Mathematics Education Innovation

Lesson 9: The converse

Goals

- Determine whether a triangle with given side lengths is a right-angled triangle using the converse of Pythagoras' theorem.
- Generalise (orally) that if the side lengths of a triangle satisfy the equation $a^2 + b^2 = c^2$ then the triangle must be a right-angled triangle.
- Justify (orally) that a triangle with side lengths 3, 4, and 5 must be a right-angled triangle.

Learning Targets

- I can explain why it is true that if the side lengths of a triangle satisfy the equation $a^2 + b^2 = c^2$ then it must be a right-angled triangle.
- If I know the side lengths of a triangle, I can determine if it is a right-angled triangle or not.

Lesson Narrative

This lesson guides students through a proof of the converse of Pythagoras' theorem. Then students have an opportunity to decide if a triangle with three given side lengths is or is not a right-angled triangle.

Addressing

- Understand and apply Pythagoras' theorem.
- Explain a proof of Pythagoras' theorem and its converse.

Building Towards

- Explain a proof of Pythagoras' theorem and its converse.

Instructional Routines

- Co-Craft Questions
- Compare and Connect

Student Learning Goals

Let's figure out if a triangle is a right-angled triangle.

9.1 The Hands of a Clock

Warm Up: 5 minutes

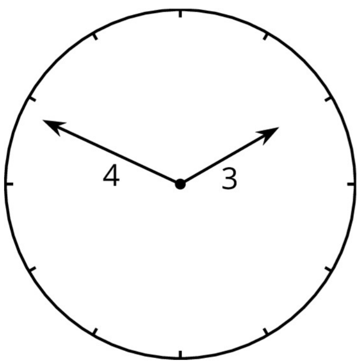
This warm-up is preparation for the argument of the converse of Pythagoras' theorem that we will construct in the next activity. The warm-up relies on Pythagoras' theorem and geometrically intuitive facts about how close or far apart the two hands of a clock can get from one another.

Launch

Arrange students in groups of 2. Give students 1 minute of quiet think time followed by partner and then whole-class discussions.

Student Task Statement

Consider the tips of the hands of an analogue clock that has an hour hand that is 3 centimetres long and a minute hand that is 4 centimetres long.

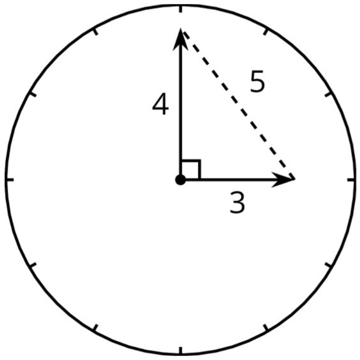


Over the course of a day:

1. What is the farthest apart the two tips get?
2. What is the closest the two tips get?
3. Are the two tips ever exactly five centimetres apart?

Student Response

1. If the two hands are pointing in opposite directions, the tips will be 7 centimetres apart.
2. If the two hands are pointing in the same direction (for example, at noon), the tips will be 1 centimetre apart.
3. Yes. Whenever the two hands make a right angle (for example, at 3:00), then by Pythagoras' theorem, the two tips will be 5 centimetres apart, since $3^2 + 4^2 = 5^2$.



Activity Synthesis

Invite students share their solutions.

Make the following line of reasoning explicit:

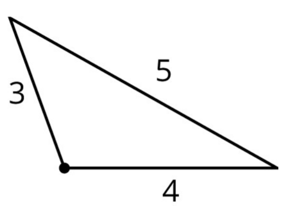
Imagine two hands starting together, where one hand stays put and the other hand rotates around the face of the clock. As it rotates, the distance between its tip and the tip of the other hand continually increases until they are pointing in opposite directions. So from one moment to the next, the tips get farther apart.

(Proving this requires mathematics beyond KS3, so for now we will just accept the results of the thought experiment as true.)

9.2 Proving the Converse

15 minutes

This activity introduces students to the *converse* of Pythagoras' theorem: In a triangle with side lengths a, b, and c, if we have $a^2 + b^2 = c^2$, then the triangle *must* be a right-angled triangle, and c must be its hypotenuse. Since up until this unit we rarely phrase things as a formal theorem, this may be the first time students have directly considered the idea that a theorem might work one way but not the other. For example, it is not clear at first glance that there is no such thing as an obtuse-angled triangle with side lengths 3, 4, and 5, as in the image. But since $3^2 + 4^2 = 5^2$, the converse of Pythagoras' theorem will say that the triangle *must* be a right-angled triangle.



The argument this activity presents for this result is based on the thought experiment that as you rotate the sides of length 3 and 4 farther apart, the distance between their endpoints also grows, from a distance of 1 when they are pointing in the same direction, to a distance of 7 when pointing in opposite direction. There is then one and only one angle along the way where the distance between them is 5, and by Pythagoras' theorem, this happens when the angle between them is a right angle. This argument generalises to an arbitrary right-angled triangle, proving that the one and only angle that gives $a^2 + b^2 = c^2$ is precisely the right angle.

The next activity will more heavily play up the distinction between Pythagoras' theorem and its converse, but it is worth emphasising here as well.

Instructional Routines

- Co-Craft Questions

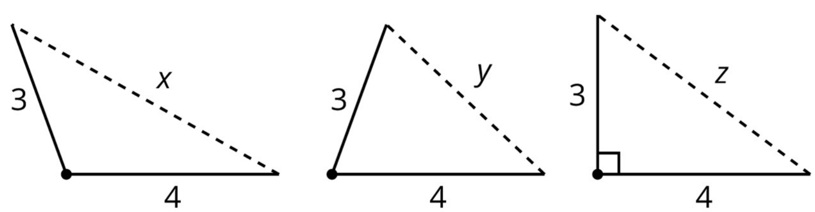
Launch

Arrange students in groups of 2. Give students 2 minutes of quiet work time. Ask partners to share their work and come to an agreement. Follow with a whole-class discussion.

Representation: Develop Language and Symbols. Use virtual or concrete manipulatives to connect symbols to concrete objects or values. For example, use GeoGebra or hands-on manipulatives to demonstrate how increasing or decreasing the angle between 2 sides affects the opposite side length. Invite students to make conjectures and generalisations for different cases.
Supports accessibility for: Conceptual processing Conversing, Writing: Co-Craft Questions. Before revealing the questions in this activity, display the image of the three triangles. Invite students to write mathematical questions that could be asked about the triangles. Invite students to compare the questions they generated with a partner before selecting 2–3 to share with the whole class. Listen for and amplify questions about the smallest or largest possible length of an unknown side. Also, amplify questions about the values of x, y, and z relative to one other. For example, "What is the largest possible value of x?", "What is the smallest possible value of y?", and "Which unknown side is the smallest or largest?" If no student asks these questions, ask students to adapt a question to align with the learning goals of this lesson. Then reveal and ask students to work on the actual questions of the task. This routine will help develop students' meta-awareness of language as they generate questions about the possible lengths of an unknown side of a triangle, given two of its side lengths.
Design Principle(s): Maximise meta-awareness

Student Task Statement

Here are three triangles with two side lengths measuring 3 and 4 units, and the third side of unknown length.



Sort the following six numbers from smallest to largest. Put an equal sign between any you know to be equal. Be ready to explain your reasoning.

1 5 7 x y z

Student Response

$1 < y < 5 = z < x < 7$.

As in the warm-up, the distance between the ends of the two sides has to be between 1 and 7 in all three cases. Since the third triangle is a right-angled triangle, we can apply Pythagoras' theorem to see that $z = 5$, since $3^2 + 4^2 = 5^2$. Finally, it must be that $x > z > y$, since as you rotate the side of length 3 away from the bottom side, the distance between them gets farther.

Are You Ready for More?

A related argument also lets us distinguish acute-angled from obtuse-angled triangles using only their side lengths.

Decide if triangles with the following side lengths are acute-angled, right-angled, or obtuse-angled. In right-angled or obtuse-angled triangles, identify which side length is opposite the right or obtuse angle.

- $x = 15, y = 20, z = 8$
- $x = 8, y = 15, z = 13$
- $x = 17, y = 8, z = 15$

Student Response

Take the two smaller sides and call them a and b. Call the longest side c. Calculate $a^2 + b^2$. If this equals c^2, it's a right-angled triangle. If c^2 is too big, the triangle is obtuse. If c^2 is too small, the triangle is acute.

- obtuse, since $8^2 + 15^2$ is 17^2. 20 is too big, so the triangle is obtuse, and the side of length 20 is opposite the obtuse angle.
- acute, since $8^2 + 13^2$ is 233. 15^2 is too small (it's 225), so the triangle is acute.
- right, since $8^2 + 15^2 = 17^2$. The side of length 17 is opposite the right angle.

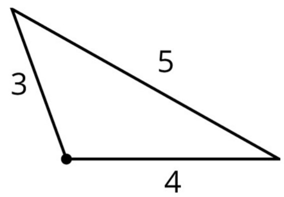
Activity Synthesis

Select some groups to share the reasoning. It is important to get out the following sequence of ideas:

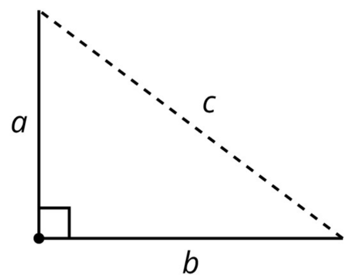
- Just as we saw with the clock problem, the length of the third side continually increases from 1 to 7 as the angles increases between 0° and 180°.

- Because of this, there is one and only one angle along the way that gives a third side length of 5.
- By Pythagoras' theorem, if the angle is a right angle, the third side length is 5.

Combining these, we see that the one and only angle that gives a third side length of 5 is the right angle. That is, *every* triangle with side lengths 3, 4, and 5 is a right-angled triangle with hypotenuse 5. Triangles like the one displayed here are thus impossible.



As a class, discuss how the argument could have been run with any two starting side lengths a and b instead of specifically 3 and 4.



Though we do not need to write it as formally, the length of the third side continually increases from $a - b$ (or $b - a$) up to $a + b$ as the angle increases between 0° and 180°. When the angle is a right angle, the third side has length equal to the value of c that makes $a^2 + b^2 = c^2$, and as in the previous discussion, that is the *only* angle that gives this length.

We conclude that the *only* way we could have $a^2 + b^2 = c^2$ is if the triangle is a right-angled triangle, with hypotenuse c. This result is called the *converse of Pythagoras' theorem.* Together, Pythagoras' theorem and its converse provide an amazing relationship between algebra and geometry: The algebraic statement $a^2 + b^2 = c^2$ is completely equivalent to the geometric statement that the triangle with side lengths a, b, and c is a right-angled triangle.

9.3 Calculating Shorter sides of Right-angled Triangles

10 minutes

The purpose of this activity is for students to apply Pythagoras' theorem and its converse in mathematical contexts. In the first problem, students apply Pythagoras' theorem to find unknown side lengths. In the second problem, students use the fact that by changing side lengths so that they satisfy $a^2 + b^2 = c^2$, the resulting triangle is guaranteed to be a right-angled triangle.

Instructional Routines

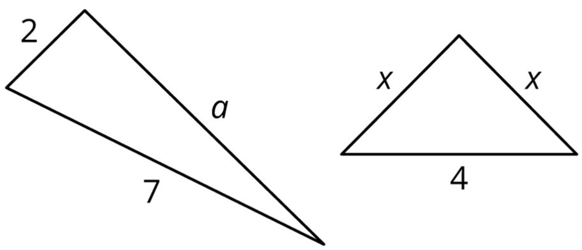
- Compare and Connect

Launch

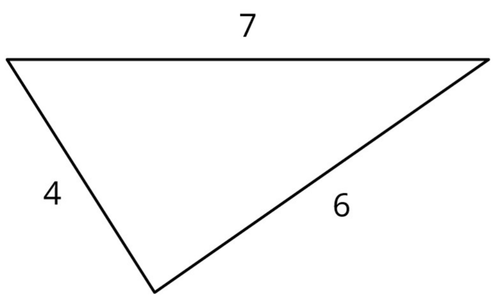
Provide students with access to calculators.

Student Task Statement

1. Given the information provided for the right-angled triangles shown here, find the unknown shorter side lengths to the nearest tenth.



2. The triangle shown here is not a right-angled triangle. What are two different ways you change *one* of the values so it would be a right-angled triangle? Sketch these new right-angled triangles, and clearly label the right angle.



Student Response

1. $a = \sqrt{45}$, $x = \sqrt{8}$. For the first triangle, $a^2 + 2^2 = 7^2$, which means $a^2 = 49 - 4 = 45$ and $a = \sqrt{45}$. For the second triangle, $x^2 + x^2 = 4^2$, which means $2x^2 = 16$ and $x = \sqrt{8}$.

2. Answers vary. Sample response: If 4 and 6 were shorter sides of a right-angled triangle, then the hypotenuse would be the value of c in the equation $4^2 + 6^2 = c^2$. This means $c^2 = 52$ and $c = \sqrt{52} \approx 7.2$.

Activity Synthesis

The goal of this discussion is for students to use Pythagoras' theorem to reason about the side lengths of right-angled triangles. Select 2–3 students to share their work for the two problems.

For the second problem, make a list of all possible changes students figured out that would make the triangle a right-angled triangle by changing the length of just one side. Depending on where students decided the right angle is located and which side they selected to change, there are nine possible solutions. Can students find them all?

Engagement: Develop Effort and Persistence. Break the class into small discussion groups and then invite a representative from each group to report back to the whole class. *Supports accessibility for: Attention; Social-emotional skills Speaking, Listening: Compare and Connect.* Ask students to prepare a visual display to show two different ways to change the triangle so that it would be a right-angled triangle. As students investigate each other's work. Listen for and amplify the language students use to explain how they used the converse of Pythagoras' theorem to justify that the triangle is a right-angled triangle. Encourage students to explain how the various equations of the form $a^2 + b^2 = c^2$ informed how they sketched the right-angled triangles. For example, the equation $4^2 + b^2 = 6^2$ implies that the length of the hypotenuse is 6, whereas the equation $4^2 + b^2 = 7^2$ implies that the length of the hypotenuse is 7. This will foster students' meta-awareness and support constructive conversations as they compare strategies for creating right-angled triangles and make connections between equations of the form $a^2 + b^2 = c^2$and the right-angled triangles they represent.
Design Principles(s): Cultivate conversation; Maximise meta-awareness

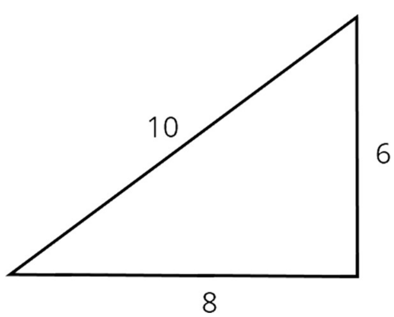
Lesson Synthesis

The purpose of this discussion is to reinforce Pythagoras' Theorem and its converse.

- "What has to be true in order to be sure a triangle is a right-angled triangle?" (The sum of the squares of the two shorter sides must equal the square of the longest side.)

Remind students that this is a result of the *converse* of Pythagoras' theorem. Pythagoras' theorem is an example of a theorem where the converse is also true. That is, Pythagoras' theorem states that for a right-angled triangle with sides a, b, and c, with c the length of the hypotenuse, the relationship $a^2 + b^2 = c^2$ is always true. The converse of Pythagoras' theorem states that if a, b, and c are side lengths of a triangle and $a^2 + b^2 = c^2$, then the angle opposite the side of length c is a right angle.

To illustrate this, display the image shown here of a triangle with sides 6, 8, and 10.



Ask students to decide if this is a right-angled triangle or not. After some quiet work time, poll the class for which type of triangle they think it is. Remind students that while the angle across from the side of length 10 looks like a right angle, we can't be sure it is just from the image. However, since $6^2 + 8^2 = 10^2$ is true, we know, by the converse of Pythagoras' theorem, that the triangle is a right-angled triangle and that the angle across from the side of length 10 is a right angle.

9.4 Is It a Right-angled triangle?

Cool Down: 5 minutes

Student Task Statement

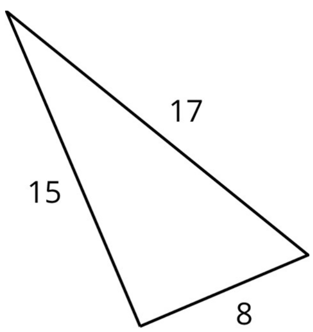
The triangle has side lengths 7, 10, and 12. Is it a right-angled triangle? Explain your reasoning.

Student Response

No. If this was a right-angled triangle, then $7^2 + 10^2$ would equal 12^2, which it does not.

Student Lesson Summary

What if it isn't clear whether a triangle is a right-angled triangle or not? Here is a triangle:



Is it a right-angled triangle? It's hard to tell just by looking, and it may be that the sides aren't drawn to scale.

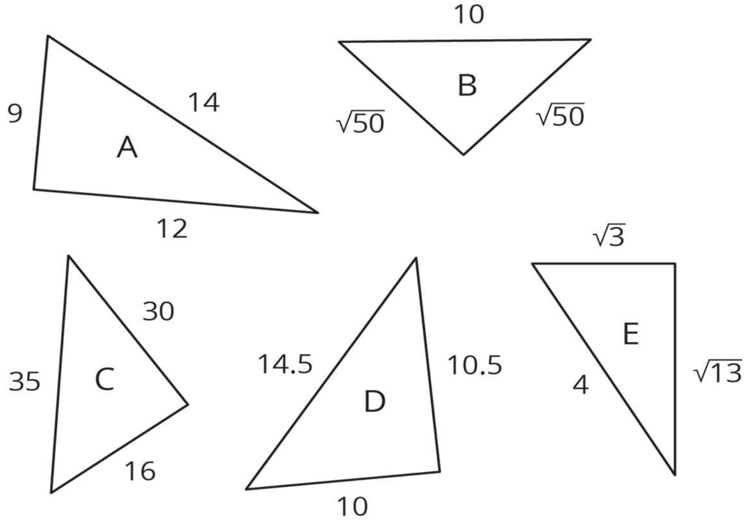
If we have a triangle with side lengths a, b, and c, with c being the longest of the three, then the *converse* of Pythagoras' theorem tells us that any time we have $a^2 + b^2 = c^2$, we *must* have a right-angled triangle. Since $8^2 + 15^2 = 64 + 225 = 289 = 17^2$, any triangle with side lengths 8, 15, and 17 *must* be a right-angled triangle.

Together, Pythagoras' theorem and its converse provide a one-step test for checking to see if a triangle is a right-angled triangle just using its side lengths. If $a^2 + b^2 = c^2$, it is a right-angled triangle. If $a^2 + b^2 \neq c^2$, it is not a right-angled triangle.

Lesson 9 Practice Problems

1. **Problem 1 Statement**

Which of these triangles are definitely right-angled triangles? Explain how you know. (Note that not all triangles are drawn to scale.)



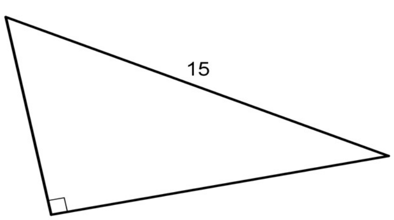
Solution

B, D, and E are right-angled triangles. A and C are not.

- A: $9^2 + 12^2 = 14^2$ is false
- B: $\sqrt{50}^2 + \sqrt{50}^2 = 10^2$ is true
- C: $16^2 + 30^2 = 35^2$ is false
- D: $10^2 + 10.5^2 = 14.5^2$ is true
- E: $\sqrt{3}^2 + \sqrt{13}^2 = 4^2$ is true

2. **Problem 2 Statement**

A right-angled triangle has a hypotenuse of 15 cm. What are possible lengths for the two shorter sides of the triangle? Explain your reasoning.



Solution

Answers vary. Sample responses: $\sqrt{200}$ and 5; $\sqrt{125}$ and 10. If the shorter sides of the triangle are a and b, then we can set up the equation $a^2 + b^2 = 15^2$. This means a^2 and b^2 must sum to 225. If $a^2 = 25$, then $b = 200$. If $a^2 = 100$, then $b^2 = 125$.

3. **Problem 3 Statement**

In each part, a and b represent the length of a shorter side of a right-angled triangle, and c represents the length of its hypotenuse. Find the missing length, given the other two lengths.

a. $a = 12, b = 5, c = ?$

b. $a = ?, b = 21, c = 29$

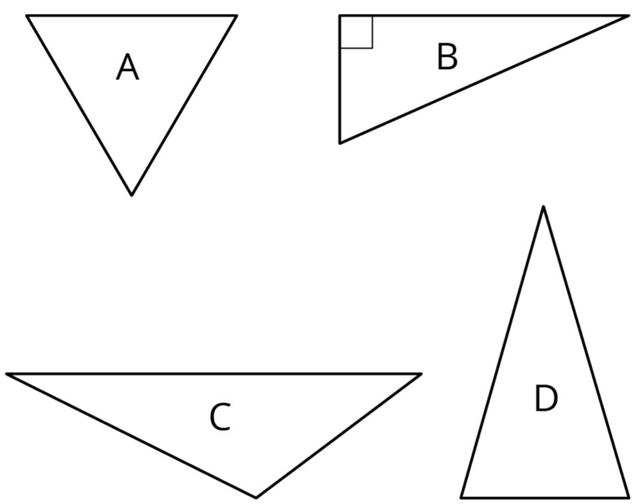
Solution

a. $c = 13$. If $a = 12$ and $b = 5$ then $12^2 + 5^2 = c^2$.

b. $a = 20$. If $b = 21$ and $c = 29$ then $a^2 + 21^2 = 29^2$.

4. **Problem 4 Statement**

For which triangle does Pythagoras' theorem express the relationship between the lengths of its three sides?



Solution

B

5. **Problem 5 Statement**

Andre makes a trip to Mexico. He exchanges some pounds for pesos at a rate of 20 pesos per pound. While in Mexico, he spends 9000 pesos. When he returns, he exchanges his pesos for pounds (still at 20 pesos per pound). He gets back $\frac{1}{10}$ the amount he started with. Find how many pounds Andre exchanged for pesos and explain your reasoning. If you get stuck, try writing an equation representing Andre's trip using a variable for the number of pounds he exchanged.

Solution

500 pounds. Sample reasoning: $\frac{20x-9000}{20} = \frac{x}{10}$, where x represents the number of pounds he exchanged. Rewrite the equation as $20x - 9000 = 2x$, and then solve to find $x = 500$.