Lesson 5: Two equations for each relationship

Goals

- Use the word "reciprocal" to explain (orally and in writing) that there are two related constants of proportionality for a proportional relationship.
- Write two equations that represent the same proportional relationship, i.e., $y = kx$ and $x = (\frac{1}{k})y$, and explain (orally) what each equation means.

Learning Targets

- I can find two constants of proportionality for a proportional relationship.
- I can write two equations representing a proportional relationship described by a table or story.

Lesson Narrative

In previous lessons students saw that a proportional relationships can be viewed in two ways, depending on which quantity you regard as being proportional to the other. In this lesson they write equations for these two ways, and they see why the two constants of proportionality associated with each way are reciprocals of each other.

The activities in this lesson use familiar contexts, but not identical situations from previous lessons: measurement conversions and water flowing at a constant rate. Students are expected to use methods developed earlier: organise data in a table, write and solve an equation to determine the constant of proportionality, and generalise from repeated calculations to arrive at an equation. After students write or use an equation, they interpret their answers in the context of the situation.

Building On

- Analyse patterns and relationships.

Addressing

- Analyse proportional relationships and use them to solve real-world and mathematical problems.
- Recognise and represent proportional relationships between quantities.
- Identify the constant of proportionality (unit rate) in tables, graphs, equations, diagrams, and verbal descriptions of proportional relationships.
- Represent proportional relationships by equations. For example, if total cost t is proportional to the number n of items purchased at a constant price p, the relationship between the total cost and the number of items can be expressed as $t = pn$.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Stronger and Clearer Each Time
- Co-Craft Questions
- Discussion Supports
- Think Pair Share

Student Learning Goals

Let's investigate the equations that represent proportional relationships.

5.1 Missing Figures

Warm Up: 5 minutes

This warm-up encourages students to look for regularity in how the tiles in the image are growing and describe this pattern using ratios as a review of their work earlier in KS3. Students may use each colour to reason about missing figures while others may reason about the way the tiles are arranged.

Launch

Arrange students in groups of 2. Display the image for all to see and tell students that the collection of tiles is growing, but we can only see the second and fourth images. Ask students to look for a pattern in the sequence of figures and to give a signal when they have thought of one. Give students 1 minute of quiet think time, and then time to discuss their patterns and arrangements for Figures 1 and 3 with a partner. Tell students to use what they discussed in figuring out their answers to the next questions.

Student Task Statement

Here are the second and fourth figures in a pattern.

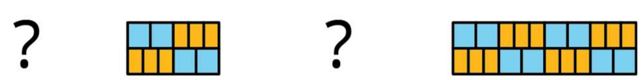

figure 1 figure 2 figure 3 figure 4

1. What do you think the first and third figures in the pattern look like?
2. Describe the 10th figure in the pattern.

Student Response

1. Figure 1 has 2 blue tiles and 3 yellow tiles and Figure 3 has 6 blue tiles and 9 yellow tiles.

2. Figure 10 would have 20 blue tiles and 30 yellow tiles.

Activity Synthesis

Invite students to share their responses and reasoning. Record and display the different ways of thinking for all to see. If possible, record the relevant reasoning on or near the images themselves. After each explanation, ask the class if they agree or disagree and to explain alternative ways of thinking, referring back to what is happening in the images each time.

Students may use each colour to reason about missing figures while others may reason about the way the tiles are arranged. Emphasise both insights in addition to the usefulness of Figure 1 as students share their strategies.

5.2 Metres and Centimetres

10 minutes

This activity is intended to build confidence and facility in writing an equation for a proportional relationship. Students build on their understanding that measurement conversions can be represented by proportional relationships, which they studied in an earlier lesson and will revisit in future lessons. Students are expected to use methods developed earlier: organise data in a table, write and solve an equation to determine the constant of proportionality, and use repeated reasoning to arrive at an equation. In addition, they are expected to identify the relationship between the constant of proportionality when going from centimetres to metres and vice versa (reciprocals). Since students have already explored a similar relationship (centimetres and millimetres) in a previous lesson, this activity may go very quickly.

As students work, look for students writing an equation like $100k = 1$ (for table 2) as a step to finding the constant of proportionality. Encourage them to say how they would solve the equation. Ask students to say why using this equation makes sense in the scenario.

Instructional Routines

- Discussion Supports

Launch

Introduce the task: "In a previous lesson, you examined the relationship between millimetres and centimetres. Today we will examine the relationship between centimetres and metres."

Student Task Statement

There are 100 centimetres (cm) in every metre (m).

length (m)	length (cm)
1	100
0.94	
1.67	
57.24	
x	

length (cm)	length (m)
100	1
250	
78.2	
123.9	
y	

1. Complete each of the tables.
2. For each table, find the constant of proportionality.
3. What is the relationship between these constants of proportionality?
4. For each table, write an equation for the proportional relationship. Let x represent a length measured in metres and y represent the same length measured in centimetres.

Student Response

1. Tables:

length (m)	length (cm)
1	100
0.94	94
1.67	167
57.24	5 724
x	$100x$

length (cm)	length (m)
100	1
250	2.5
78.2	0.782
123.9	1.239

y	$0.01y$

2. The constant of proportionality for the first table is 100, and the second is 0.01 or $\frac{1}{100}$.

3. The constants of proportionality are reciprocals.

4. $y = 100x$ and $x = 0.01y$.

Are You Ready for More?

1. How many cubic centimetres are there in a cubic metre?

2. How do you convert cubic centimetres to cubic metres?

3. How do you convert the other way?

Student Response

1. 1 000 000

2. multiply by $\frac{1}{1\,000\,000}$

3. multiply by 1 000 000

Activity Synthesis

Invite students to share their answers. These questions can guide the discussion:

- "How can we find an equation for each table?"
- "Where does the constant of proportionality occur in the table and equation?"
- "What is the relationship between the two constants of proportionality? How can you use the equations to see why this should be true?"

The equations can help students see why the constants of proportionality are reciprocals:

$$y = 100x$$

$$\left(\frac{1}{100}\right)y = \frac{1}{100} \times (100x)$$

$$\left(\frac{1}{100}\right)y = x$$

$$x = \left(\frac{1}{100}\right)y$$

This line of reasoning illustrated above should be accessible to students, because it builds on earlier work with expressions and equations.

Ask students to interpret the meaning of the equations in the context: "What do the equations tell us about the conversion from metres to centimetres and back?"

- Given the length in metres, to find the length in centimetres, multiply the number of centimetres by 100.
- Given the length in centimetres, to find the length in metres, multiply the number of metres by $\frac{1}{100}$.

Engagement: Develop Effort and Persistence. Encourage and support opportunities for peer interactions. Prior to the whole-class discussion, invite students to share their work with a partner. Display sentence frames to support student conversation such as: “First, I _____ because….,” “I noticed _____ so I….,” “Why did you . . .?,” and “I agree/disagree because….”
Supports accessibility for: Language; Social-emotional skills Speaking: Discussion supports. Provide students with sentence frames such as “I found the constant of proportionality by _____” to help explain their reasoning. Revoice student ideas to model mathematical language use by restating a response as a question in order to clarify, and apply appropriate language.
Design Principle(s): Maximise linguistic and cognitive awareness

5.3 Filling a Water Cooler

15 minutes

The theme continues by asking students to make sense of the two rates associated with a given proportional relationship. Here, students are asked to reason from an equation rather than a table, although they may find it helpful to create a table or graph. In this particular example, students work with both number of gallons per minute and number of minutes per gallon.

Monitor for students who are using different ways to decide if the cooler was filling faster before or after the flow rate was changed.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Stronger and Clearer Each Time
- Think Pair Share

Launch

Give students 5 minutes quiet work time followed by sharing with a partner and a whole-class discussion.

Action and Expression: Internalise Executive Functions. To support development of organisational skills, check in with students within the first 2–3 minutes of work time.

Check to make sure students have identified the correct equations that represent the relationship between w and t and can justify their reasoning.
Supports accessibility for: Memory; Organisation Writing, conversing: Stronger and Clearer Each Time. Use this routine to help students improve their written mathematical argument for whether the cooler is filling faster before or after Priya changed the rate of water flow. Give students time to meet with 2–3 partners to share and get feedback on their initial drafts. Display feedback prompts that will help students strengthen their ideas and clarify their language. For example, "Can you explain how . . .", "Is there another way to say . . . ?", and "How do you know . . . . ? Invite students to go back and revise or refine their written argument based on the feedback from peers. This will help students understand situations in which two different rates are associated with the same proportional relationship through communicating their reasoning with a partner.
Design Principle(s): Optimise output (for justification); Cultivate conversation

Anticipated Misconceptions

For the first question, if students struggle to identify the correct equations, encourage them to create two tables of values for the situation. Encourage them to create rows for both unit rates, in order to foster connections to prior learning.

Student Task Statement

It took Priya 5 minutes to fill a cooler with 8 gallons of water from a tap that was flowing at a steady rate. Let w be the number of gallons of water in the cooler after t minutes.

1. Which of the following equations represent the relationship between w and t? Select **all** that apply.

   a. $w = 1.6t$

   b. $w = 0.625t$

   c. $t = 1.6w$

   d. $t = 0.625w$

2. What does 1.6 tell you about the situation?

3. What does 0.625 tell you about the situation?

4. Priya changed the rate at which water flowed through the tap. Write an equation that represents the relationship of w and t when it takes 3 minutes to fill the cooler with 1 gallon of water.

5. Was the cooler filling faster before or after Priya changed the rate of water flow? Explain how you know.

Student Response

1. Both $w = 1.6t$ and $t = 0.625w$ represent the relationship of w and t.

2. 1.6 tells you that water is flowing at 1.6 gallons per minute.

3. 0.625 tells you that it takes 0.625 minutes for a gallon of water to flow out of the tap (or into the cooler).

4. $t = 3w$ or $w = \frac{1}{3}t$

5. The cooler filled faster before Priya changed the rate of water flow. Before the change, it took 0.625 minutes to get one gallon, but after, it took 3 minutes to get one gallon. (Alternatively, before the change she got 1.6 gallons per minute, but after the change she only got $\frac{1}{3}$ of a gallon per minute.)

Activity Synthesis

Select students to share their answers. Elicit both responses from the class, and be sure to identify connections between them.

Select students who used different explanations to share their answers to the last question.

5.4 Feeding Shrimp

Optional: 10 minutes

This activity provides an additional opportunity for students to represent a proportional relationship with two related equations in a new context. This situation builds on the earlier work they did with feeding a crowd, but includes more complicated calculations. Students interpret the meaning of the constants of proportionality in the context of the situation and use the equations to answer questions.

The first few questions ask about 1 shrimp. The question about feeding 10 shrimp helps prepare students for work they will do in the next lesson with multiple quantities that are in proportional relationships to each other.

Instructional Routines

- Co-Craft Questions
- Think Pair Share

Launch

Arrange students in groups of 2. Give students 6 minutes of partner work time followed by whole-class discussion.

Action and Expression: Internalise Executive Functions. Chunk this task into more manageable parts to support students who benefit from support with organisational skills in problem solving. For example, present one question at a time and monitor students to ensure they are making progress throughout the activity.
Supports accessibility for: Organisation; Attention Speaking, Reading: Co-Craft Questions. Use this routine to help students interpret the language of proportional relationships, and to increase awareness of language used to talk about proportional relationships. Display only the first sentence of this problem ("At an aquarium, a shrimp is fed $\frac{1}{5}$ gram of food each feeding and is fed 3 times each day."), and ask students to write down possible mathematical questions that could be asked about the situation. Invite students to compare their questions before revealing the remainder of the question. Listen for and amplify any questions involving relationships between the two quantities in this task (amount of food and number of feedings).
Design Principle(s): Maximise meta-awareness; Support sense-making

Student Task Statement

At an aquarium, a shrimp is fed $\frac{1}{5}$ gram of food each feeding and is fed 3 times each day.

1. How much food does a shrimp get fed in one day?
2. Complete the table to show how many grams of food the shrimp is fed over different numbers of days.

number of days	food in grams
1	
7	
30	

3. What is the constant of proportionality? What does it tell us about the situation?

4. If we switched the columns in the table, what would be the constant of proportionality? Explain your reasoning.

5. Use d for number of days and f for amount of food in grams that a shrimp eats to write *two* equations that represent the relationship between d and f.

6. If a tank has 10 shrimp in it, how much food is added to the tank each day?

7. If the aquarium manager has 300 grams of shrimp food for this tank of 10 shrimp, how many days will it last? Explain or show your reasoning.

Student Response

1. $\frac{3}{5}$ grams

2.

number of days	food in grams
1	$\frac{3}{5}$
7	$4\frac{1}{5}$
30	18

3. $\frac{3}{5}$ The shrimp is fed $\frac{3}{5}$ grams of food per day.

4. $\frac{5}{3}$ since the reciprocal of $\frac{3}{5}$ is $\frac{5}{3}$.

5. $f = \frac{3}{5}d$ and $d = \frac{5}{3}f$

6. 6 grams

7. 50 days (300 ÷ 6 = 50)

Activity Synthesis

Invite students to share their answers. Ask students which equation was most useful to answer each of the last two questions.

Lesson Synthesis

Briefly revisit the first two activities to summarise the important points of the lesson. For example:

- We examined the proportional relationship between metres and centimetres. Why were we able to write two equations for this situation? What were they? What were the constants of proportionality?
- We examined a proportional relationship where we knew how long it took to fill a water cooler with a certain amount of water. What were the constants of proportionality for this relationship? What equations did we determine would represent this situation?
- In each case, what was the relationship between the two constants of proportionality and between the two equations?

5.5 Flight of the Albatross

Cool Down: 5 minutes

Student Task Statement

An albatross is a large bird that can fly 400 kilometres in 8 hours at a constant speed. Using d for distance in kilometres and t for number of hours, an equation that represents this situation is $d = 50t$.

1. What are two constants of proportionality for the relationship between distance in kilometres and number of hours? What is the relationship between these two values?

2. Write another equation that relates d and t in this context.

Student Response

1. 50 and $\frac{1}{50}$. These numbers are reciprocals.

2. $t = \frac{1}{50}d$

Student Lesson Summary

If Kiran rode his bike at a constant 10 miles per hour, his distance in miles, d, is proportional to the number of hours, t, that he rode. We can write the equation $d = 10t$ With this equation, it is easy to find the distance Kiran rode when we know how long it took because we can just multiply the time by 10.

We can rewrite the equation:

$$\begin{aligned} d &= 10t \\ \left(\frac{1}{10}\right)d &= t \\ t &= \left(\frac{1}{10}\right)d \end{aligned}$$

This version of the equation tells us that the amount of time he rode is proportional to the distance he travelled, and the constant of proportionality is $\frac{1}{10}$. That form is easier to use when we know his distance and want to find how long it took because we can just multiply the distance by $\frac{1}{10}$.

When two quantities x and y are in a proportional relationship, we can write the equation $y = kx$ and say, "y is proportional to x." In this case, the number k is the corresponding constant of proportionality. We can also write the equation $x = \frac{1}{k}y$ and say, "x is proportional to y." In this case, the number $\frac{1}{k}$ is the corresponding constant of proportionality. Each one can be useful depending on the information we have and the quantity we are trying to figure out.

Lesson 5 Practice Problems

1. **Problem 1 Statement**

   The table represents the relationship between a length measured in metres and the same length measured in kilometres.

   a. Complete the table.

   b. Write an equation for converting the number of metres to kilometres. Use x for number of metres and y for number of kilometres.

metres	kilometres
1 000	1
3 500	
500	
75	
1	
x	

Solution

metres	kilometres
1 000	1
3 500	3.5
500	0.5
75	0.075
1	0.001
x	$0.001x$

$y = 0.001x$ or equivalent

2. **Problem 2 Statement**

Concrete building blocks weigh 28 pounds each. Using b for the number of concrete blocks and w for the weight, write two equations that relate the two variables. One equation should begin with $w =$ and the other should begin with $b =$.

Solution

$w = 28b$ and $b = \frac{1}{28}w$

3. **Problem 3 Statement**

A store sells rope by the metre. The equation $p = 0.8L$ represents the price p (in pounds) of a piece of nylon rope that is L metres long.

a. How much does the nylon rope cost per metre?

b. How long is a piece of nylon rope that costs £1.00?

Solution

a. £0.80.

b. 1.25 metres or $\frac{5}{4}$ metres or $1\frac{1}{4}$ metres.

4. **Problem 4 Statement**

The table represents a proportional relationship. Find the constant of proportionality and write an equation to represent the relationship.

a	y
2	$\frac{2}{3}$
3	1
10	$\frac{10}{3}$
12	4

Constant of proportionality: ________

Equation: $y =$

Solution

Constant of proportionality: $\frac{1}{3}$

Equation: $y = \frac{1}{3}a$

5. **Problem 5 Statement**

On a map of Chicago, 1 cm represents 100 m. Select **all** statements that express the same scale.

a. 5 cm on the map represents 50 m in Chicago.

b. 1 mm on the map represents 10 m in Chicago.

c. 1 km in Chicago is represented by 10 cm the map.

d. 100 cm in Chicago is represented by 1 m on the map.

Solution ["B", "C"]