Lesson 14: Fractional lengths in triangles and prisms

Goals

- Apply dividing by fractions to calculate the base or height of a triangle, given its area and the other measurement.
- Determine the volume of a rectangular prism by counting how many 12 inch or 13 inch cubes it takes to build, and explain (orally and in writing) the solution method.
- Generalise that the volume of a rectangular prism with fractional edge lengths can be found by multiplying the edge lengths.

Learning Targets

- I can explain how to find the volume of a rectangular prism using cubes that have a unit fraction as their edge length.
- I can use division and multiplication to solve problems involving areas of triangles with fractional bases and heights.
- I know how to find the volume of a rectangular prism even when the edge lengths are not whole numbers.

Lesson Narrative

In this transitional lesson, students conclude their work with area and begin to explore volume of rectangular prisms. First, they extend their work on area to include triangles, using division to find the length of a base or a height in a triangle when the area is known. Second, they undertake a key activity for extending their understanding of how to find the volume of a prism.

In KS2, students learned that the volume of a prism with whole-number edge lengths is the product of the edge lengths. Now they consider the volume of a prism with dimensions $1\frac{1}{2}$ inch by 2 inches by $2\frac{1}{2}$ inches. They picture it as being packed with cubes whose edge length is $\frac{1}{2}$ inch, making it a prism that is 3 cubes by 4 cubes by 5 cubes, for a total of 60 cubes, because $3 \times 4 \times 5 = 60$. At the same time, they see that each of these $\frac{1}{2}$-inch cubes has a volume of $\frac{1}{8}$ cubic inches, because we can fit 8 of them into a unit cube. They conclude that the volume of the prism is $60 \times \frac{1}{8} = 7\frac{1}{2}$ cubic inches.

In the next lesson, by repeating this reasoning and generalising, students see that the volume of a rectangular prism with fractional edge lengths can also be found by multiplying its edge lengths directly (e.g., $\left(1\frac{1}{2}\right) \times 2 \times \left(2\frac{1}{2}\right) = 7\frac{1}{2}$).

Addressing

- Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems.

- Find the volume of a right rectangular prism with fractional edge lengths by packing it with unit cubes of the appropriate unit fraction edge lengths, and show that the volume is the same as would be found by multiplying the edge lengths of the prism. Apply the formulas $V = lwh$ and $V = bh$ to find volumes of right rectangular prisms with fractional edge lengths in the context of solving real-world and mathematical problems.

- Interpret and calculate quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem. For example, create a story context for $\left(\frac{2}{3}\right) \div \left(\frac{3}{4}\right)$ and use a visual fraction model to show the quotient; use the relationship between multiplication and division to explain that $\left(\frac{2}{3}\right) \div \left(\frac{3}{4}\right) = \frac{8}{9}$ because $\frac{3}{4}$ of $\frac{8}{9}$ is $\frac{2}{3}$. In general, $\left(\frac{a}{b}\right) \div \left(\frac{c}{d}\right) = \frac{ad}{bc}$. How much chocolate will each person get if 3 people share $\frac{1}{2}$ lb of chocolate equally? How many $\frac{3}{4}$ cup servings are in $\frac{2}{3}$ of a cup of yogurt? How wide is a rectangular strip of land with length $\frac{3}{4}$ mi and area $\frac{1}{2}$ square mi?

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Collect and Display
- Discussion Supports
- Think Pair Share

Required Materials

$\frac{1}{2}$ inch cubes

Geometry toolkits

tracing paper, graph paper, coloured pencils, scissors, and an index card to use as a straightedge or to mark right angles.

When compasses are required they are listed as a separate Required Material.

Required Preparation

For the Volumes of Cubes and Prisms activity, prepare 20 half-inch cubes for every group of 3–4 students. Wooden ones are available inexpensively at craft stores. If you have access to centimetre cubes, you could use those instead. Tell students that we will consider them half-inch cubes for the purposes of that activity.

Student Learning Goals

Let's explore area and volume when fractions are involved.

14.1 Area of Triangle

Warm Up: 5 minutes

This warm-up allows students to review calculation of triangular area and prepares them to use division of fractions to solve area problems involving triangles later. Students calculate the area of a triangle given fractional base and height measurements.

Instructional Routines

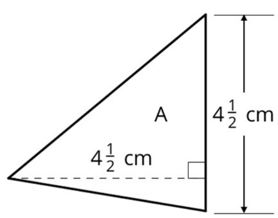
- Think Pair Share

Launch

Arrange students in groups of 2. Give students 2 minutes of quiet work time, followed by 1 minute of partner discussion. Before students begin, review the formula for the area of a triangle. Consider displaying a drawing of a triangle with one side labelled as a base and a corresponding height shown and labelled as such.

Student Task Statement

Find the area of Triangle A in square centimetres. Show your reasoning.



Student Response

The area of any triangle is $A = \frac{1}{2} \times \text{base} \times \text{height}$. The area of this triangle is $10\frac{1}{8}$ cm^2, because $\frac{1}{2} \times \left(4\frac{1}{2}\right) \times \left(4\frac{1}{2}\right) = 10\frac{1}{8}$.

Activity Synthesis

Ask a student to share their solution and reasoning. Record it for all to see. Poll the class to see whether students agree or disagree. Ask if others had alternative ways of reasoning or calculating.

Tell students that they will solve more problems involving the area of triangles in this lesson.

14.2 Bases and Heights of Triangles

10 minutes

In this activity, students apply their knowledge of division of fractions to answer questions about bases and heights of triangles, which they studied previously. In the warm-up, students recalled how to find the area of a triangle given a pair of base and height. Here, they find a missing length given the area of a triangle and a fractional base or height.

The formula for the area of a triangle $A = \frac{1}{2} \times b \times h$ presents a different multiplication situation than students have seen in this unit—there are three factors at play. Students are likely to approach in a number of ways. They may:

- Draw a duplicate of the triangle and make a parallelogram with the same base and height, double the given area (to represent the area of the parallelogram), and then divide by the known length to find the unknown length
- Without drawing, multiply the given area by 2 to find the value of $b \times h$, and then divide it by the known length
- Perform division twice, i.e., dividing the area by $\frac{1}{2}$ and then by the known base or length, or vice versa
- Multiply the known length and $\frac{1}{2}$ first, so there are only two factors to work with (e.g., for $\frac{1}{2} \times b \times \frac{8}{3} = 8$, they may apply the commutative and associative properties of operations and write $\frac{1}{2} \times \frac{8}{3} \times b = 8$ and then $\frac{4}{3} \times b = 8$)

Monitor for these or other approaches as students work. Select students who use different strategies to share later.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Discussion Supports

Launch

Keep students in groups of 2. Give students 4–5 minutes of quiet work time and 2 minutes to discuss their responses and complete the activity with their partner. Keep the formula for the area of a triangle and a labelled drawing of a triangle displayed.

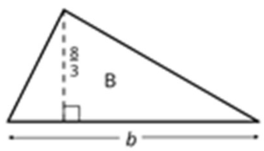
Action and Expression: Develop Expression and Communication. Activate or supply background knowledge. During the launch, take time to review the following terms from previous lessons that students will need to access for this activity: base and height of a triangle, formula for area of a triangle.
Supports accessibility for: Memory; Language

Anticipated Misconceptions

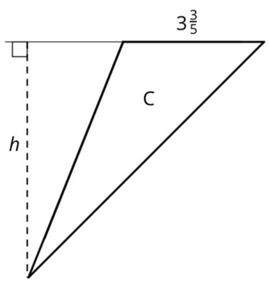
Some students may not know how to find a missing factor when three—instead of two—factors are involved. Ask if there is any step to take or any calculation to make first so they are only working with two factors.

Student Task Statement

1. The area of Triangle B is 8 square units. Find the length of b. Show your reasoning.



2. The area of Triangle C is $\frac{54}{5}$ square units. What is the length of h? Show your reasoning.



Student Response

1. 6 units. $\frac{1}{2} \times \frac{8}{3} \times b = 8$, so $b = 6$.

2. 6 units. $\frac{1}{2} \times \left(3\frac{3}{5}\right) \times h = \frac{54}{5}$, so $h = 6$.

Activity Synthesis

Ask previously selected students to share their solutions and reasoning. Sequence their presentations so that students who reasoned concretely (e.g., by duplicating the triangle to build a parallelogram) present first and those who reasoned symbolically (e.g., only by manipulating expressions or equations) present last. After each student shares, ask all students to indicate whether they reasoned the same way.

Clarify that there is no single way to solve problems such as these. Point out specific points in the solving process where division of fractions enabled them to complete their reasoning. Emphasise that what we learned about fractions and operations in this unit can help us reason more effectively about problems in other areas of mathematics.

Speaking: Discussion Supports. Use this to routine amplify students' use of mathematical language to communicate reasoning about finding the area of triangles. When students share their solutions and reasoning, remind them to use words such as quadrilateral, parallelogram, base, height, area, and rearrange. Invite students to chorally repeat the

phrases that include these words in context. Students used these words in a previous unit, so this will help students be more precise with their language.
Design Principle(s): Optimise output (for explanation)

14.3 Volumes of Cubes and Prisms

20 minutes (there is a digital version of this activity)

This activity extends students' understanding about the volume of rectangular prisms from earlier years. Previously, students learned that the volume of a rectangular prism with whole-number edge lengths can be found by computing the number of unit cubes that can be packed into the prism. Here, they draw on the same idea to find the volume of a prism with fractional edge lengths. The edge length of the cubes used as units of measurement are not 1 unit long, however. Instead, they have a unit fraction ($\frac{1}{2}$, $\frac{1}{4}$, etc.) for their edge length. Students calculate the number of these smaller cubes in a prism and use it to find the volume in a standard unit of volume measurement (cubic inches, in this case).

By reasoning repeatedly with small cubes (with $\frac{1}{2}$-inch edge lengths), students notice that the volume of a rectangular prism with fractional edge lengths can also be found by directly multiplying the edge lengths in inches.

Instructional Routines

- Collect and Display

Launch

Tell students to look at the image of the 1 inch cube in the task (or display it for all to see). Ask students:

- "This cube has an edge length of 1 inch. What is its volume in cubic inches?" (1 cubic inch.) "How do you know?" ($1 \times 1 \times 1 = 1$)
- "How do we find the volume of a cube with an edge length of 2 inches?" ($2 \times 2 \times 2 = 8$. Or, since we can pack it with eight 1 inch cubes, we can tell its area is 8×1 or 8 cubic inches.)

If no students mentioned using the 1 inch cube to find the volume of the 2 inch cube, bring it up. Consider telling students that we can call a cube with edge length of 1 inch a "1 inch cube."

Arrange students in groups of 3–4. Give each group 20 cubes and 5 minutes to complete the first set of questions. Ask them to pause for a brief class discussion afterwards.

Invite students to share how they found the volume of a cube with $\frac{1}{2}$ inch edge length and the prism composed of 4 stacked cubes. For the $\frac{1}{2}$ inch cube, if students do not mention one of the two ways shown in the Possible Responses, bring it up. For the tower, if they don't mention multiplying the volume of a $\frac{1}{2}$ inch cube, which is $\frac{1}{8}$ cubic inch, ask if that is a possible way to find the volume of the prism.

Next, give students 7–8 minutes to complete the rest of the activity.

For classes using the digital materials, an applet is provided, but using physical cubes is preferred and recommended. Adapted from an applet made in GeoGebra by Susan Addington.

Conversing, Representing: Collect and Display. As students work on the first set of questions, listen for the words and phrases students use as they discuss how to find the volume of each prism (length, width, height, and volume). Display collected language for all to see, and include any diagrams that students use to represent their thinking. Continue to update the display throughout the lesson. Remind students to borrow language from the display as needed, as this will help students produce language related to volume.
Design Principle(s): Support sense-making

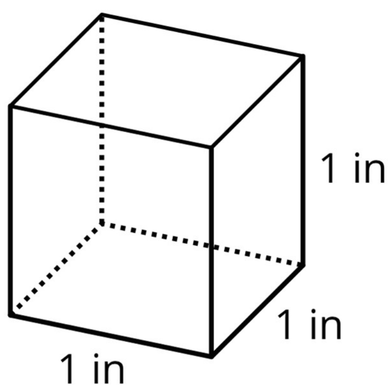
Anticipated Misconceptions

Students may need a reminder to label volume in cubic units.

Student Task Statement

Your teacher will give you cubes that have edge lengths of $\frac{1}{2}$ inch.

1. Here is a drawing of a cube with edge lengths of 1 inch.



   a. How many cubes with edge lengths of $\frac{1}{2}$ inch are needed to fill this cube?

   b. What is the volume, in cubic inches, of a cube with edge lengths of $\frac{1}{2}$ inch? Explain or show your reasoning.

2. Four cubes are piled in a single stack to make a prism. Each cube has an edge length of $\frac{1}{2}$ inch. Sketch the prism, and find its volume in cubic inches.

3. Use cubes with an edge length of $\frac{1}{2}$ inch to build prisms with the lengths, widths, and heights shown in the table.

   a. For each prism, record in the table how many $\frac{1}{2}$ inch cubes can be packed into the prism and the volume of the prism.

prism length (in)	prism width (in)	prism height (in)	number of $\frac{1}{2}$ inch cubes in prism	volume of prism (in^3)
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$		
1	1	$\frac{1}{2}$		
2	1	$\frac{1}{2}$		
2	2	1		
4	2	$\frac{3}{2}$		
5	4	2		
5	4	$2\frac{1}{2}$		

   b. Examine the values in the table. What do you notice about the relationship between the edge lengths of each prism and its volume?

4. What is the volume of a rectangular prism that is $1\frac{1}{2}$ inches by $2\frac{1}{4}$ inches by 4 inches? Show your reasoning.

Student Response

1.

   a. 8 cubes

   b. The volume of a $\frac{1}{2}$ inch cube is $\frac{1}{8}$ in^3 because it is $\frac{1}{8}$ of the volume of a 1 inch cube.

2. $\frac{4}{8}$ in^3 or $\frac{1}{2}$ in^3

3. a.

prism length (in)	prism width (in)	prism height (in)	number of $\frac{1}{2}$-inch cubes in prism	volume of prism (cu in)
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{8}$
1	1	$\frac{1}{2}$	4	$\frac{1}{2}$

2	1	$\frac{1}{2}$	8	1
2	2	1	32	4
4	2	$\frac{3}{2}$	96	12
5	4	2	320	40
5	4	$2\frac{1}{2}$	400	50

b. Answers vary. Sample response: I noticed that the volume is equal to $l \times w \times h$.

4. $13\frac{1}{2}$ in^3. Sample reasoning: $\left(1\frac{1}{2}\right) \times \left(2\frac{1}{4}\right) = \frac{27}{8}$, and $\frac{27}{8} \times 4 = 13\frac{1}{2}$.

Are You Ready for More?

A unit fraction has a 1 in the numerator.

- These are unit fractions: $\frac{1}{3}, \frac{1}{100}, \frac{1}{1}$.
- These are *not* unit fractions: $\frac{2}{9}, \frac{8}{1}, 2\frac{1}{5}$.

1. Find three unit fractions whose sum is $\frac{1}{2}$. An example is: $\frac{1}{8} + \frac{1}{8} + \frac{1}{4} = \frac{1}{2}$ How many examples like this can you find?
2. Find a box whose surface area in square units equals its volume in cubic units. How many like this can you find?

Student Response

Answers vary. Sample response:

1. $\frac{1}{3} + \frac{1}{12} + \frac{1}{12}$
   $\frac{1}{5} + \frac{1}{5} + \frac{1}{10}$
   $\frac{1}{6} + \frac{1}{6} + \frac{1}{6}$
2. The denominators of the fractions that work for the first question can be used as the length, width, and height of the box.

Activity Synthesis

Display a completed table for all to see. Give students a minute to check their responses. Invite a few students to share how they determined the volume of the prisms, and their observations about the relationships between the number of cubes (with $\frac{1}{2}$ inch edge lengths) and the volume of the prism in cubic inches. Highlight that the volume can be found by calculating the number of cubes and multiplying it by $\frac{1}{8}$, because each small cube has a volume of $\frac{1}{8}$ cubic inch.

If students do not notice a pattern in their table, ask them what they notice about the edge lengths of the prisms in the second, fourth, and sixth rows of the table and their corresponding volumes. Make sure they see that the volume of each prism can also be found by multiplying its side lengths. In other words, we can find the volume of a rectangular prism with fractional edge lengths the same way we find that of a prism with whole-number edge lengths.

Lesson Synthesis

In this lesson, we used division to solve for a missing base or height in a triangle. Review with students that we used a given area to find an unknown base or height.

- "How is finding an unknown base or height in a triangle different than finding an unknown side length in a rectangle?" (In a triangle, there are three factors at play.)
- "What multiplication equation can we write to help us find the height of a triangle that has a base of $\frac{5}{4}$ cm and an area of 10 sq cm?" ($\frac{1}{2} \times \frac{5}{4} \times h = 10$.)
- "How do we go about finding the unknown length?" (We can multiply the two known factors first. In this case, we can multiply $\frac{1}{2} \times \frac{5}{4}$ to get $\frac{5}{8}$, and then divide 10 by $\frac{5}{8}$. Or we can divide by the factors one at a time—divide by $\frac{1}{2}$ first, and then by $\frac{5}{4}$.)

What we know about fractions and operations can also help us find the volume of rectangular prisms when the edge lengths are not whole numbers.

- How can we use cubes with $\frac{1}{2}$ inch edge lengths to find the volume of a prism that is $\frac{1}{2}$ inch by 2 inch by $3\frac{1}{2}$ inches? (We can think of the prism as being built of these cubes; the prism will have whole numbers of cubes for its edge lengths. In this case, they are 1, 4, and 7, so there are 28 cubes with $\frac{1}{2}$ inch edge length. Each of these cubes is $\frac{1}{8}$ cubic inch because 8 of them fit in 1 cubic inch. The volume of the prism, in cubic inches, is then $28 \times \frac{1}{8}$, which is $3\frac{1}{2}$.)
- How else might we find the volume of a rectangular prism with fractional edge lengths? (We can multiply the three edge lengths in inches directly. $\frac{1}{2} \times 2 \times \frac{7}{2} = \frac{7}{2} = 3\frac{1}{2}$.)

14.4 Triangles and Cubes

Cool Down: 5 minutes

If time is limited, ask students to complete only the second problem.

Student Task Statement

1. A triangle has a base of $3\frac{2}{5}$ inches and an area of $5\frac{1}{10}$ square inches. Find the height of the triangle. Show your reasoning.

2. Answer each question and show your reasoning.

   a. How many cubes with edge lengths of $\frac{1}{3}$ inch are needed to build a cube with an edge length of 1 inch?

   b. What is the volume, in cubic inches, of one cube with an edge length of $\frac{1}{3}$ inch?

Student Response

1. $\frac{1}{2} \times \left(3\frac{2}{5}\right) \times h = 5\frac{1}{10}$, so h is $5\frac{1}{10} \div \left(\frac{1}{2} \times 3\frac{2}{5}\right)$ and the height is 3 inches.

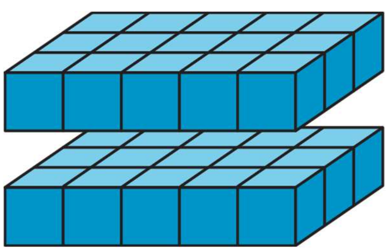
2.

   a. 27

   b. $\frac{1}{27}\,\text{in}^3$

Student Lesson Summary

If a rectangular prism has edge lengths of 2 units, 3 units, and 5 units, we can think of it as 2 layers of unit cubes, with each layer having (3×5) unit cubes in it. So the volume, in cubic units, is: $2 \times 3 \times 5$



To find the volume of a rectangular prism with fractional edge lengths, we can think of it as being built of cubes that have a unit fraction for their edge length. For instance, if we build a prism that is $\frac{1}{2}$ inch tall, $\frac{3}{2}$ inch wide, and 4 inches long using cubes with a $\frac{1}{2}$ inch edge length, we would have:

- A height of 1 cube, because $1 \times \frac{1}{2} = \frac{1}{2}$.
- A width of 3 cubes, because $3 \times \frac{1}{2} = \frac{3}{2}$.
- A length of 8 cubes, because $8 \times \frac{1}{2} = 4$.

The volume of the prism would be $1 \times 3 \times 8$, or 24 cubic units. How do we find its volume in cubic inches? We know that each cube with a $\frac{1}{2}$ inch edge length has a volume of $\frac{1}{8}$ cubic inch, because $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$. Since the prism is built using 24 of these cubes, its volume, in cubic inches, would then be $24 \times \frac{1}{8}$, or 3 cubic inches.

The volume of the prism, in cubic inches, can also be found by multiplying the fractional edge lengths in inches: $\frac{1}{2} \times \frac{3}{2} \times 4 = 3$

Lesson 14 Practice Problems

1. **Problem 1 Statement**

   Clare is using little wooden cubes with edge length $\frac{1}{2}$ inch to build a larger cube that has edge length 4 inches. How many little cubes does she need? Explain your reasoning.

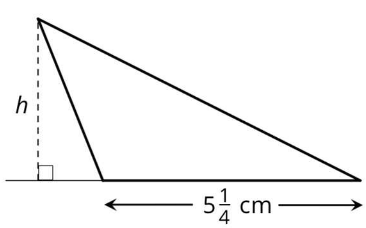
   Solution

   512. Since there are 8 half inches in 4 inches, Clare needs $8 \times 8 \times 8$ little cubes. $8 \times 8 \times 8 = 512$.

2. **Problem 2 Statement**

   The triangle has an area of $7\frac{7}{8}$ cm^2 and a base of $5\frac{1}{4}$ cm.

   What is the length of h? Explain your reasoning.

   Solution

   3 cm. One half of the base ($2\frac{5}{8}$ cm) times the height is $7\frac{7}{8}$ cm^2. So the height in cm is $\left(7\frac{7}{8}\right) \div \left(2\frac{5}{8}\right) = 3$.

3. **Problem 3 Statement**

   a. Which expression can be used to find how many cubes with edge length of $\frac{1}{3}$ unit fit in a prism that is 5 units by 5 units by 8 units? Explain or show your reasoning.

- $\left(5 \times \frac{1}{3}\right) \times \left(5 \times \frac{1}{3}\right) \times \left(8 \times \frac{1}{3}\right)$
- $5 \times 5 \times 8$
- $(5 \times 3) \times (5 \times 3) \times (8 \times 3)$
- $(5 \times 5 \times 8) \times \left(\frac{1}{3}\right)$

b. Mai says that we can also find the answer by multiplying the edge lengths of the prism and then multiplying the result by 27. Do you agree with her? Explain your reasoning.

Solution

a. $(5 \times 3) \times (5 \times 3) \times (8 \times 3)$. Sample reasoning: It takes three $\frac{1}{3}$ units to make 1 unit. In terms of the edge length of the small cube, the prism is 15 by 15 by 24.

b. Mai is correct. Sample reasoning: Because it takes three $\frac{1}{3}$ units to make 1 unit, it takes $3 \times 3 \times 3$ cubes with edge length of $\frac{1}{3}$ unit to make one cube with edge length 1 unit.

4. **Problem 4 Statement**

A builder is building a fence with $6\frac{1}{4}$ inch-wide wooden boards, arranged side-by-side with no gaps or overlaps. How many boards are needed to build a fence that is 150 inches long? Show your reasoning.

Solution

24 boards. ($150 \div 6\frac{1}{4} = 150 \times \frac{4}{25} = 24$)

5. **Problem 5 Statement**

Find the value of each expression. Show your reasoning and check your answer.

a. $2\frac{1}{7} \div \frac{2}{7}$

b. $\frac{17}{20} \div \frac{1}{4}$

Solution

a. $\frac{15}{2}$ or $7\frac{1}{2}$

b. $\frac{17}{5}$ or $3\frac{2}{5}$

6. **Problem 6 Statement**

Consider the problem: A bucket contains $11\frac{2}{3}$ gallons of water and is $\frac{5}{6}$ full. How many gallons of water would be in a full bucket?

Write a multiplication and a division equation to represent the situation. Then, find the answer and show your reasoning.

Solution

14 gallons. Equations: $\frac{5}{6} \times ? = 11\frac{2}{3}$ and $11\frac{2}{3} \div \frac{5}{6} = ?$. $11\frac{2}{3} \div \frac{5}{6} = \frac{35}{3} \times \frac{6}{5}$, which equals 14.

7. **Problem 7 Statement**

There are 80 kids in a gym. 75% are wearing socks. How many are *not* wearing socks? If you get stuck, consider using a bar model.

Solution

20. Sample reasoning: if 75% are wearing socks, then 25% are not wearing socks. 25% of a number is the same as $\frac{1}{4}$ of the number, and $\frac{1}{4}$ of 80 is 20.

8. **Problem 8 Statement**

   a. Lin wants to save £75 for a trip to the city. If she has saved £37.50 so far, what percentage of her goal has she saved? What percentage remains?

   b. Noah wants to save £60 so that he can purchase a concert ticket. If he has saved £45 so far, what percentage of his goal has he saved? What percentage remains?

Solution

a. 50% has been saved, and 50% remains, (37.50 is half of 75).

b. 75% has been saved, and 25% remains, ($\frac{1}{4}$ of 60 is 15, so 15 is 25%.)