Lesson 2: Regular tessellations

Goals

- Justify (orally and in writing) that regular triangles, squares, and hexagons are the only regular polygons that can be used to create a regular tessellation.

Lesson Narrative

In this second in a sequence of three lessons, students look at regular tessellations. Two polygons in a regular tessellation must

- not meet at all or
- share a single vertex or
- share a single side

Students show in detail that triangles, squares, and hexagons give the only possible regular tessellations.

Building On

- Solve real-life and mathematical problems involving angle sizes, area, surface area, and volume.

Addressing

- Understand congruence and similarity using physical models, transparencies, or geometry software.
- Use informal arguments to establish facts about the angle sum and exterior angle of triangles, about the angles created when parallel lines are cut by a transversal, and the angle-angle criterion for similarity of triangles. For example, arrange three copies of the same triangle so that the sum of the three angles appears to form a line, and give an argument in terms of transversals why this is so.

Instructional Routines

- Discussion Supports

Required Materials

Protractors
Clear protractors with no holes and with radial lines printed on them are recommended.

Tracing paper

Required Preparation

If using the applet in the digital version of the activity, tracing paper and protractors are not needed.

Student Learning Goals

Let's make some regular tessellations.

2.1 Regular Tessellations

Optional: 15 minutes (there is a digital version of this activity)

The goal of this activity is to introduce a *regular tessellation* of the plane and conjecture which shapes give regular tessellations. Students construct arguments for which shapes can and cannot be used to make a regular tessellation. The focus is on experimenting with shapes and noticing that in order for a shape to make a regular tessellation, we need to be able to put a whole number of those shapes together at a single vertex with no gaps and no overlaps. This greatly limits what angles the polygons can have and, as a result, there are only three regular tessellations of the plane. This conjecture will be demonstrated in the other two activities of this lesson.

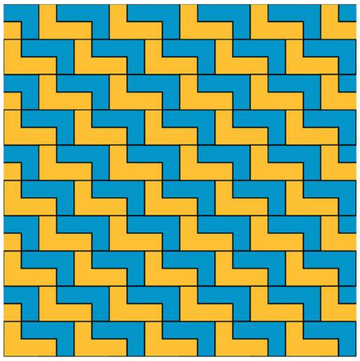
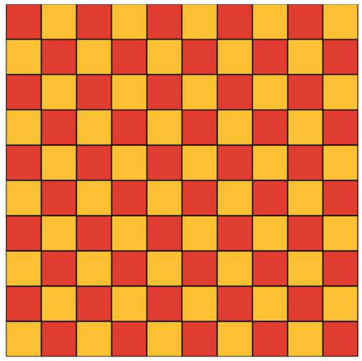
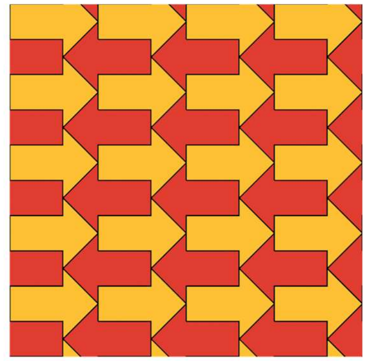
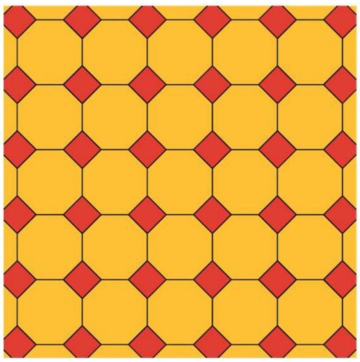
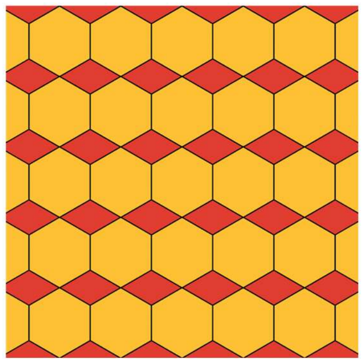
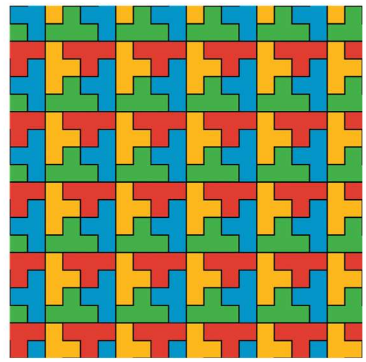
Launch

Display a table for all to see with at least two columns keeping track of which regular polygons make a tessellation and which do not. A third column could be used for extra comments (for example, about angle size of the polygon or other remarks). Here is an example of a table you might use:

shape	tessellate?	notes
octagon		
hexagon		
pentagon		
square		
triangle		

Introduce the idea of a regular tessellation:

- Only one type and size of polygon used.
- If polygons meet, they either share a single vertex or a single side.
- Show some pictures of tessellations that are not regular, and ask students to identify why they are not (e.g., several different polygons used, edges of polygons do not match up completely). Ask students which of the tessellations pictured here are regular tessellations (only the one with squares):

For the print version, make tracing paper available to all students. Tell students that they can use the tracing paper to put together several copies of the polygons.

For the digital version tell students that they can click on the shapes to make several copies.

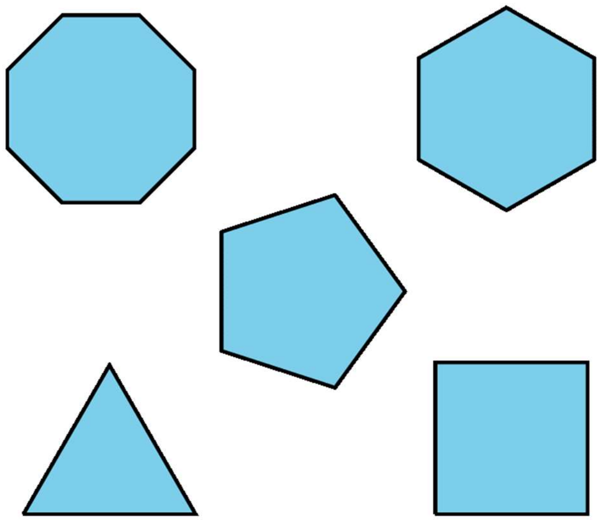
Engagement: Develop Effort and Persistence. Encourage and support opportunities for peer interactions. Invite students to talk about their ideas with a partner before writing them down. Display sentence frames to support students when they explain their strategy. For example, "The _____ can make a regular tessellation of the plane because . . ." , "I noticed _____ so I . . .", or "I agree/disagree because . . . ."
Supports accessibility for: Language; Social-emotional skills

Anticipated Misconceptions

If students working with the pentagon and octagon add other shapes to make a more complicated tessellation, remind them that a regular tessellation uses copies of a single shape.

Student Task Statement

1. For each shape (triangle, square, pentagon, hexagon, and octagon), decide if you can use that shape to make a regular tessellation of the plane. Explain your reasoning.



2. For the polygons that do not work, what goes wrong? Explain your reasoning.

Student Response

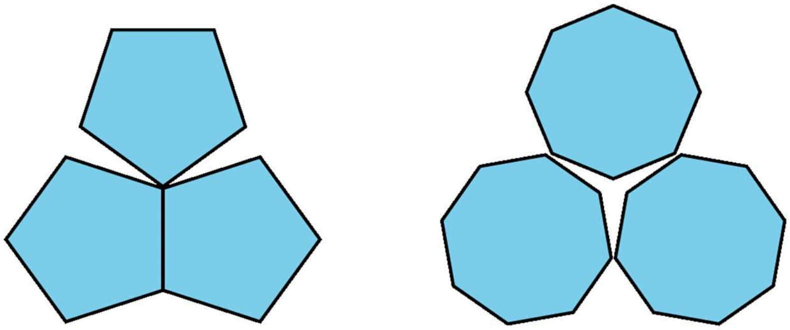
1. The triangles, squares, and hexagons all tessellate the plane. Three hexagons meet at each vertex, four squares meet at each vertex, and six triangles meet at each vertex.

2. For the pentagons, three can fit together at a vertex. There is a little extra space left, but it's not enough to fit another pentagon. Two octagons can fit together at a vertex with space left over, but cannot tessellate the plane.

Activity Synthesis

To help students think more about what shapes do and do not tessellate and why, ask:

- "Which polygons appear to tessellate the plane?" (Square, equilateral triangle, hexagon.)
- "How did you decide?" (I put as many together as I could at one vertex and checked to see if there was extra space leftover.)
- "Why does the pentagon not work to tessellate the plane?" (3 fit together at one vertex, but there is extra space, not enough for a fourth.)
- "Why does the octagon not work?" (2 fit together, but there is not enough space for a third.)



During the discussion, fill out the table, indicating that it is possible to make a tessellation with equilateral triangles, squares, and hexagons, but not with pentagons or octagons.

2.2 Equilateral Triangle Tessellation

Optional: 15 minutes

The goal of this activity is to verify, via angle calculations, that equilateral triangles (and hence) regular hexagons can be used to make regular tessellations of the plane. Students have encountered the equilateral triangle plane tessellations earlier in year 9 when working on an isometric grid. In order to complete their investigation of regular tessellations of the plane, it remains to be shown that no other polygons work. This will be done in the next activity.

Students are required to reason abstractly and quantitatively in this activity. Tracing paper indicates that six equilateral triangles can be put together sharing a single vertex. Showing that this is true for abstract equilateral triangles requires careful reasoning about angle sizes.

Instructional Routines

- Discussion Supports

Launch

In the previous task, equilateral triangles, squares, and hexagons appeared to make regular tessellations of the plane. Tell students that the goal of this activity is to use geometry to verify that they do.

Refer students to regular polygons printed in the previous activity for a visual representation of an equilateral triangle.

Representation: Internalise Comprehension. Represent the same information through different modalities by using physical objects to represent abstract concepts. Some students may benefit by using virtual or concrete manipulatives (GeoGebra, tracing paper, or equilateral triangle cut-outs) to connect abstract concepts to concrete objects.
Supports accessibility for: Conceptual processing; Visual-spatial processing

Anticipated Misconceptions

Students may know that an equilateral triangle has 60-degree angles but may not be able to explain why. Consider prompting these students for the sum of the three angles in an equilateral triangle.

Students may not see a pattern of hexagons within the triangle tessellation. Consider asking these students what shape they get when they put 6 equilateral triangles together at a single vertex.

Student Task Statement

1. What is the size of each angle in an equilateral triangle? How do you know?
2. How many triangles can you fit together at one vertex? Explain why there is no space between the triangles.
3. Explain why you can continue the pattern of triangles to tessellate the plane.
4. How can you use your triangular tessellation of the plane to show that regular hexagons can be used to give a regular tessellation of the plane?

Student Response

1. 60 degrees. The sum of the angles is 180 degrees, and they are all congruent, so each must be a 60-degree angle.
2. 6 because $6 \times 60 = 360$
3. Each place where two or more triangles meet in the pattern, the rest of the six triangles at that vertex can be filled out.
4. Each set of 6 triangles meeting in a single vertex makes a regular hexagon. These hexagons tile the plane.

Activity Synthesis

Consider asking the following questions to lead the discussion of this activity:

- "How did you find the angles in an equilateral triangle?" (The sum of the angles is 180 degrees, and they are all congruent so each is 60 degrees.)
- "Why is there no space between six triangles meeting at a vertex?" (The angles total 360 degrees, which is a full circle.)
- "How does your tessellation with triangles relate to hexagons?" (You can group the triangles meeting at certain vertices into hexagons, which tessellate the plane.)
- "Are there other tessellations of the plane with triangles?" (Yes. You can make infinite rows of triangles that can be placed on top of one another—and displaced relative to one another.)

Consider showing students an isometric grid, used earlier in year 9 for experimenting with transformations, and ask them how this relates to tessellations. (It shows a tessellation with equilateral triangles.)

Point out that this activity provides a mathematical justification for the “yes” in the table for triangles and hexagons.

Speaking: Discussion Supports. Use this routine to support whole-class discussion. After a response to one of the discussion questions is shared with the class, invite students to restate and/or revoice what they heard using mathematical language. Consider providing students time to restate what they hear to a partner, before selecting one or two students to share with the class. Ask the original speaker if their peer was accurately able to restate their thinking. This will provide more students with an opportunity to practice using language about tessellations.
Design Principle(s): Support sense-making

2.3 Regular Tessellation for Other Polygons

Optional: 15 minutes (there is a digital version of this activity)

The goal of this activity is to show that *only* triangles, squares, and hexagons give regular tessellations of the plane. The method used is experimentation with other regular polygons. The key observation is that the angles on regular polygons get larger as we add more sides, which is a good example of observing structure. Since three is the smallest number of polygons that can meet at a vertex in a regular tessellation, this means that once we pass six sides (hexagons), we will not find any further regular tessellations. The activities in this lesson now show that there are three and only three regular tessellations of the plane: triangles, squares, and hexagons.

Instructional Routines

- Discussion Supports

Launch

Ask students “Are there some other regular polygons, in addition to equilateral triangles, squares, and hexagons, that can be used to give regular tessellations of the plane?” Some students may suggest regular polygons with more sides than the ones they have seen already, others may think that there are no other possibilities. Tell students that for this activity, they are going to investigate polygons with 7, 8, 9, 10, and 11 sides to see if they do or do not tessellate and why.

Print version: Provide access to tracing paper and protractors and tell students that they can use these to explore their conjectures.

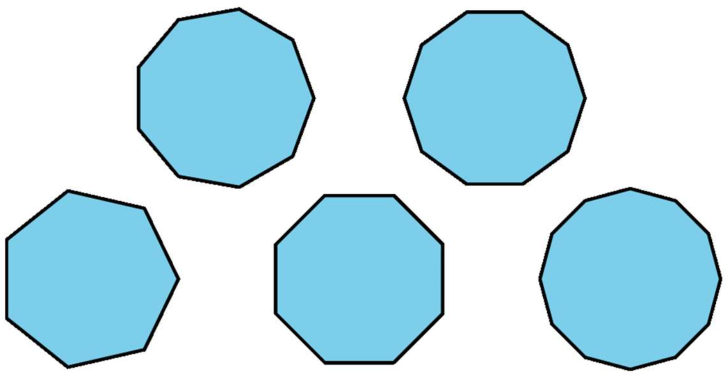
Digital version: Tell students to use the app to explore their conjectures.

Action and Expression: Internalise Executive Functions. Chunk this task into more manageable parts to support students who benefit from support with organisational skills

in problem solving. For example, present one question at a time and monitor students to ensure they are making progress throughout the activity.
Supports accessibility for: Organisation; Attention

Student Task Statement

1. Can you make a regular tessellation of the plane using regular polygons with 7 sides? What about 9 sides? 10 sides? 11 sides? 12 sides? Explain.



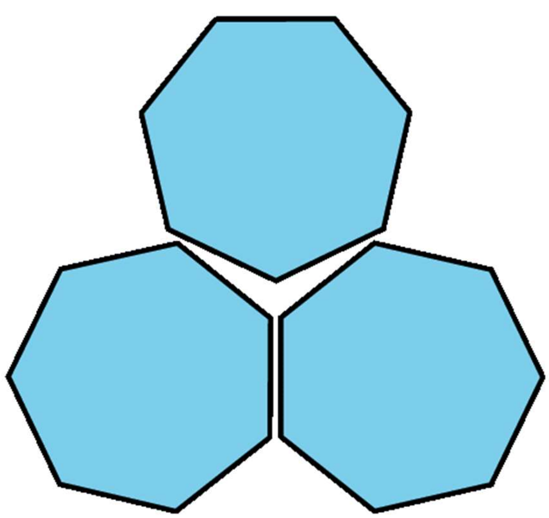
2. How does the size of each angle in a square compare to the size of each angle in an equilateral triangle? How does the size of each angle in a regular 8-sided polygon compare to the size of each angle in a regular 7-sided polygon?

3. What happens to the angles in a regular polygon as you add more sides?

4. Which polygons can be used to make regular tessellations of the plane?

Student Response

1. None of the other polygons will tessellate the plane. For each, three cannot be brought together at one vertex. Here is the attempt at doing so with the regular polygon with 7 sides.



2. The angles in a square are 90 degrees, greater than the 60-degree angles in a triangle. The angles in a regular 8-sided polygon are greater than the angles in a regular 7-sided polygon.

3. The angles increase in size. Imagine opening up a polygon with 6 sides to add a 7th equal side. In order to fit the new side, all of the other sides must be spread out or opened up, increasing the size of the angles.

4. Only the triangle, square, and hexagon. As more sides are added, the angles get greater. 120 degrees is the biggest divisor of 360 that can be the size of an interior angle of a regular polygon.

Activity Synthesis

Consider asking the following questions:

- "How many triangles meet at each vertex in a regular tessellation with triangles?" (6)
- "What about squares?" (4)
- "Hexagons?" (3)
- "Why can't there be any regular tessellations with polygons of more than 6 sides?" (Only two could meet at a vertex, but this isn't possible since the angles have to add up to 360 degrees.)

There are only three regular tessellations of the plane. Ask students if they have encountered these tessellations before and if so where. For example:

- triangles (isometric grid)
- squares (chessboard, coordinate grid, floor and ceiling tiles)
- hexagons (beehives, tiles)

Speaking, Listening: Discussion Supports. During the synthesis, focus on the question about why a regular tessellation with more than six sides wouldn't work. Ask students to prove their justification to this question to their partner. They should try to convince them as if they were someone who really doesn't believe them (a sceptic). Encourage listeners to press speakers to use specific language about "angles" and "degrees". If time permits, have listeners repeat the process with a new partner, taking on the role of speaker. This will provide students with practice using specific geometric terms to justifying their reasoning. *Design Principle(s): Optimise output (for justification); Maximise meta-awareness*