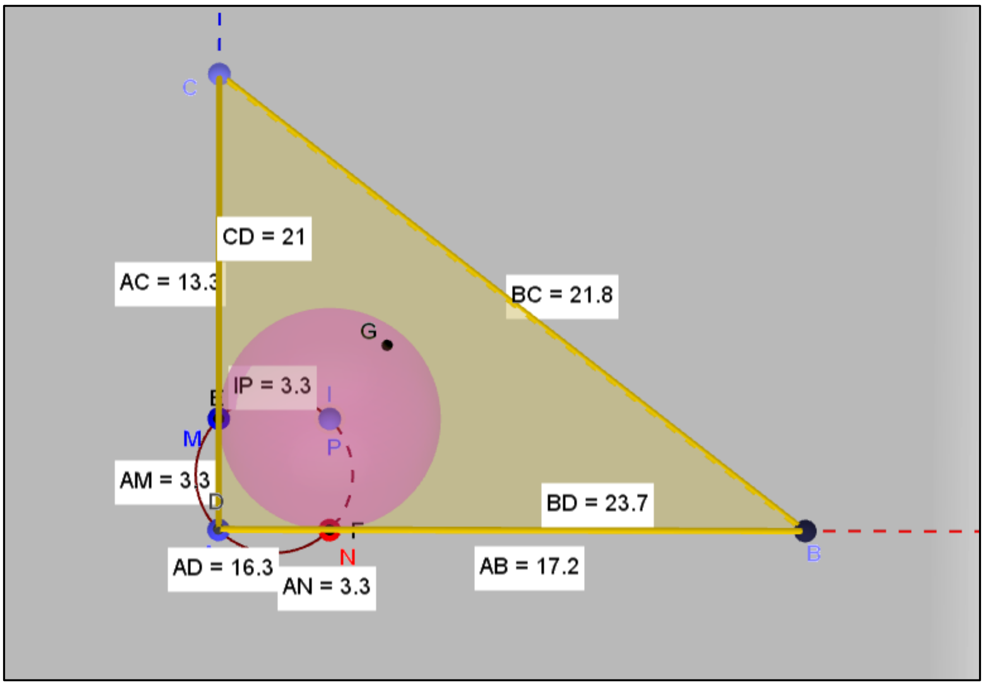
Step 3: Then we try to change the position of A, B and C to make D projects on A. It can be done by increasing the lengths of AB and AC first (then the projection of D will lie in △ABC)

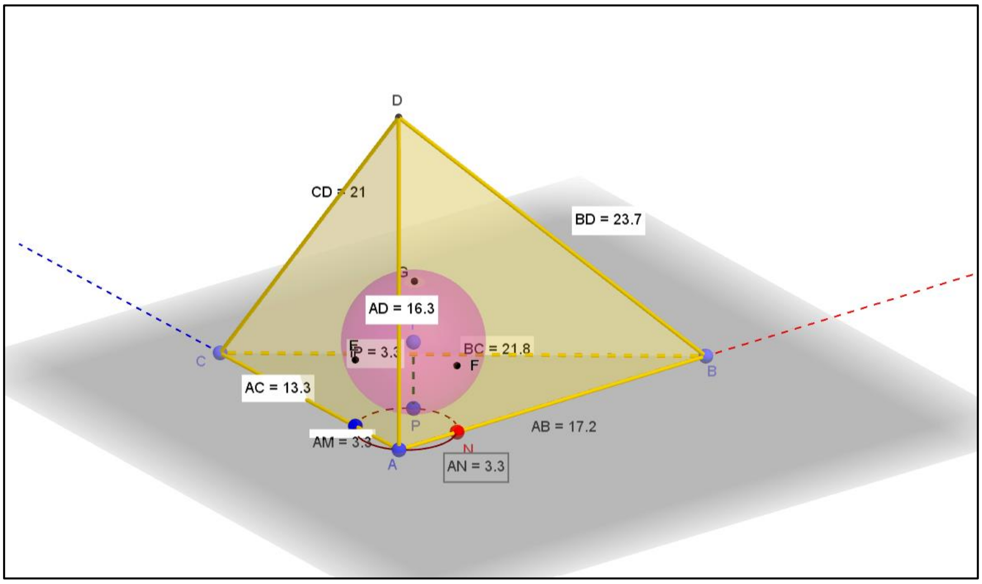


Afterwards, drag the point A to make AM = AN = IP = radius of the insphere. Once it is done, the tetrahedron ABCD will be a trirectangular tetrahedron with ∠BAC = ∠DAB = ∠DAC = 90°.

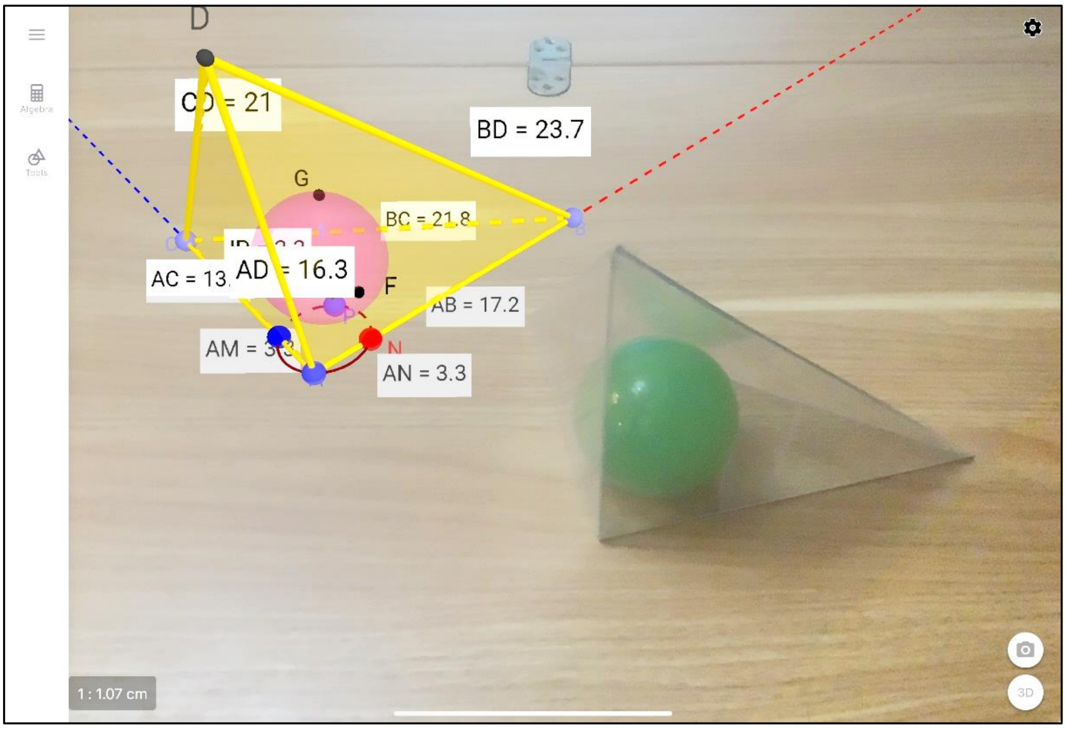


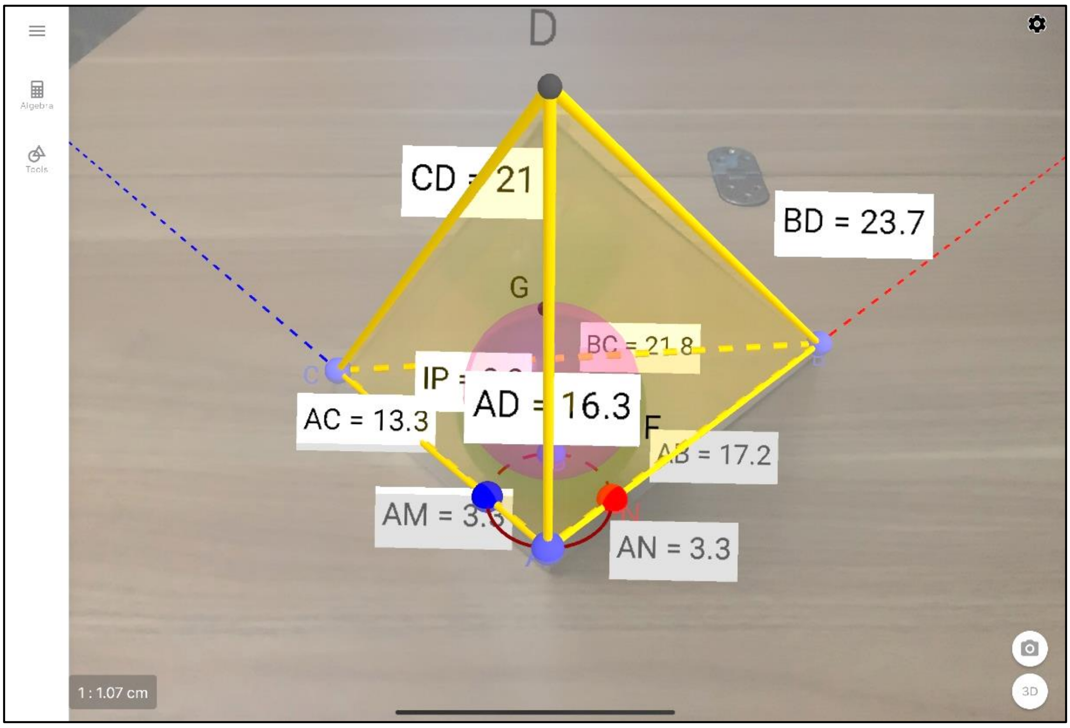
Step 4: Finally, adjust the size of the tetrahedron ABCD by dragging the points B and C (the tetrahedron ABCD will keep trirectangular through dragging). Then we can make it to become a real 3-D model!





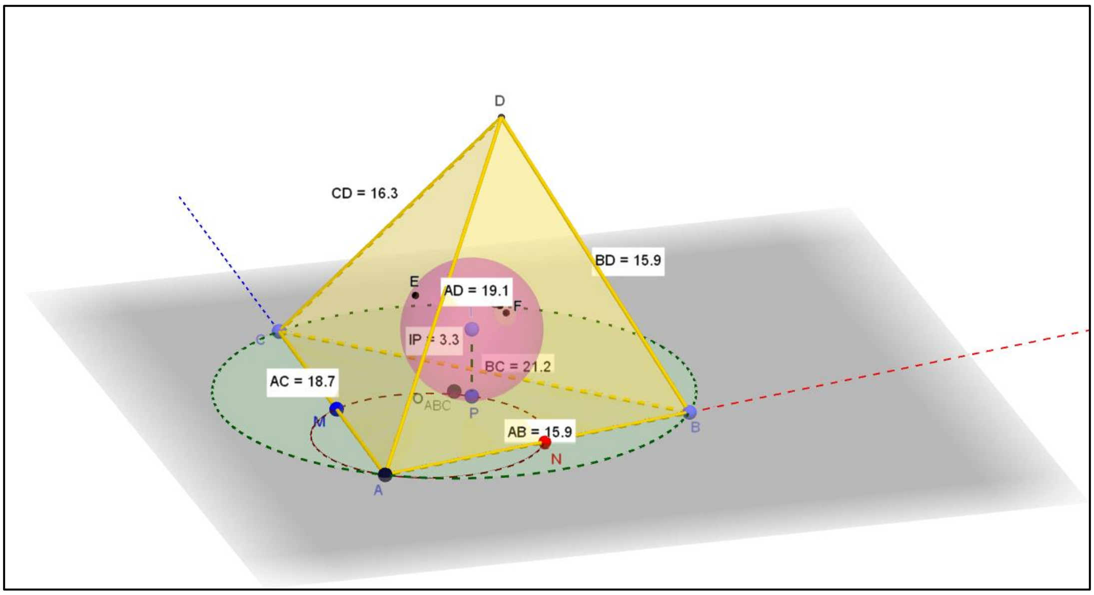
We made the 3-D model and checked it with GeoGebra AR, it is observed that the virtual (AR) model and the real 3-D model are identical and the job is done.

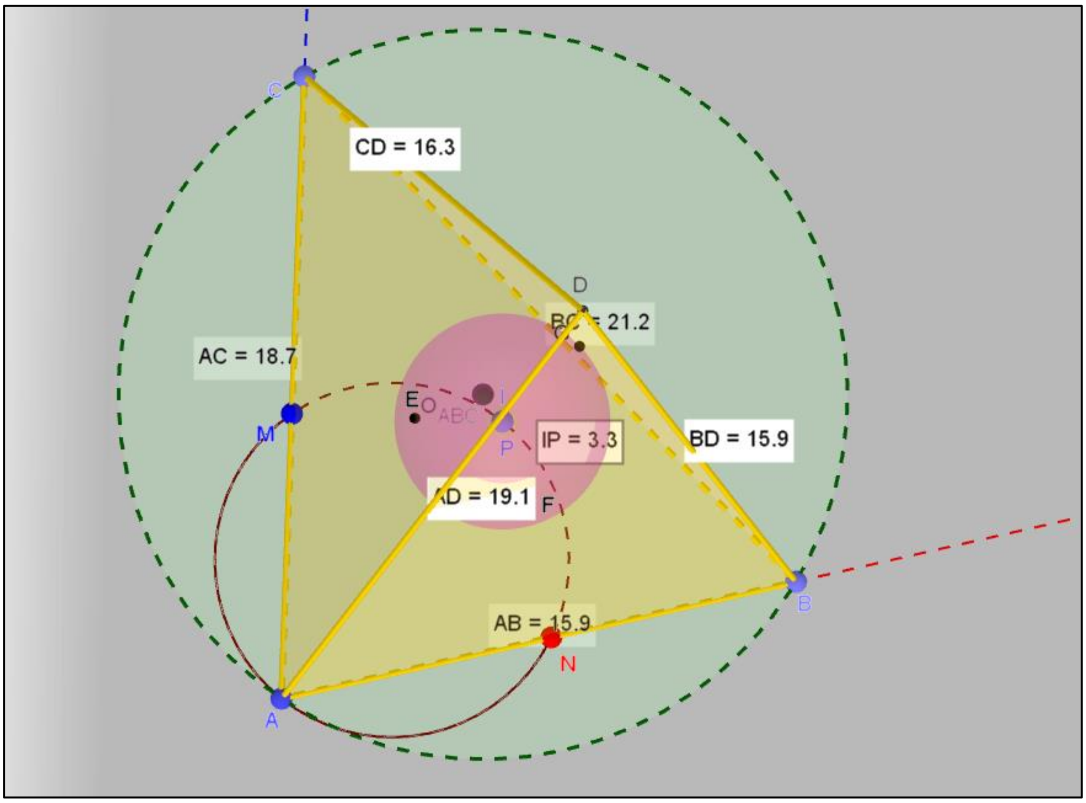




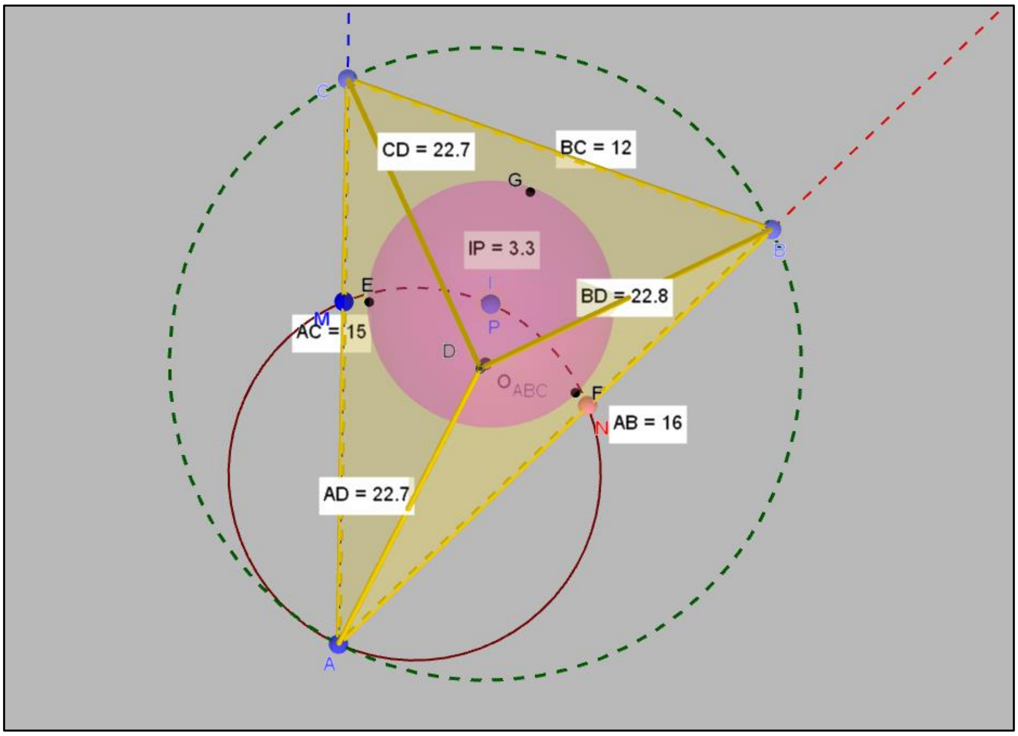
(2) Making a 3-D model of tetrahedron with equal slant edges with given insphere

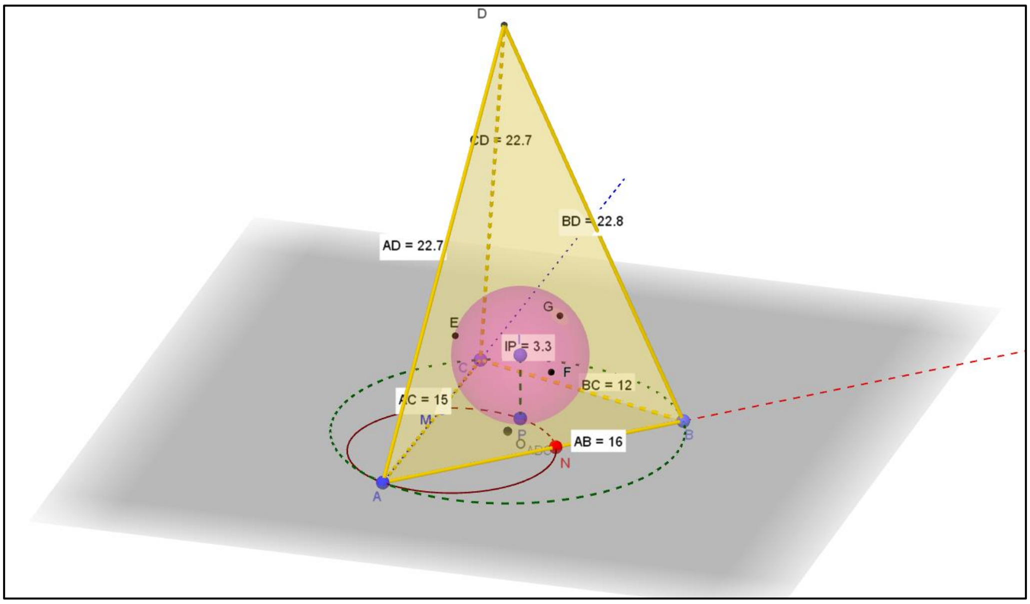
Step 1: We open the App and construct the circumcircle passing through A, B and C. Then mark the center of the circumcircle be O_{ABC} (both can be done by built-in tools in GeoGebra). Then we rotate the view of the tetrahedron to view it from the top.



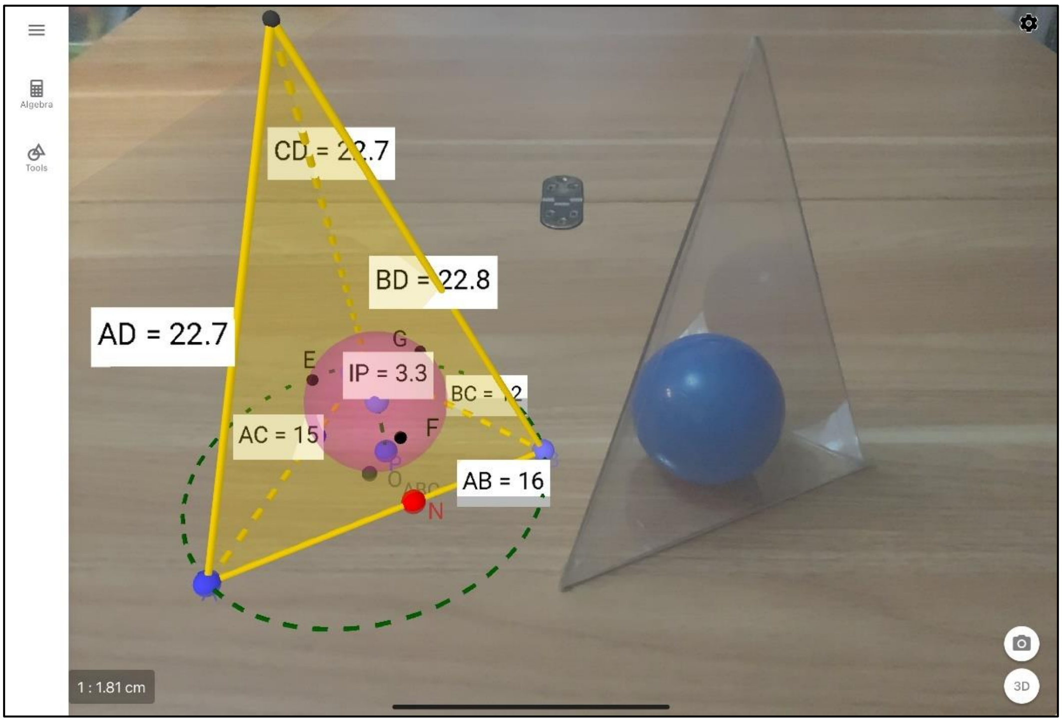


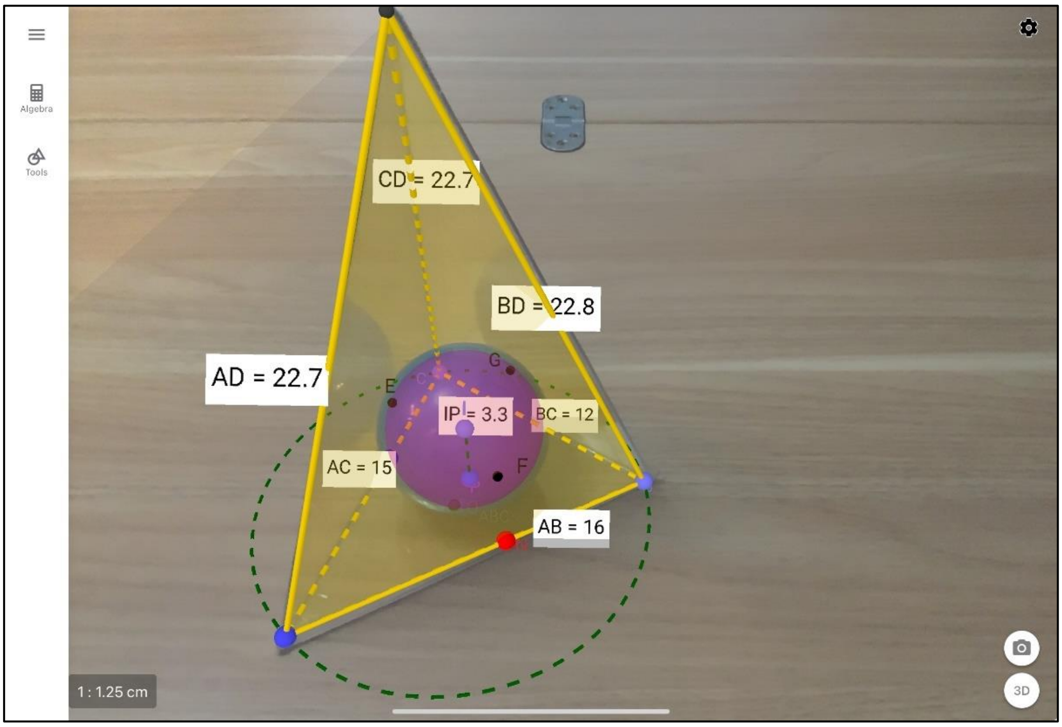
Step 3: Then we try to change the position of A, B and C to make D projects on O_{ABC}. It can be done by increasing the lengths of AB, AC and ∠MAN. Once it is done, then the lengths AD, BD and CD will be equal.





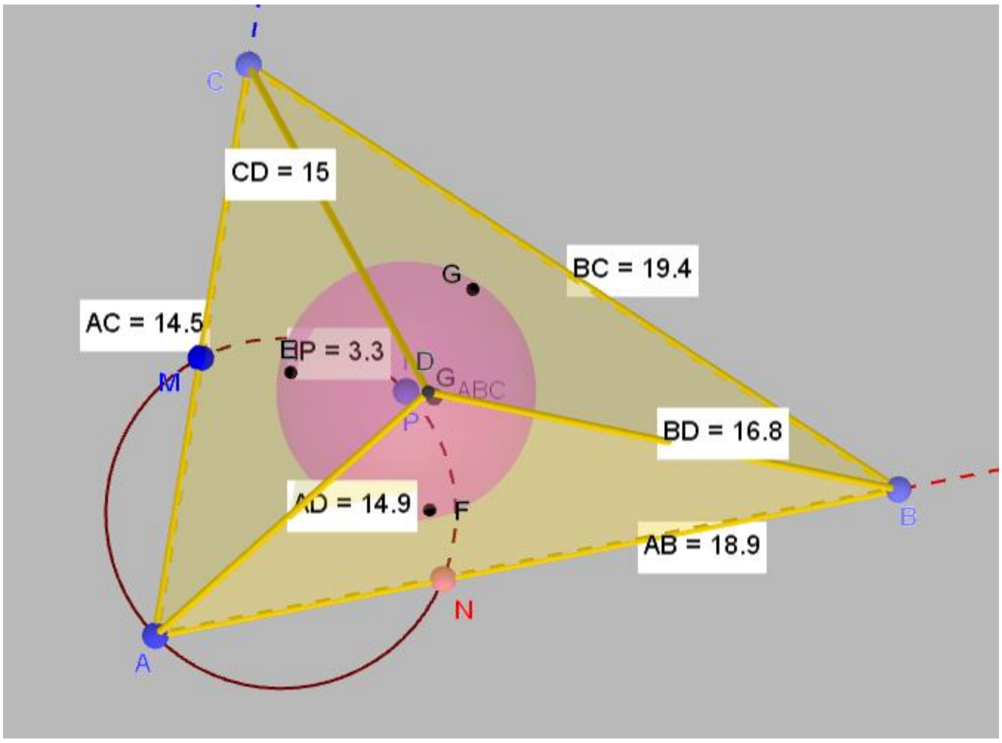
We made the 3-D model and checked it with GeoGebra AR, it is observed that the virtual (AR) model and the real 3-D model are identical and the job is done again.

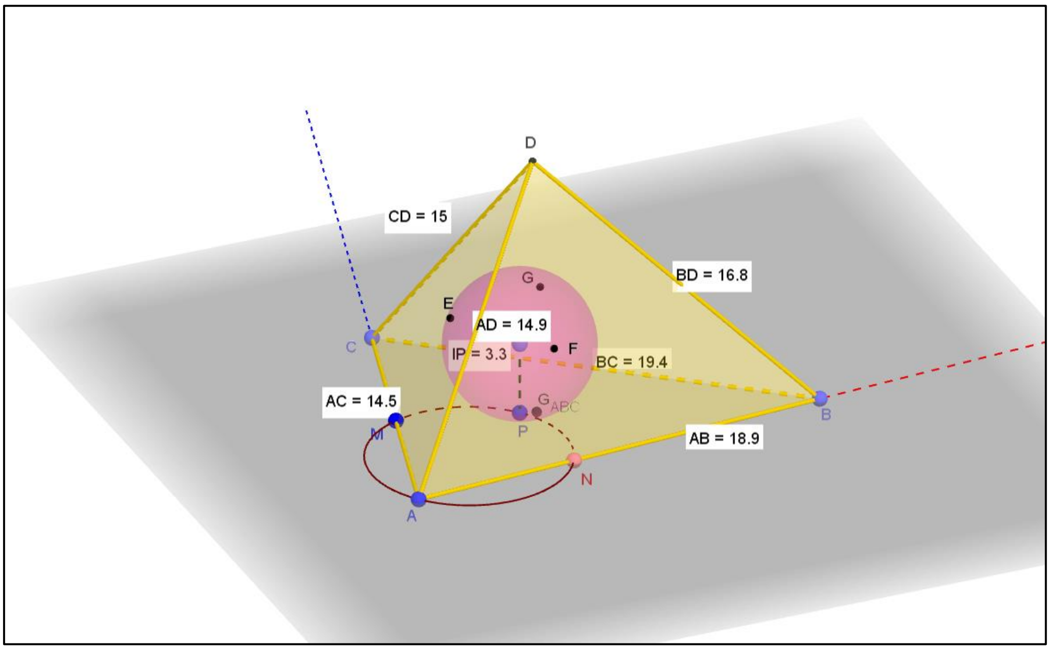




(3) Making a 3-D model of right triangular pyramid with given insphere

First, we mark the centroid of the △ABC be G_{ABC} (it can be done by input the command “Centroid(Polygon(A, B, C))” in the input bar of GeoGebra). Then we rotate the view of the tetrahedron to view it from the top. Then we make D projects on G_{ABC}. Once it is done, then ABCD will be a right triangular pyramid.

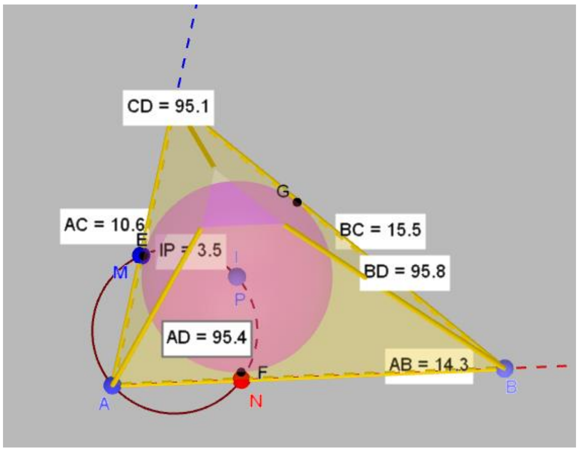
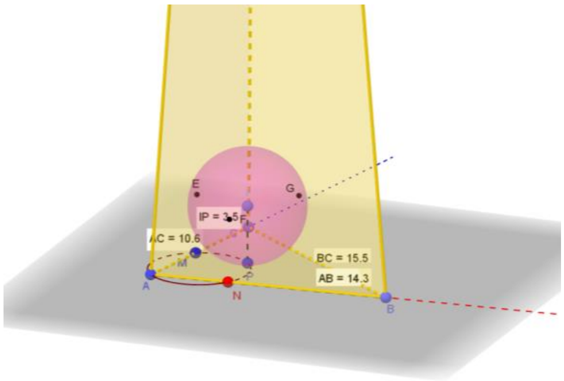
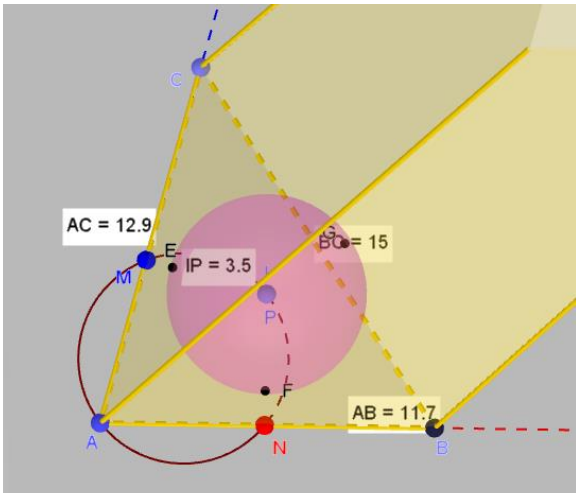
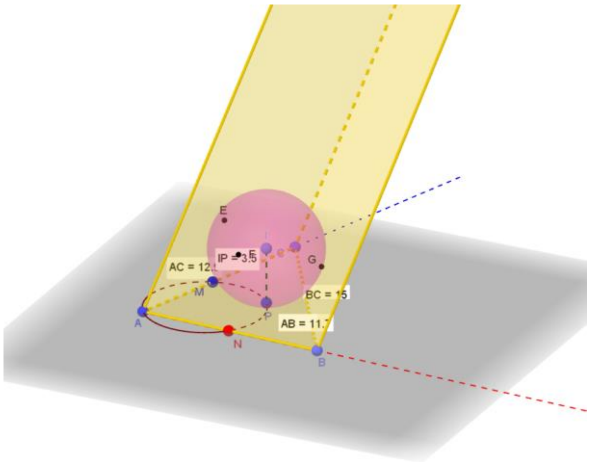




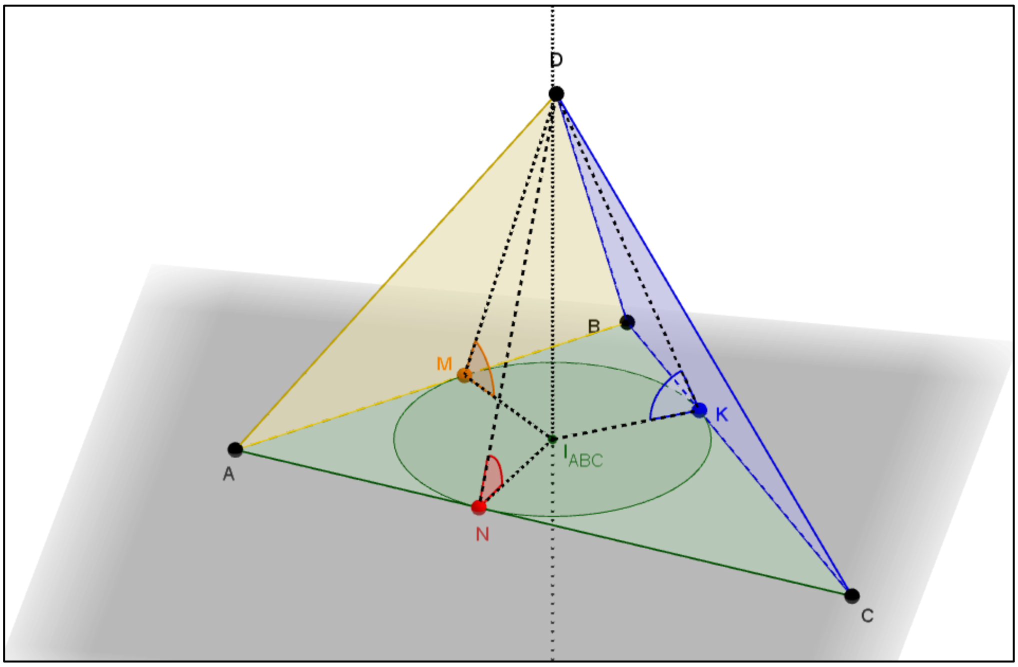
Remark:

In order to form a tetrahedron with given insphere, the size of the base of the tetrahedron should be large enough.

Be more specific, consider the base of the tetrahedron △ABC and denote the radius of the incircle of △ABC be r. Then the radius of the insphere of the tetrahedron must be less than r. But the projection of the insphere on △ABC is not necessary to be inscribed in △ABC.

Top view of the tetrahedron	3-D view of the tetrahedron
	
	

Bonus: What tetrahedron can we formed when D projects on the incenter of $\triangle ABC$ instead?



If D projects on I_{ABC}, which is the incenter of $\triangle ABC$, then we have

$DI_{ABC} \perp MI_{ABC}$ and $DI_{ABC} \perp NI_{ABC}$ and $DI_{ABC} \perp KI_{ABC}$

and $MI_{ABC} = NI_{ABC} = KI_{ABC}$ (incenter)

Therefore, $DMI_{ABC} \cong DNI_{ABC} \cong DKI_{ABC}$ (S.A.S)

Furthermore,

$AB \perp MI_{ABC}$ and $AC \perp NI_{ABC}$ and $BC \perp KI_{ABC}$ (tangent $\perp$ radius)

$DM \perp AB$ and $DN \perp AC$ and $DK \perp BC$ (theorem of three perpendiculars)

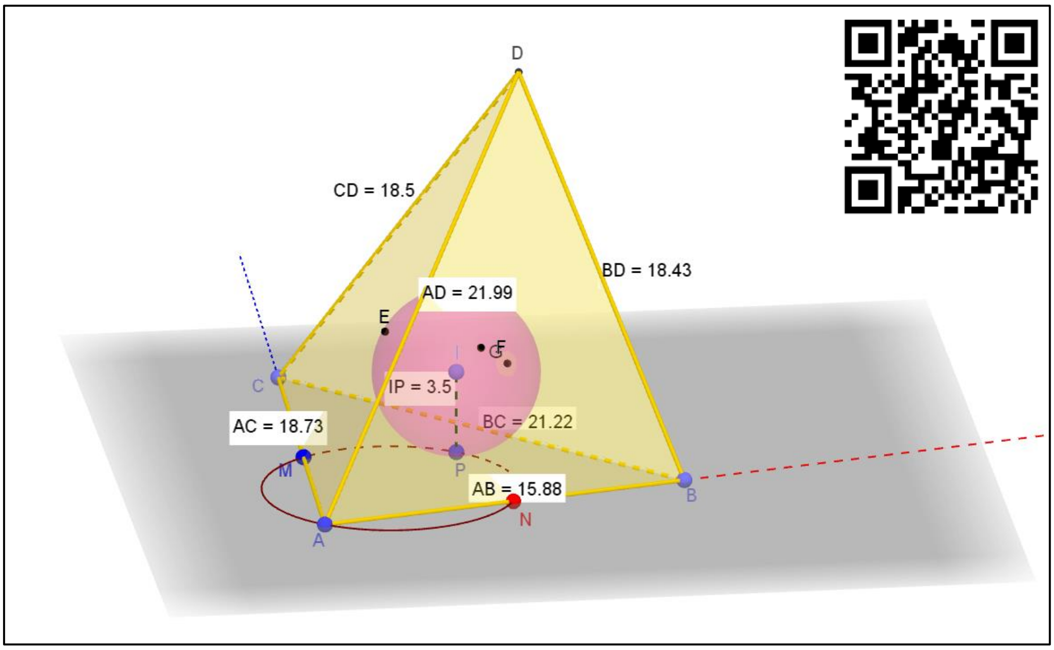
To conclude, $\angle DMI_{ABC}$, $\angle DNI_{ABC}$ and $\angle DKI_{ABC}$ are angles between the lateral faces ($\triangle ABD$, $\triangle ACD$ and $\triangle BCD$) and the base ($\triangle ABC$). As $DMI_{ABC} \cong DNI_{ABC} \cong DKI_{ABC}$, we have $\angle DMI_{ABC} = \angle DNI_{ABC} = \angle DKI_{ABC}$. In other words, the angles between the lateral faces and the base will be equal.

7. Summary

In this project, we show that the incenter of a tetrahedron is very difficult to be located by the angle bisectors between the faces of the tetrahedron. Therefore, we find another way to locate the incenter of the tetrahedron by using 3-D trigonometry instead.

Furthermore, from the work of locating the incenter of the tetrahedron, we discover a way to construct a tetrahedron with given insphere. We use the straightedge and compass construction from 2-D to 3-D and make the construction in GeoGebra 3-D calculator. It let us develop an App to help uses to construct their own tetrahedron with given insphere, with the aid of GeoGebra augmented reality (AR) technology.

GeoGebra App: https://www.geogebra.org/3d/ezaz4g8r



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