Lesson 6: Finding side lengths of triangles

Goals

- Comprehend the term "Pythagoras' theorem" (in written and spoken language) as the equation $a^2 + b^2 = c^2$ where a and b are the lengths of the shorter sides and c is the length of the hypotenuse of a right-angled triangle.
- Describe (orally) patterns in the relationships between the side lengths of triangles.
- Determine the exact side lengths of a triangle in a coordinate grid and express them (in writing) using square root notation.

Learning Targets

- I can explain what Pythagoras' theorem says.

Lesson Narrative

This is the first of three lessons in which students investigate relationships between the side lengths of right and non-right-angled triangles leading to Pythagoras' theorem.

In the warm-up for this lesson, students notice and wonder about 4 triangles. While there is a lot to notice, one important aspect is whether the triangle is a right-angled triangle or not. This primes them to notice patterns of right and non-right-angled triangles in the other activities in the lesson. In the next two activities, students systematically look at the side lengths of right and non-right-angled triangles for patterns. By the end of this lesson, they see that for right-angled triangles with shorter sides a and b and **hypotenuse** c, the side lengths are related by $a^2 + b^2 = c^2$. In the next lesson they will prove Pythagoras' theorem.

Building On

- Classify two-dimensional figures in a hierarchy based on properties.
- Draw, construct, and describe geometrical figures and describe the relationships between them.
- Use square root and cube root symbols to represent solutions to equations of the form $x^2 = p$ and $x^3 = p$, where p is a positive rational number. Evaluate square roots of small perfect squares and cube roots of small perfect cubes. Know that $\sqrt{2}$ is irrational.

Addressing

- Understand and apply Pythagoras' theorem.
- Apply Pythagoras' theorem to determine unknown side lengths in right-angled triangles in real-world and mathematical problems in two and three dimensions.

Building Towards

- Understand and apply Pythagoras' theorem.

- Explain a proof of Pythagoras' theorem and its converse.

Instructional Routines

- Collect and Display
- Compare and Connect
- Which One Doesn't Belong?

Student Learning Goals

Let's find triangle side lengths.

6.1 Which One Doesn't Belong: Triangles

Warm Up: 5 minutes

In this warm-up, students compare four triangles. To give all students access the activity, each triangle has one obvious reason it does not belong. One key thing for them to notice is whether the triangle is a right-angled triangle or not.

Instructional Routines

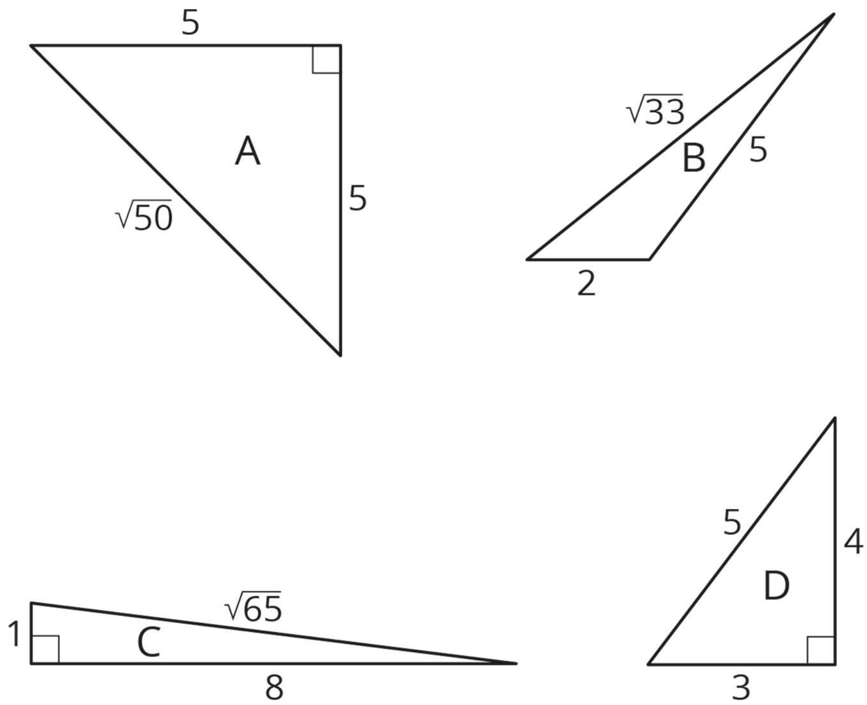
- Which One Doesn't Belong?

Launch

Arrange students in groups of 2–4. Display the image of the four triangles for all to see. Ask students to indicate when they have noticed one triangle that does not belong and can explain why. Give students 1 minute of quiet think time and then time to share their thinking with their group. After everyone has conferred in groups, ask the group to offer at least one reason *each* figure doesn't belong.

Student Task Statement

Which triangle doesn't belong?



Student Response

Answers vary. Sample responses:

A is the only one that is upside down (the base is not on the bottom). Or is the only one that is isosceles.

B is the only one that isn't a right-angled triangle.

C is the only long skinny one. Or it is the only one that doesn't have a side length of 5.

D is the only one where all three side lengths are whole numbers.

Activity Synthesis

Ask each group to share one reason why a particular triangle does not belong. Record and display the responses for all to see. After each response, ask the class if they agree or disagree. Since there is no single correct answer to the question of which one does not belong, attend to students' explanations and ensure the reasons given make sense.

If no student brings up the fact that Triangle B is the only one that is not a right-angled triangle, be sure to point that out.

6.2 A Table of Triangles

15 minutes

In this activity, students calculate the side lengths of the triangles by both drawing in tilted squares and reasoning about segments that must be congruent to segments whose lengths are known. Students then record both the side length and the area of the squares in tables and look for patterns. The purpose of this task is for students to think about the relationships between the squares of the side lengths of triangles as a lead up to Pythagoras' theorem at the end of this lesson.

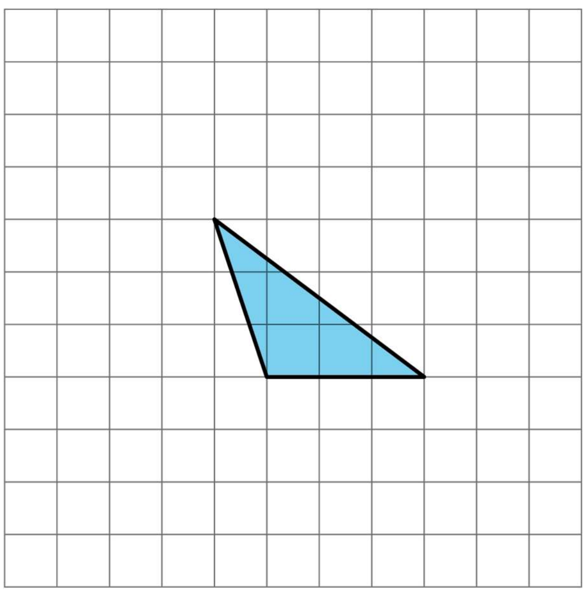
Note that students do not have to draw squares to find every side length. Some squares are intentionally positioned so that students won't be able to draw squares and must find other ways to find the side lengths. Some segments are congruent to others whose lengths are already known.

Instructional Routines

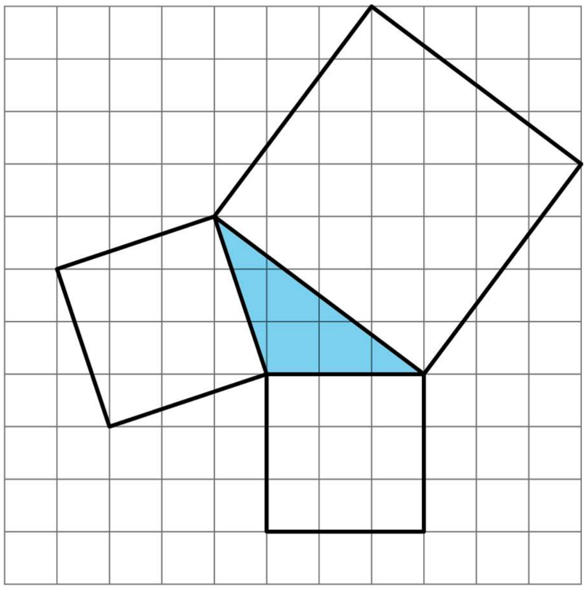
- Collect and Display

Launch

Arrange students in groups of 2–3. Display the image of the triangle on a grid for all to see and ask students to consider how they would find the value of each of the side lengths of the triangle.



After 1–2 minutes of quiet think time, ask partners to discuss their strategies and then calculate the values. Select 2–3 groups to share their strategies and the values for the side lengths they found ($\sqrt{9} = 3$, $\sqrt{10}$, $\sqrt{25} = 5$). Next, show the same image but with three squares drawn in, each using one of the sides of the triangle as a side length.



This directly reflects work students have done previously for finding the length of a diagonal on a grid. Students may point out that for the side that is not diagonal, the square is not needed. This is true, but, if no student points it out, note that $3 = \sqrt{9}$, and so the strategy of drawing in a square still works.

Tell students they will use their strategies to determine the side lengths of several triangles in the activity. Alert them to the fact that it's possible to figure out some of the side lengths without having to draw a square. Encourage groups to divide up the work completing the tables and discuss strategies to find the rest of the unknown side lengths.

Engagement: Develop Effort and Persistence. Connect a new concept to one with which students have experienced success. For example, reference examples from the previous lessons on finding the length of a diagonal of a grid by drawing squares to provide an entry point into this activity.
Supports accessibility for: Social-emotional skills; Conceptual processing Conversing, Reading: Collect and Display. As students work in groups on the task, circulate and listen as they discuss what they notice about the values in the table for Triangles E and Q that does not apply to the other triangles. Write down the words and phrases students use on a visual display. As students review the language collected, encourage students to clarify the meaning of a word or phrase. For example, a phrase such as: "The values of a^2 and b^2 add up to c^2." can be restated as "The sum of a^2 and b^2 is c^2. Encourage students to refer back to the visual display during whole-class discussions throughout the lesson and unit. This routine will provide feedback to students in a way that supports sense-making while simultaneously increasing meta-awareness of language.
Design Principle(s): Support sense-making; Maximise meta-awareness

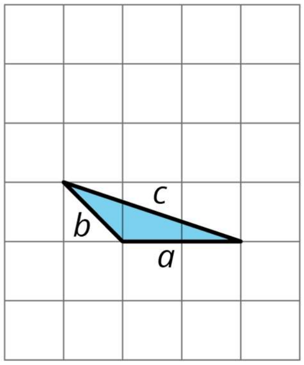
Anticipated Misconceptions

Some students may use the language of hypotenuse for all of the triangles in the activity. If you hear this, remind students that hypotenuse only applies to right-angled triangles.

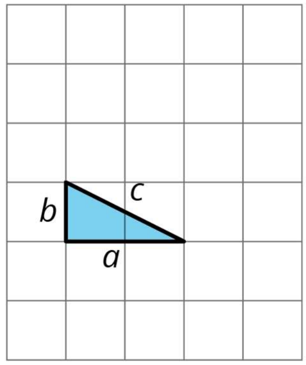
Student Task Statement

1. Complete the tables for these three triangles:

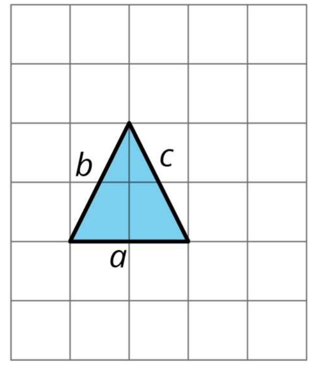
D



E



F

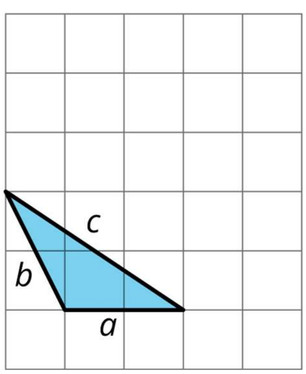


triangle	a	b	c
D			
E			
F			
triangle	a^2	b^2	c^2
D			
E			
F			

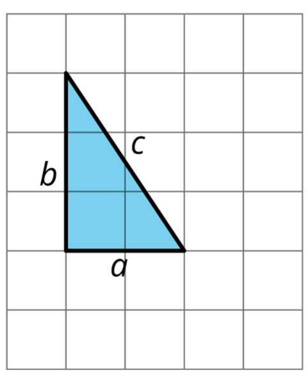
2. What do you notice about the values in the table for Triangle E but not for Triangles D and F?

3. Complete the tables for these three more triangles:

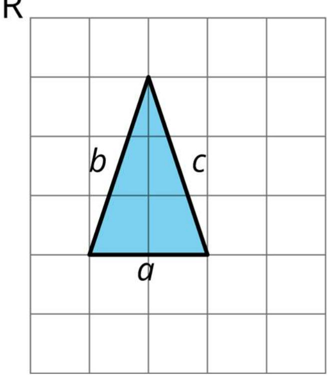
P



Q



R



triangle	a	b	c
P			
Q			
R			
triangle	a^2	b^2	c^2
P			
Q			
R			

4. What do you notice about the values in the table for Triangle Q but not for Triangles P and R?

5. What do Triangle E and Triangle Q have in common?

Student Response

1. Answers vary. Sample responses:

	a	b	c
triangle D	2	$\sqrt{2}$	$\sqrt{10}$
triangle E	2	1	$\sqrt{5}$
triangle F	2	$\sqrt{5}$	$\sqrt{5}$
	a^2	b^2	c^2
triangle D	4	2	10
triangle E	4	1	5
triangle F	4	5	5

2. The sum of $a^2 = 4$ and $b^2 = 1$ equals $c^2 = 5$.

3.

	a	b	c
triangle P	2	$\sqrt{5}$	$\sqrt{13}$
triangle Q	2	3	$\sqrt{13}$
triangle R	2	$\sqrt{10}$	$\sqrt{10}$
	a^2	b^2	c^2
triangle P	4	5	13

triangle Q	4	9	13
triangle R	4	10	10

4. The sum of $a^2 = 4$ and $b^2 = 9$ equals $c^2 = 13$.

5. Triangle E and Triangle Q are both right-angled triangles.

Activity Synthesis

Invite groups to share their responses to the activity and what they noticed about the relationships between specific triangles. Hopefully, someone noticed that $a^2 + b^2 = c^2$ for triangles E and Q and someone else noticed they are right-angled triangles. If so, ask students if any of the other triangles are right-angled triangles (they are not). If students do not see these patterns, don't give it away. Instead, tell students that we are going to look at more triangles to find a pattern.

6.3 Meet Pythagoras' Theorem

10 minutes

In this task, students can use squares or count grid units to find side lengths and check whether Pythagoras' identity $a^2 + b^2 = c^2$ holds or not. If students don't make the connection that it works for the two right-angled triangles but not the other one, this should be brought to their attention. In the synthesis of this activity or the lesson synthesis, the teacher formally states Pythagoras' theorem and lets students know they will prove it in the next lesson.

Instructional Routines

- Compare and Connect

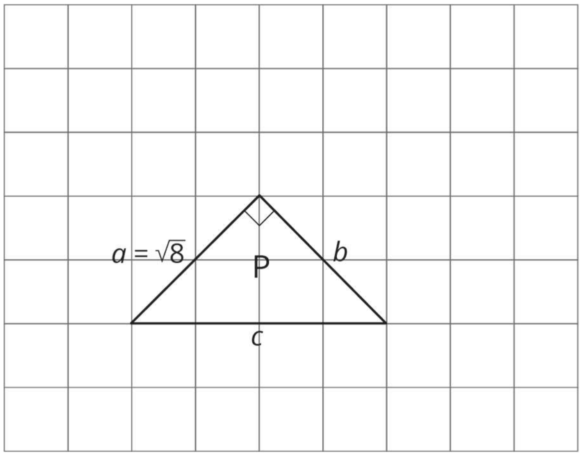
Launch

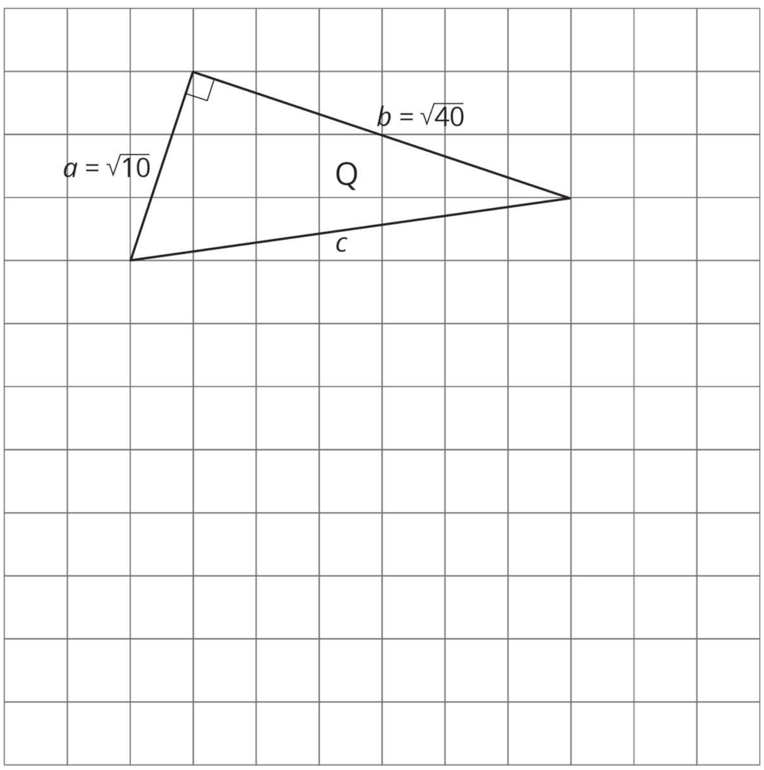
Arrange students in groups of 2. Give students 4 minutes of quiet work time followed by partner and then whole-class discussions.

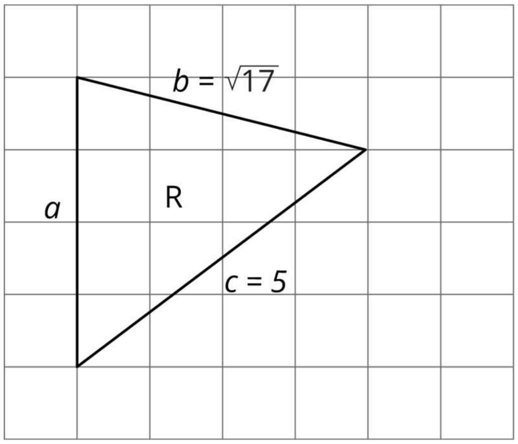
Student Task Statement

1. Find the missing side lengths. Be prepared to explain your reasoning.

2. For which triangles does $a^2 + b^2 = c^2$?


$a = \sqrt{8}$
b
P
c


$b = \sqrt{40}$
$a = \sqrt{10}$
Q
c

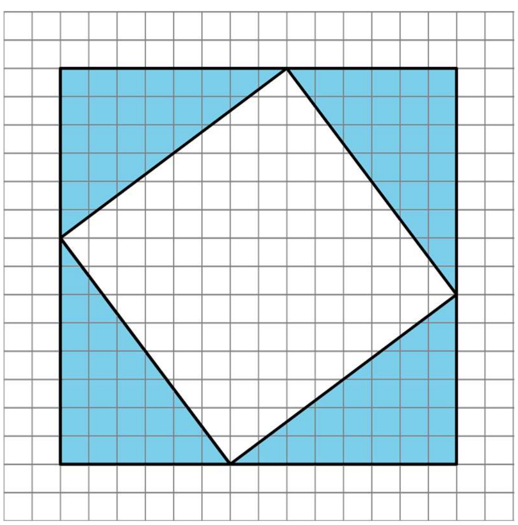


Student Response

1. For triangle P: $c = 4$, which we can see by counting. $b = \sqrt{8}$, which we can see by the fact that it is an isosceles triangle or by drawing a square on the side and finding its area.
   For triangle Q: $c = \sqrt{50}$ which we can see by drawing a square on the side and finding its area.
   For triangle R: $a = 4$ which we can see by counting.
2. $a^2 + b^2 = c^2$ for triangles P and Q, but not triangle R.

Are You Ready for More?

If the four shaded triangles in the figure are congruent right-angled triangles, does the inner quadrilateral have to be a square? Explain how you know.



Student Response

Yes. To prove the inner quadrilateral is a square, we need to show that it has 4 sides of equal length and 4 right angles.

Equal length sides: Since the four triangles are congruent, all four hypotenuses are the same length, so all four sides of the inner quadrilateral have equal length.

Right angles: Each triangle has a 90 degree angle and two others. Call the other angles x and y and label them on all of the congruent triangles. Since the angles in a triangle sum to 180 degrees, we know $x + y + 90 = 180$. There is also straight angle consisting of x, y, and any angle from the inner quadrilateral, which we'll call z. Now $x + y + z = 180$ and $x + y + 90 = 180$, so $z = 90$.

Activity Synthesis

Ask selected students to share their reasoning. Make sure the class comes to an agreement. Then tell students that Pythagoras' theorem says:

If a, b, and c are the sides of a right-angled triangle, where c is the hypotenuse, then

$a^2 + b^2 = c^2$

It is important for students to understand that it only works for *right-angled triangle*s. Tell them we will prove that this is always true in the next lesson.

Representation: Develop Language and Symbols. Create a display of important terms and vocabulary. Invite students to suggest language or diagrams to include that will support their understanding of Pythagoras' theorem and hypotenuse.
Supports accessibility for: Memory; Language Speaking, Listening: Compare and Connect. Use this routine to help students consider audience when preparing a visual display of their work. Ask students to prepare a visual display that shows how they found the missing side lengths. Some students may wish to include notes, details or drawings to help communicate their thinking. Invite students to investigate each other's work. Listen for and amplify the language students use to describe how they used squares to determine the side lengths of the triangle. Encourage students to make connections between the values of a^2, b^2, and c^2 and the squares in the diagram. For example, the value of a^2 represents the area of the square with side length a, and b^2 represents the area of the square with side length b. As a result, the equation $a^2 + b^2 = c^2$ suggests that the area of the square with side length c is the sum of the areas of the square with side length a and the square with side length b. This will foster students' meta-awareness and support constructive conversations as they compare strategies for finding the exact side lengths of triangles and make connections between quantities and the areas they represent.
Design Principles(s): Cultivate conversation; Maximise meta-awareness

Lesson Synthesis

In this lesson we looked at the relationship between the side lengths of different triangles. We saw a pattern for right-angled triangles that did not hold for non-right-angled triangles. Ask students:

- "What was the relationship we saw for the right-angled triangles we looked at?" (The sum of the squares of the shorter sides was equal to the square of the hypotenuse.)

If time allows, draw a few right-angled triangles with labelled side lengths marked a, b, and c and display for all to see. Ask students to check that Pythagoras' theorem is true for these triangles. As students work, check to make sure they understand that for $a^2 + b^2$, a and b need to be squared first, and then added. Some students may confuse exponents with multiplying by 2, and assume they can "factorise" the expression.

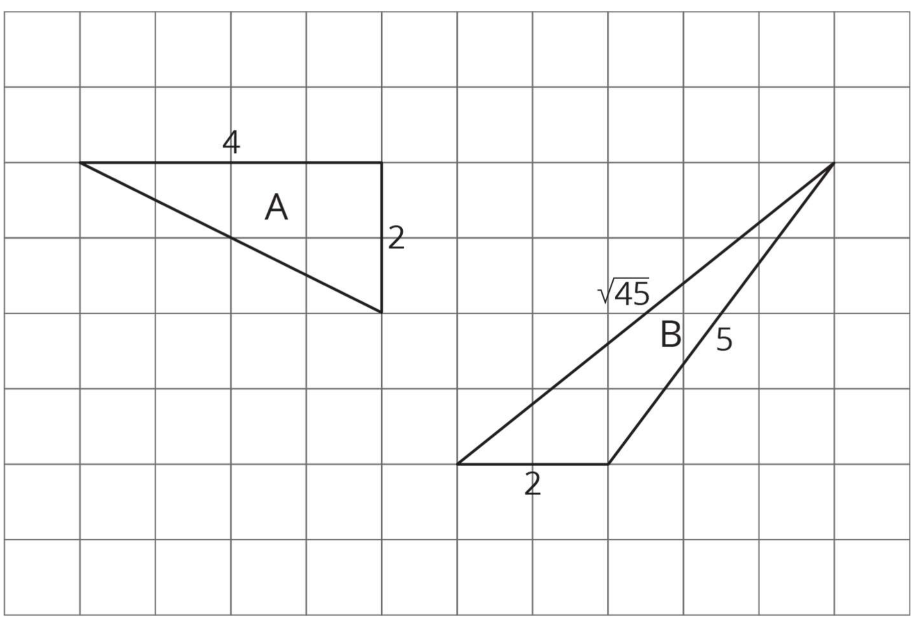
In the next lesson, we will actually prove that what we saw in these examples is *always* true for right-angled triangles.

6.4 Does a Squared Plus b Squared Equal c Squared?

Cool Down: 5 minutes

Student Task Statement

For each of the following triangles, determine if $a^2 + b^2 = c^2$, where a, b, and c are side lengths of the triangle. Explain how you know.



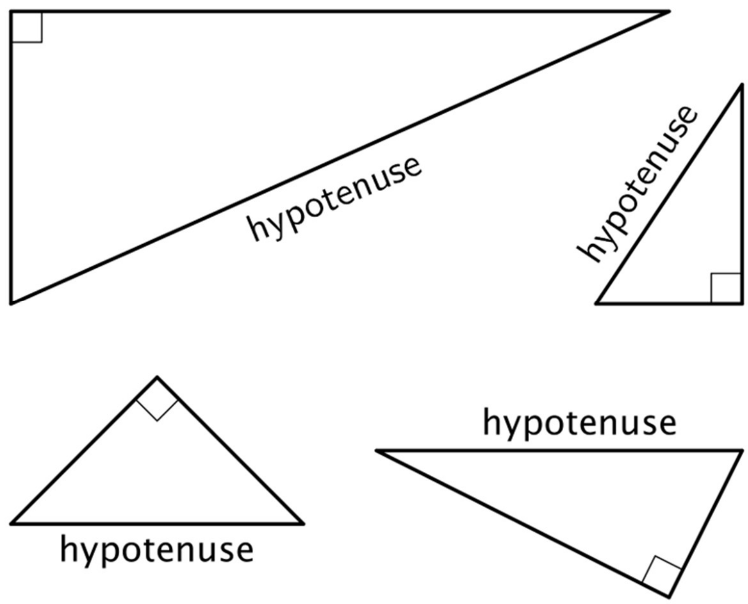
Student Response

It is true for A because it is a right-angled triangle. You can also find the third side length by constructing a square on it and checking.

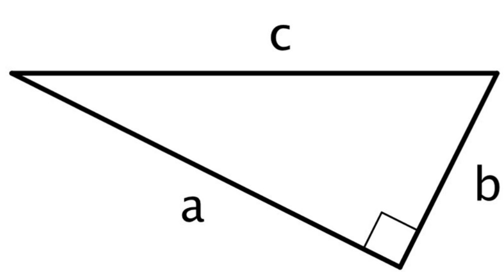
It is not true for B. You can see this by squaring the side lengths.

Student Lesson Summary

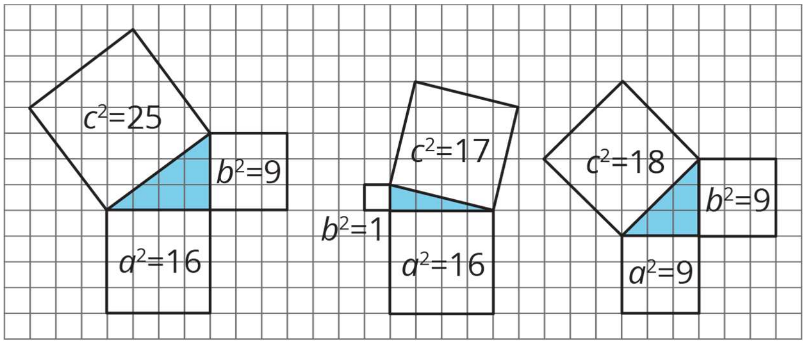
A *right-angled triangle* is a triangle with a right angle. In a right-angled triangle, the side opposite the right angle is called the **hypotenuse**, and the two other sides are called its **shorter sides**. Here are some right-angled triangles with the hypotenuse labelled:



We often use the letters a and b to represent the lengths of the shorter sides of a triangle and c to represent the length of the longest side of a right-angled triangle. If the triangle is a right-angled triangle, then a and b are used to represent the lengths of the shorter sides, and c is used to represent the length of the hypotenuse (since the hypotenuse is always the longest side of a right-angled triangle). For example, in this right-angled triangle, $a = \sqrt{20}$, $b = \sqrt{5}$, and $c = 5$.

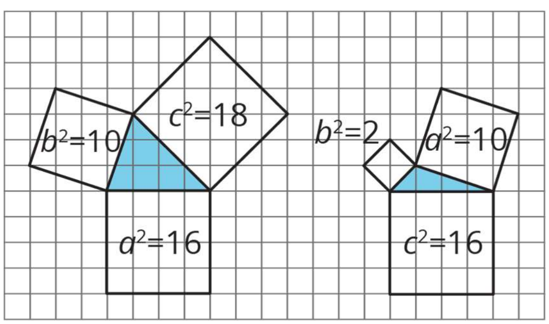


Here are some right-angled triangles:



Notice that for these examples of right-angled triangles, the square of the hypotenuse is equal to the sum of the squares of the shorter sides. In the first right-angled triangle in the diagram, $9 + 16 = 25$, in the second, $1 + 16 = 17$, and in the third, $9 + 9 = 18$. Expressed another way, we have $a^2 + b^2 = c^2$ This is a property of all right-angled triangles, not just these examples, and is often known as Pythagoras' theorem. The name comes from a mathematician named Pythagoras who lived in ancient Greece around 2 500 BCE, but this property of right-angled triangles was also discovered independently by mathematicians in other ancient cultures including Babylon, India, and China. In China, a name for the same relationship is the Shang Gao theorem. In future lessons, you will learn some ways to explain why Pythagoras' theorem is true for *any* right-angled triangle.

It is important to note that this relationship does not hold for *all* triangles. Here are some triangles that are not right-angled triangles, and notice that the lengths of their sides do not have the special relationship $a^2 + b^2 = c^2$. That is, $16 + 10$ does not equal 18, and $2 + 10$ does not equal 16.



Glossary

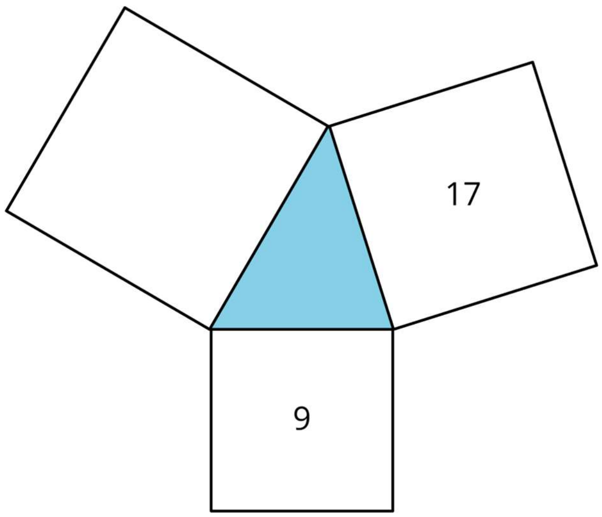
- hypotenuse
- Pythagoras' theorem

Mathematics Education Innovation

Lesson 6 Practice Problems

1. **Problem 1 Statement**

   Here is a diagram of an acute-angled triangle and three squares.



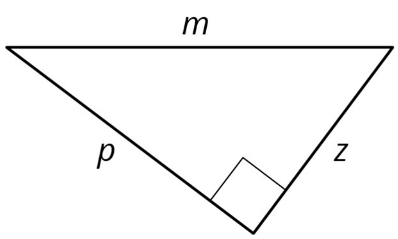
   Priya says the area of the large unmarked square is 26 square units because $9 + 17 = 26$. Do you agree? Explain your reasoning.

   Solution

   No, I disagree. Priya's pattern only works for right-angled triangles, and this is an acute-angled triangle.

2. **Problem 2 Statement**

   m, p, and z represent the lengths of the three sides of this right-angled triangle.



   Select **all** the equations that represent the relationship between m, p, and z.

   a. $m^2 + p^2 = z^2$

   b. $m^2 = p^2 + z^2$

   c. $m^2 = z^2 + p^2$

   d. $p^2 + m^2 = z^2$

e. $z^2 + p^2 = m^2$

f. $p^2 + z^2 = m^2$

Solution ["B", "C", "E", "F"]

3. **Problem 3 Statement**

The lengths of the three sides are given for several right-angled triangles. For each, write an equation that expresses the relationship between the lengths of the three sides.

a. 10, 6, 8

b. $\sqrt{5}, \sqrt{3}, \sqrt{8}$

c. $5, \sqrt{5}, \sqrt{30}$

d. $1, \sqrt{37}, 6$

e. $3, \sqrt{2}, \sqrt{7}$

Solution

a. $6^2 + 8^2 = 10^2$

b. $\sqrt{5}^2 + \sqrt{3}^2 = \sqrt{8}^2$

c. $5^2 + \sqrt{5}^2 = \sqrt{30}^2$

d. $1^2 + 6^2 = \sqrt{37}^2$

e. $\sqrt{2}^2 + \sqrt{7}^2 = 3^2$

4. **Problem 4 Statement**

Order the following expressions from least to greatest.

$25 \div 10$

$250,000 \div 1,000$

$2.5 \div 1,000$

$0.025 \div 1$

Solution

– $2.5 \div 1,000$

- $0.025 \div 1$
- $25 \div 10$
- $250,000 \div 1,000$

5. **Problem 5 Statement**

Which is the best explanation for why -$\sqrt{10}$ is irrational?

a. -$\sqrt{10}$ is irrational because it is not rational.

b. -$\sqrt{10}$ is irrational because it is less than zero.

c. -$\sqrt{10}$ is irrational because it is not a whole number.

d. -$\sqrt{10}$ is irrational because if I put -$\sqrt{10}$ into a calculator, I get -3.16227766, which does not make a repeating pattern.

Solution D

6. **Problem 6 Statement**

A teacher tells her students she is just over 1 and $\frac{1}{2}$ billion seconds old.

a. Write her age in seconds using scientific notation (standard form).

b. What is a more reasonable unit of measurement for this situation?

c. How old is she when you use a more reasonable unit of measurement?

Solution

a. 1.5×10^9

b. Years

c. She is about 48 years old. There are 31 536 000 seconds in a year. $1.5 \times 10^9 \div$ is 31 536 000 about 47.6.