Lesson 9: Linear models

Goals

- Compare and contrast (orally and in writing) different linear models of the same data, and determine (in writing) the range of values for which a given model is a good fit for the data.
- Create a model of a non-linear data using a linear function, and justify (orally and in writing) whether the model is a good fit for the data.

Learning Targets

- I can decide when a linear function is a good model for data and when it is not.
- I can use data points to model a linear function.

Lesson Narrative

In this lesson, students use linear functions to model real-world situations. In the candle activity, they are given data for an almost linear relationship and develop a linear model. They use their model to make predictions and discuss the reasonableness of the model. In the shadow activity, it is difficult to tell from the information given if a linear model is appropriate, but when they are given more information, it becomes clear that the relationship is *not* linear. In the garbage recycling activity, different linear models apply to different time periods.

None of the given data are perfectly fitted by a linear function, and students have to determine whether a linear approximation is reasonable and for which values it would be reasonable. Students should start to see both the value of linear models and their limitations. The garbage recycling activity leads into the next lesson, which is about modelling with piece-wise linear functions.

Addressing

- Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from two (x, y) values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values.

Instructional Routines

- Three Reads
- Discussion Supports
- Poll the Class

Required Materials

Straightedges

A rigid edge that can be used for drawing line segments. Sometimes a ruler is okay to use as a straightedge, but sometimes it is preferable to use an unruled straightedge, like a blank index card.

Required Preparation

Provide access to straightedges to each student.

Student Learning Goals

Let’s model situations with linear functions.

9.1 Candlelight

Warm Up: 10 minutes (there is a digital version of this activity)

In this warm-up, students work with data to determine if the situation represented by the data could be modelled by a linear function. Students are given 3 different data points and use what they know about linear functions and proportional relationships to estimate when the candle will burn out. Students are then asked to determine if this situation could be modelled by a linear function. The focus of the discussion should be around the last question and how students justify their reasoning.

Instructional Routines

- Poll the Class

Launch

Arrange students in groups of 2. Give students 1–2 minutes of quiet work time and ask them to pause after the first question. Poll the class for their response to the first question, and display the range of responses for all to see. Then, ask them to continue and discuss their response to the second question with their partner. If they don't agree, partners should work to understand each other’s thinking. Follow with a whole-class discussion.

If using the digital activity, follow the structure above, as the prompts are the same. However, the digital activity allows students to plot points quickly without having to set up the axes from scratch. This means students may conclude the graph is not quite linear purely from a visual. Make sure these students can explain and understand their peers’ rationale in answering the questions using more than just the plotted points. If any students attempt to guess a linear equation that fits the data, ask them to share during the discussion.

Student Task Statement

A candle is burning. It starts out 12 inches long. After 1 hour, it is 10 inches long. After 3 hours, it is 5.5 inches long.

1. When do you think the candle will burn out completely?

2. Is the height of the candle a function of time? If yes, is it a linear function? Explain your thinking.

Student Response

1. Answers vary. Sample response: Since it burns about 2 inches every hour, it will burn out between 5 and 6 hours after it was lit.

2. The height of a candle is a function of time, because at any given time, the candle will have one and only one height. It is not exactly linear, although it looks reasonable to approximate it as a linear function since the rate of burning is almost constant (2 inches per hour).

Activity Synthesis

The purpose of this discussion is for students to justify how this situation can be modelled by a linear equation. Select students who answered yes to the last question and ask:

- "Was the data exactly linear? If not, what made you decide that you could treat it as such?"
- "Which data points did you use to predict when the candle would burn out?"
- "What was the gradient between the first two data points? What was the gradient between the last two data points? What does it mean that their gradients are different?"

Tell students that although the data is not precisely linear, it does makes sense to model the data with a linear function because the points resemble a line when graphed. We can then use different data points to help predict when the candle would burn out. Answers might vary slightly, but it results in a close approximation.

Conclude the discussion by asking students to reconsider the range of values posted earlier for the first question and ask if they think that range is acceptable or if it needs to change (for example, students may now think the range should be smaller after considering the different gradients).

9.2 Shadows

10 minutes (there is a digital version of this activity)

The purpose of this activity is for students to determine if a given set of data can be modelled by a linear function. Students first view a set of pictures and data for the length of a shadow at 0, 20, and 60 minutes. Then, they make a prediction about how long a shadow will be after 95 minutes. Students then compare their estimate with the actual length of the shadow and make conclusions about the model they used to make their estimate.

Monitor for students using different strategies to make their prediction. For example, students may use different pairs of points to make their prediction or they may try to use all three. The discussion for this activity focuses on how, if we are given two input-output pairs, we can always find a linear function with these inputs and outputs, but that doesn't mean a linear function is actually appropriate for the situation. In this case, we can see when we get more data that a linear function is not appropriate.

Instructional Routines

- Three Reads

Launch

Arrange students in groups of 2. Tell students to close their books or devices, and display the image and given data for all to see. Give students 1–2 minutes of quiet think time to estimate the length of the shadow after 95 minutes and discuss their responses with their partner. Encourage partners to discuss their estimation strategy and why their estimate makes sense. Invite groups to share their estimate and reasoning with the whole class.

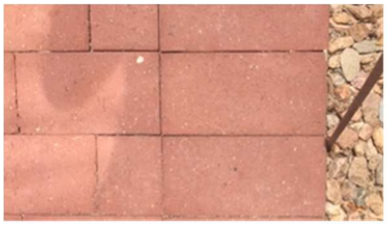
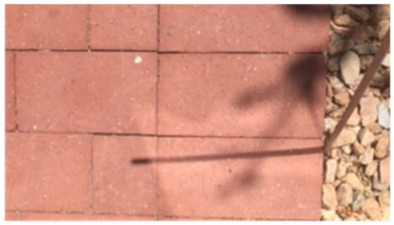
Tell students to open their books or devices and give work time for the remaining questions. Follow with a whole-class discussion.

If using the digital activity, follow the directions above. In this lesson, the digital activity allows students to plot their points and test their thinking with a dynamic applet, however, the mathematics is truly the same.

Engagement: Develop Effort and Persistence. Encourage and support opportunities for peer interactions. Display sentence frames to support students when they explain their estimation with their partner. For example, "I predict _____ because . . ." and "How did you get . . .?"
Supports accessibility for: Language; Social-emotional skills Reading, Speaking, Listening: Three Reads. Use this routine to support reading comprehension without solving for students. In the first read, students read the problem and review the images with the goal of comprehending the situation (e.g., a photo of a stick's shadow was taken at different times; the length of the shadow was different at different times). In the second read, identify the important quantities by asking students what can be counted or measured (e.g., at 20 minutes, the shadow was 10.5 cm long; at 60 minutes, the shadow was 26 cm long). After the third read, ask students to brainstorm possible strategies to complete the task. This will help students connect the language in the problem and the reasoning needed to solve the problem, while keeping the intended level of cognitive demand in the task.
Design Principle(s): Support sense-making

Student Task Statement

When the Sun was directly overhead, the stick had no shadow. After 20 minutes, the shadow was 10.5 cm long. After 60 minutes, it was 26 cm long.

1. Based on this information, estimate how long it will be after 95 minutes.
2. After 95 minutes, the shadow measured 38.5 cm. How does this compare to your estimate?
3. Is the length of the shadow a function of time? If so, is it linear? Explain your reasoning.

Student Response

1. Answers vary. Sample response: If we model this with the linear function that goes through (0,0) and (20,10.5), we would predict that the length would be growing at a rate of $\frac{10.5}{20}$ centimetres per minute. After 95 minutes, this would give a prediction of about 50 centimetres since $95 \times \frac{10.5}{20} \approx 49.88$.
2. Answers vary. Sample response: The prediction we made overestimated the length by about 11.5 centimetres.
3. The length of the shadow is a function of time, since every time determines and only one length. It is not a linear function of the time, since the points (0,0), (20,10.5), (60,26) and (95,38.5) do not lie on any one line. Based on the answer to the last part, it is not even very well approximated by a linear function.

Activity Synthesis

Select groups that had different strategies for making their original prediction to share their reasoning about whether or not a linear model is a good fit for predicting the length of the shadow. In particular, make sure it is pointed out how much using the rate of change determined by the first two points over-predicts the length of the shadow after 95 minutes.

Tell students that if we only use two data points, it is always possible to model a situation with a linear function. We need additional data to help us determine if a linear model is appropriate. In this case, mathematicians have also used the geometry of Earth travelling around the Sun to provide a better model for the length of the shadow as a function of time that is not a linear function.

9.3 Recycling

10 minutes

The purpose of this activity is for students to approximate different parts of a graph with an appropriate line segment. This graph is from a previous activity, but students interact with it differently by sketching a linear function that models a certain part of the data. They take this model and consider its ability to predict input and output for other parts of the graph. This helps students think about subsets of data that might have different models from other parts of the data.

Identify students who draw in different lines for the first question to share during the Activity Synthesis.

Instructional Routines

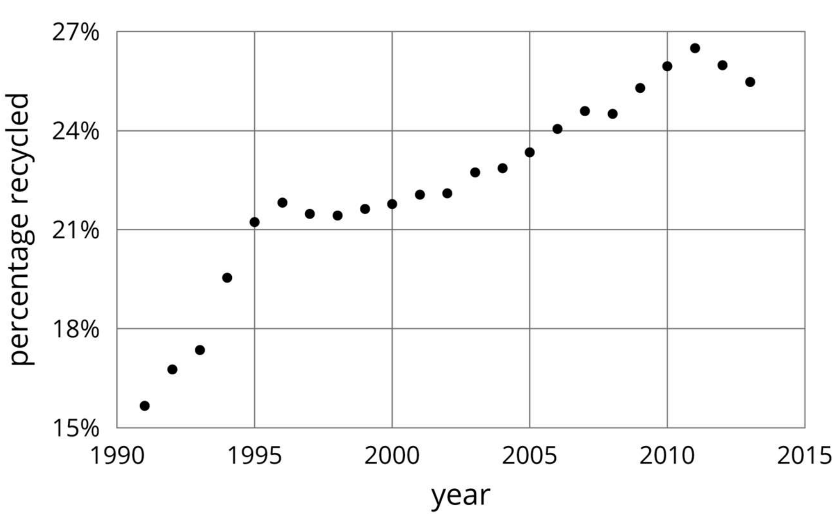
- Discussion Supports

Launch

Arrange students in groups of 2. Provide students with access to straightedges. Give students 3–5 minutes of quiet work time and then time to share their responses with their partner. Follow with a whole-class discussion.

Student Task Statement

In an earlier lesson, we saw this graph that shows the percentage of all garbage in the U.S. that was recycled between 1991 and 2013.

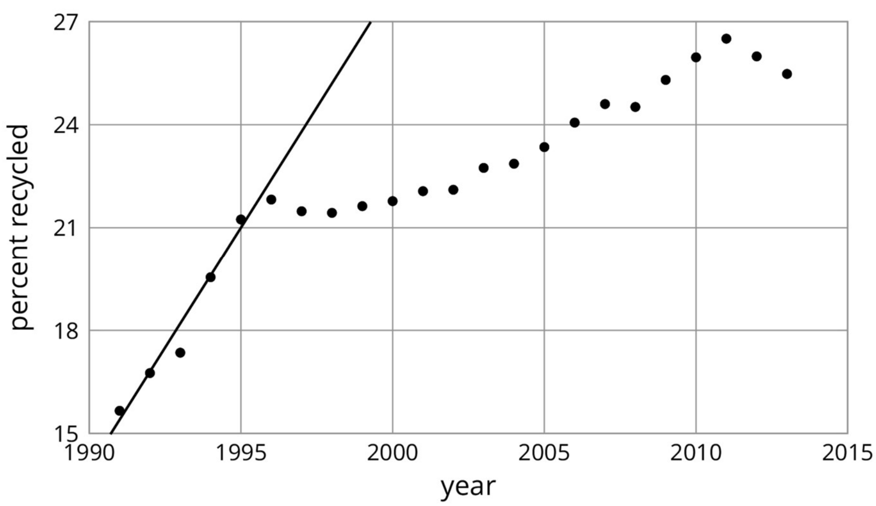


1. Sketch a linear function that models the change in the percentage of garbage that was recycled between 1991 and 1995. For which years is the model good at predicting the percentage of garbage that is produced? For which years is it not as good?

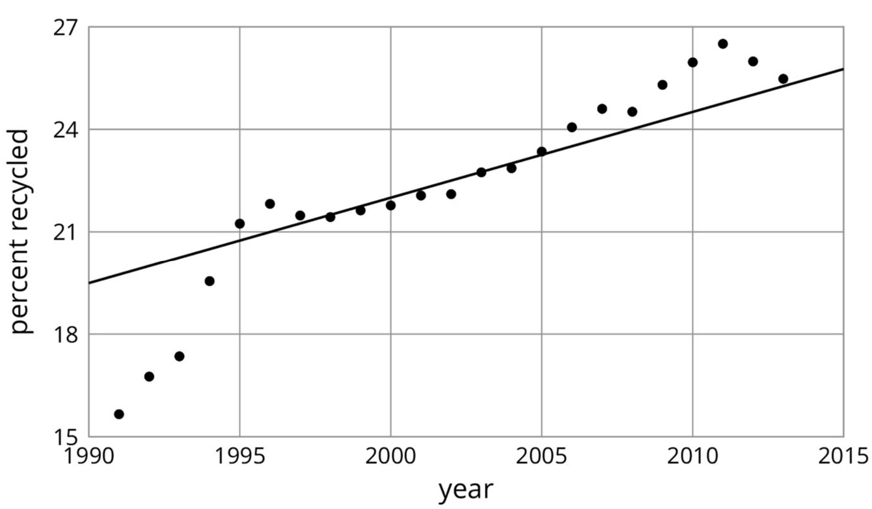
2. Pick another time period to model with a sketch of a linear function. For which years is the model good at making predictions? For which years is it not very good?

Student Response

1. Answers vary. The one displayed is a reasonable approximation of the data between 1991 and 1996, and a bad approximation from then on.



2. Answers vary. The one displayed is a reasonable approximation of the data between 1994 and 2013, but a bad approximation from 1991 to 1993.



Activity Synthesis

The purpose of this discussion is for students to understand that although you might find a good model for one part of a graph, that does not mean that model will work for other parts.

Select students previously identified to share their models. Display these for all to see throughout the entire discussion. Questions for discussion:

- "How much does your model for 1991 to 1995 overestimate 1996? 1997?"
- "If we drew in a single line to model 1997 to 2013, what would that line predict well? What would that line predict poorly?" (A single line modelling those years would reasonably predict the percent recycled from 1997 to 2010, but it wouldn't be able to show how the percent recycled from 2011 to 2013 is decreasing.)

Conclude the discussion by telling students that there is a trade-off in number of years to include in the interval and accuracy. We could "connect the dots" and be accurate about everything, but then our model has limited use and is complicated with so many parts. (Just imaging writing an equation for each piece!)

Speaking: Discussion Supports. Use this routine to support whole-class discussion. For each model that is shared, ask students to summarise what they heard using mathematical language. Consider providing students time to restate what they hear to a partner, before selecting one or two students to share with the class. Ask the original speaker if their peer was accurately able to restate their thinking. Call students' attention to any words or phrases that helped to clarify the original statement. This will provide more students with an opportunity to speak, and better understand each model.
Design Principle(s): Support sense-making

Lesson Synthesis

Tell students that a mathematical *model* is a mathematical object like an equation, a function, or a geometric figure that we use to represent a real-life situation. Sometimes a situation can be modelled by a linear function. We have to use judgment about whether this is a reasonable thing to do based on the information we are given. We must also be aware that the model makes imprecise predictions, or may only be appropriate for certain ranges of values.

Give students 1–2 minutes to think of a situation that may seem linear but actually is not. Invite them to share their situations. For example, the height of humans may look linear for short periods of time, but eventually growth stops, so we wouldn't want to use a linear model for height over a large period of time.

9.4 Board Game Sales

Cool Down: 5 minutes

Student Task Statement

A small company is selling a new board game, and they need to know how many to produce in the future.

After 12 months, they sold 4 thousand games; after 18 months, they sold 7 thousand games; and after 36 months, they sold 15 thousand games.

1. Could this information be reasonably estimated using a single linear model?
2. If so, use the model to estimate the number of games sold after 48 months. If not, explain your reasoning.

Student Response

1. Yes.
2. Answers vary. After 48 months there should be between 16 and 22 thousand sales depending on the data points used for the model.

Student Lesson Summary

Water has different boiling points at different heights above sea level. At 0 m above sea level, the boiling point is 100° C. At 2 500 m above sea level, the boiling point is 91.3° C. If we assume the boiling point of water is a linear function of height above sea level, we can use these two data points to calculate the gradient of the line: $m = \frac{91.3-100}{2\,500-0} = \frac{-8.7}{2\,500}$

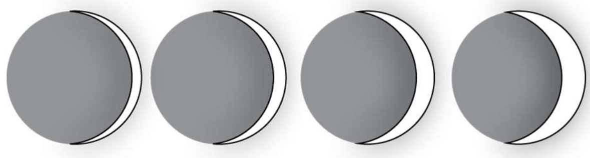
This gradient means that for each increase of 2 500 m, the boiling point of water decreases by 8.7° C. Next, we already know the y-intercept is 100° C from the first point, so a linear equation representing the data is $y = \frac{-8.7}{2\,500}x + 100$

This equation is an example of a mathematical *model*. A mathematical model is a mathematical object like an equation, a function, or a geometric figure that we use to represent a real-life situation. Sometimes a situation can be modelled by a linear function. We have to use judgment about whether this is a reasonable thing to do based on the information we are given. We must also be aware that the model may make imprecise predictions, or may only be appropriate for certain ranges of values.

Testing our model for the boiling point of water, it accurately predicts that at a height of 1 000 m above sea level (when $x = 1\,000$), water will boil at 96.5° C since $y = \frac{-8.7}{2\,500} \times 1000 + 100 = 96.5$. For greater heights, the model is not as accurate, but it is still close. At 5 000 m above sea level, it predicts 82.6° C, which is 0.6° C off the actual value of 83.2° C. At 9 000 m above sea level, it predicts 68.7° C, which is about 3° C less than the actual value of 71.5° C. The model continues to be less accurate at even greater heights since the relationship between the boiling point of water and height isn't linear, but for the heights at which most people live, it's pretty good.

Lesson 9 Practice Problems

1. **Problem 1 Statement**



On the first day after the new moon, 2% of the Moon's surface is illuminated. On the second day, 6% is illuminated.

a. Based on this information, predict the day on which the Moon's surface is 50% illuminated and 100% illuminated.

b. The Moon's surface is 100% illuminated on day 14. Does this agree with the prediction you made?

c. Is the percentage illumination of the Moon's surface a linear function of the day?

Solution

a. Answers vary. Sample response: A simple approach is to attempt a linear model starting at Day 1. If the illumination is increased by 4% every day, then after 11 more days (after Day 2) it reaches 50%. In 13 more days, illumination reaches 100%. This gives a prediction of Day 13 for 50% and Day 26 for 100%.

b. No

c. No (The linear model did a very bad job of approximating the data.)

2. **Problem 2 Statement**

In a science lesson, Jada uses a graduated cylinder with water in it to measure the volume of some marbles. After dropping in 4 marbles so they are all under water, the water in the cylinder is at a height of 10 millilitres. After dropping in 6 marbles so they are all under water, the water in the cylinder is at a height of 11 millilitres.

a. What is the volume of 1 marble?

b. How much water was in the cylinder before any marbles were dropped in?

c. What should be the height of the water after 13 marbles are dropped in?

d. Is the relationship between volume of water and number of marbles a linear relationship? If so, what does the gradient of a line representing this relationship mean? If not, explain your reasoning.

Solution

a. 0.5 ml

b. 8 ml

c. 14.5 ml

d. Yes. The gradient of the line represents the volume of 1 marble.

3. **Problem 3 Statement**

Solve each of these equations. Explain or show your reasoning.

$2(3x + 2) = 2x + 28$

$5y + 13 = -43 - 3y$

$4(2a + 2) = 8(2 - 3a)$

Solution

a. $x = 6$. Responses vary. Sample response: Multiply out by the 2 on the left side, add -4 to each side, add -$2x$ to each side, then divide each side by 4.

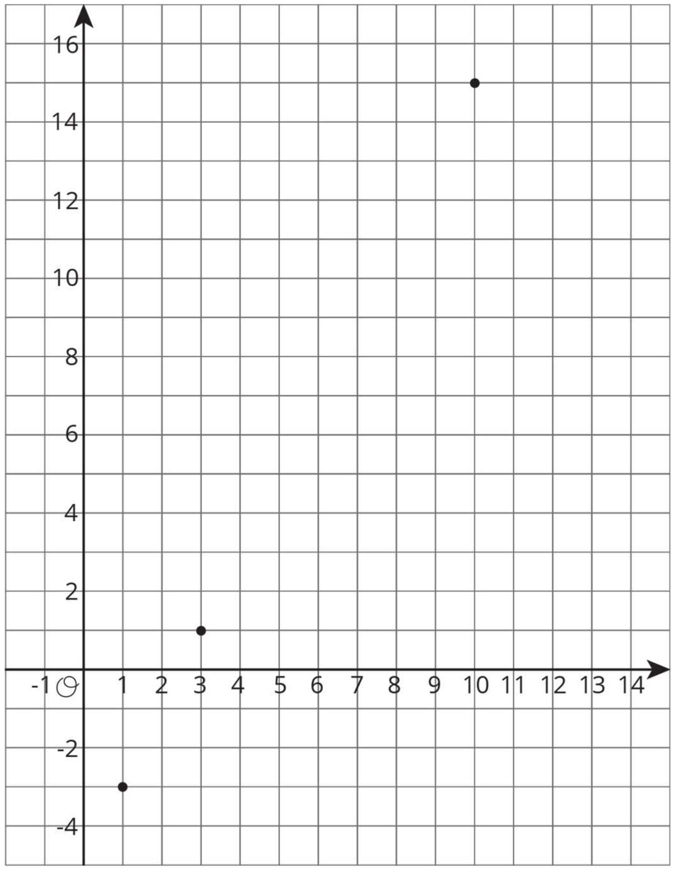
b. $y = -7$. Responses vary. Sample response: Add $3y$ to each side, subtract 13 from each side, then divide each side by 8.

c. $a = \frac{1}{4}$. Responses vary. Sample response: Divide each side by 4, distribute 2 on the right side, subtract 2 from each side, add $6a$ to each side, then divide each side by 8.

4. **Problem 4 Statement**

For a certain city, the high temperatures (in degrees Celsius) are plotted against the number of days after the new year.

Based on this information, is the high temperature in this city a linear function of the number of days after the new year?



Solution

Answers vary. Sample response: Although this data does fit a linear model, it does not make sense to use a linear model for this situation. For example, after only 2 months, the highest temperature would be more than the boiling point of water, which is unlikely.

5. **Problem 5 Statement**

The school designed their vegetable garden to have a perimeter of 32 feet with the length measuring two feet more than twice the width.

a. Using ℓ to represent the length of the garden and w to represent its width, write and solve a system of equations that describes this situation.

b. What are the dimensions of the garden?

Solution

a. $2\ell + 2w = 32$, $\ell = 2w + 2$

b. $\ell = 11\frac{1}{3}$, $w = 4\frac{2}{3}$