Five Space-Filling Polyhedra

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Five space-filling polyhedra

GUY INCHBALD

Introduction

Solid shapes which pack together to fill space cover a large and varied range. I recently found five such polyhedra which I have not seen described elsewhere. The names which I have used for them here are rather ferocious I am afraid, but that just happens to be the way the names of polyhedra work.

Nets and coordinates for the polyhedra are given in Appendices A and B respectively.

The bisymmetric hendecahedra

A polyhedron with eleven faces is called a hendecahedron, from the Greek for eleven. The one shown in Figure 1 has two planes of symmetry, i.e. it is bisymmetric. This hendecahedron also has eleven vertices; polyhedra with the same number of faces as vertices are not very common. It has 2 large rhombic faces, a small rhombic face (which in the proportions used here is square), 4 congruent isosceles triangular faces which meet along edges at right angles, and 4 congruent kite-shaped faces. (See Appendix B for coordinates.)

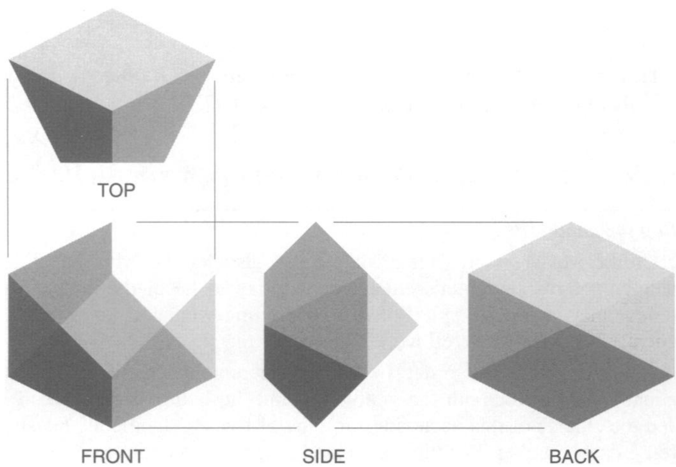


FIGURE 1 The bisymmetric hendecahedron

Figures 2 and 3 shows how four hendecahedra together form a kind of hexagonal boat shape which will stack in interlocking layers. This boat shape is also a 'translation unit' – it can be regularly stacked in a lattice to fill space, without any rotation or reflection. This lattice is similar to the body-centred cubic, but scaled vertically by a factor which is here one-half (but see below). In Figure 4 the way the hendecahedra themselves stack together to form a space-filling 'honeycomb' can be seen.

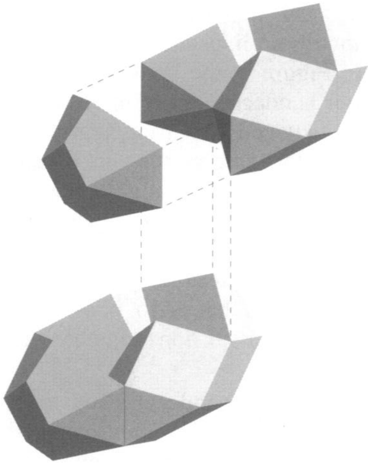


FIGURE 2

The hendecahedra form interlocking hexagonal 'boat' shapes.

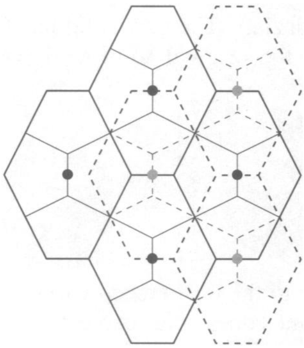


FIGURE 3

One layer (dashed) over another, showing the centre of each translation unit

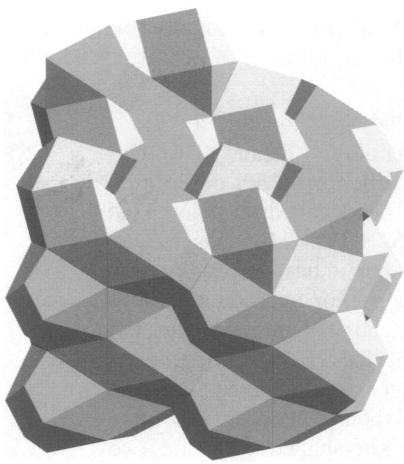


FIGURE 4

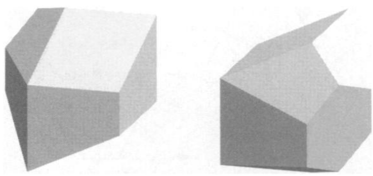
A general stack

The particular polyhedron described here has an arbitrary height (chosen for convenience of its coordinates): it can be distorted vertically by stretching to give an infinite series of shapes which are all space-fillers. In

Figure 4, vertical lines can be seen running through the stack. Figure 2 shows how these lines are made from the edges where two triangular faces meet. In Figure 3, the lines are viewed end-on and appear as the corners where four hendecahedra from each layer meet. It is obvious from this that the angle between two triangular faces must be a right angle, but what should the other angles seen in Figure 3 be? These other angles can be varied by rotating the triangular faces together about their vertical edges, whilst maintaining the right angles and the overall symmetry of the shapes, up to the points at which faces merge or disappear. This rotation generates space-filling hendecahedra varying continuously from ones with broad front rhombs and blunt backs to ones with narrow front rhombs and sharper backs. The two distortions together yield a doubly-varying range of space-fillers.

By cutting a hendecahedron in half horizontally and inserting a (pentagonal) prismatic centre section, an elongated bisymmetric hendecahedron is formed (Figure 5). The square face has become hexagonal, and the triangles are now trapezia.* The new solid has fourteen vertices with coordinates obtained from the original solid as described in Appendix B.

FIGURE 5
Two views of the elongated bisymmetric hendecahedron



The shape fills space in a similar way to the unstretched variant. It may be distorted vertically in two independent zones: one being the prismatic centre zone and the other the tapering top and bottom ends corresponding to an unelongated shape. Together with rotation of the trapezoidal faces, this yields a threefold range of distortions which still fill space.

The sphenoid hendecahedra

'Sphenoid' means wedge-shaped, which is an apt description for the hendecahedron shown in Figure 6. This also has eleven vertices, but it has only one plane of symmetry: top and bottom halves have reflective symmetry, but left and right halves are different, as can be seen in the end views. The sphenoid hendecahedron has three sizes of kite-shaped face and two types of isosceles triangle, all

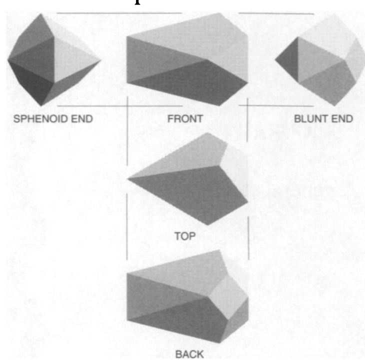


FIGURE 6
The sphenoid hendecahedron

* The term *trapezium* has two quite different interpretations – I use the term to describe a quadrilateral with one pair of parallel sides.

coming in pairs. In the proportions used here, the rhombic face is square. (See Appendix B for coodinates.) Unexpectedly the numbers of 3- and 4-sided faces are the same as for the bisymmetric variant, and indeed both hendecahedra have the same topology – they can be simply distorted into one another.

Figure 7 shows how six units pack like flower petals to form a 'floret'. A second floret fits up to it from below and is the other way up. The two florets form a translation unit which packs in a simple hexagonal lattice, here with a height equal to the unit length.

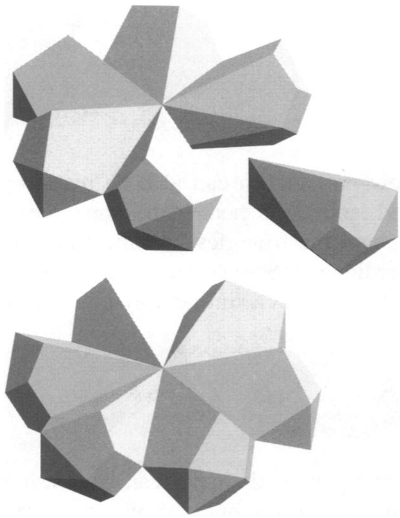


FIGURE 7

Six hendecahedra form a 'floret'. These stack in layers, alternate ways up.

Florets will stack in layers, alternate ways up, to form faceted columns. In Figure 8 one floret is laid upon another, and in Figure 9 it can be seen how florets also fit together side by side to form a layer. The layer is not quite symmetrical, alternate layers being right- and left-handed about the junction of three florets. Perhaps a little harder to visualise is the way the whole structure of layers and columns interlocks with no gaps, as illustrated in Figure 10.

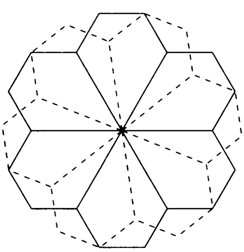


FIGURE 8 One floret (dashed) on top of another

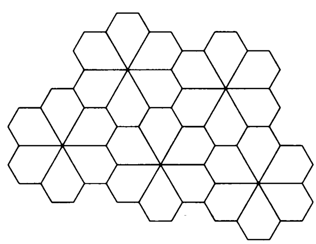


FIGURE 9 A layer of florets

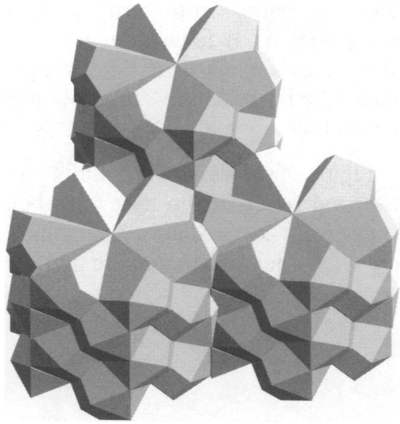


FIGURE 10

The florets form columns which pack together.

A new polyhedron can also be made by elongation (Figure 11): in the elongated sphenoid hendecahedron, the square face again becomes a hexagon and the triangles become trapezia. It also has fourteen vertices, with coordinates obtained from the original solid as described in Appendix B, and fills space in a similar way to its unstretched cousin.

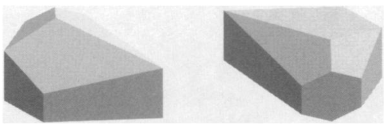


FIGURE 11
Two views of the elongated sphenoid hendecahedron

The two sphenoid shapes can be distorted vertically in the same way as the bisymmetric ones, but have no analogue of the rotational distortion.

The dodecahemioctahedron

There are a number of solids with faces which pass through their centre, so that the face has no inside or outside but can be seen from both sides in different places. These central faces are typically parallel to the faces of some normal convex polyhedron, but number half as many. Figure 12 shows the 4 hexagonal faces corresponding to half an octahedron which make this solid 'hemi-octa' (though parts of the hexagons cannot be seen from either side). Neither the hexagons nor the octahedron are regular, the octahedron being slightly flattened or 'oblate'. The solid also

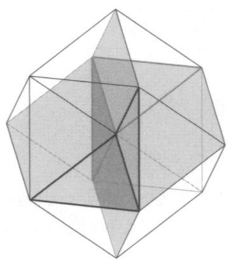


FIGURE 12

The dodecahemioctahedron. Two hexagonal faces are shown shaded.

has 4 rhombic faces and 8 (isosceles) triangular ones, making in all 12 ordinary faces. Hence the name dodecahemioctahedron.

Figures 13 and 14 illustrate two other ways of deriving its shape: Figure 13 as four oblate octahedra joined face to face around a central vertex, and Figure 14 as a cube cut into six square pyramids meeting at the centre, with two opposing pyramids removed and four new pyramids stuck onto the square bases of the remaining ones. It can also be thought of as a rhombic dodecahedron with two oblate octahedra removed leaving dimples behind.

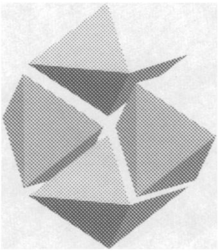


FIGURE 13
As four 'oblate' octahedra

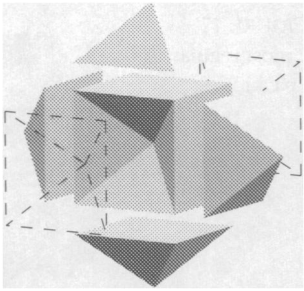


FIGURE 14
As a cube with two square pyramids removed and four more added

The corner of one dodecahemioctahedron exactly fits the dimple of another (Figure 15). A series of units can be fitted together in two basic ways, as in Figure 16. In various combinations, these give rise to several different packings which fill space. These packings are not true lattices, since they have 'false' edges, where the edge of one or more units lies across the face of another.

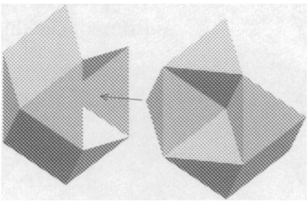


FIGURE 15
The corner of one dodecahemioctahedron fits into the dimple of another.

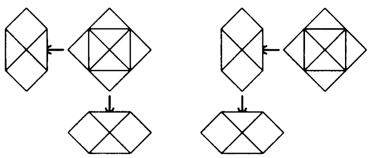


FIGURE 16
The two basic ways of fitting units together.

The most regular packing I have found is shown in Figure 17. The units form layers, each of which has alternate rows of peaks and dimples. The packing does not have full cubic symmetry, since the rows of peaks and dimples give each layer a directional 'grain'. But there is no distinction between the three main axes. Another packing, shown in Figure 18, forms distinct layers of peaks alternating with layers of dimples. This pattern is only seen in one plane, so the packing has a definite way up or orientation.

The dodecahemioctahedron is pristine, which means it cannot be distorted in any way and still fill space.

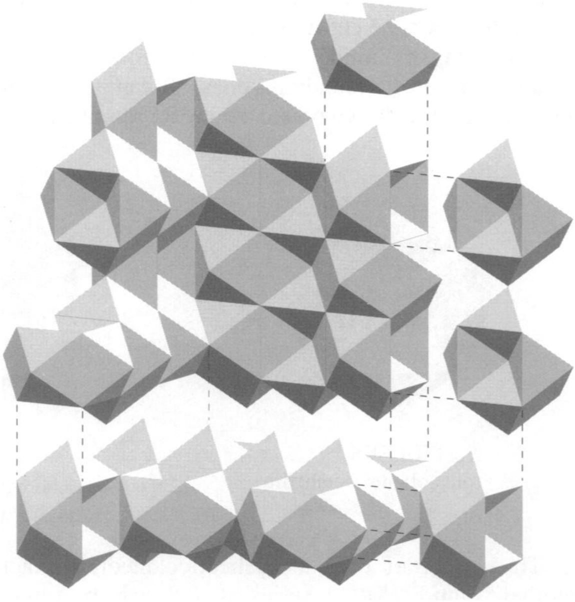


FIGURE 17
The most regular packing?

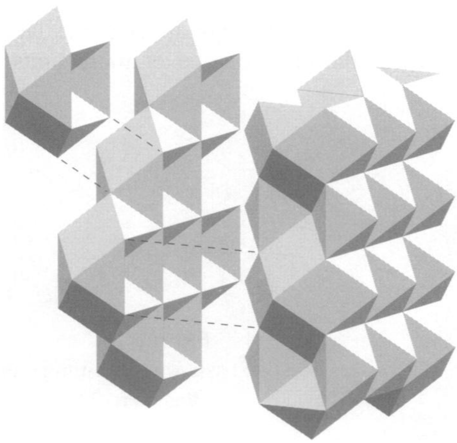


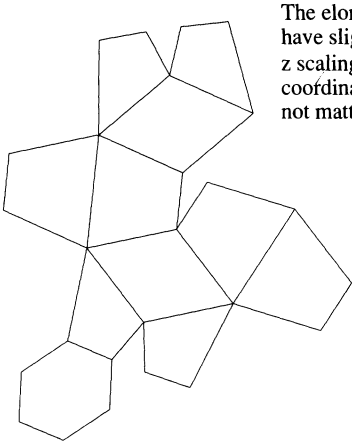
FIGURE 18
A different packing

Acknowledgements

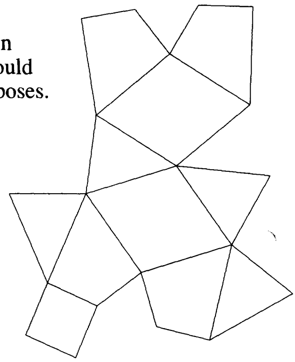
The 3-D views of polyhedra were created by WimpPoly and Polydraw software from Fortran Friends, PO Box 64, Didcot OX11 0TH. I am indebted to the author K. M. Crennell for advice, enthusiasm and pre-release software.

Appendix A: Nets

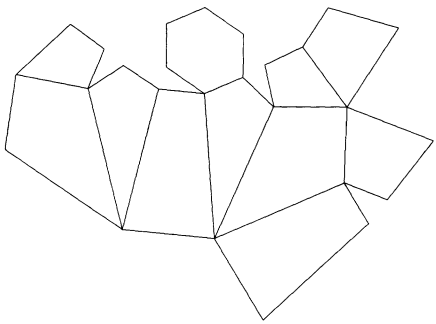
The elongated hendecahedra have slightly different z scalings from the given coordinates, but this should not matter for most purposes.



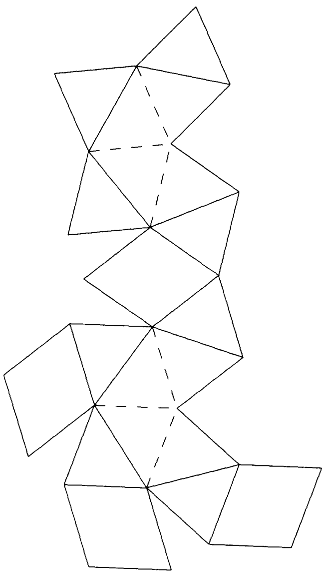
Elongated bisymmetric hendecahedron



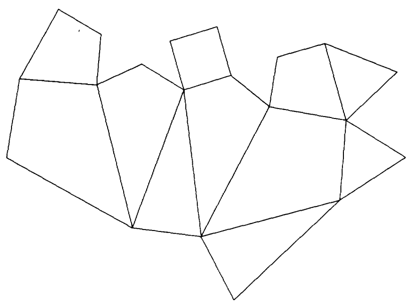
Bisymmetric hendecahedron



Elongated sphenoid hendecahedron



Dodecahemioctahedron



Sphenoid hendecahedron

Nets for making up

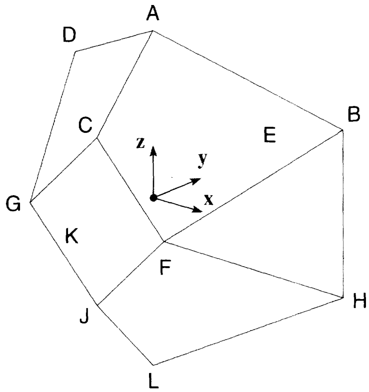
A set of nets is available, especially designed for easy assembly. They come on thin card, suitable for making up directly or for photocopying (though photocopies can be too distorted to make up accurately). Please send £1.50 per set, or £8.00 for 10 sets, to:

Stardust, Park View, Queenhill, Upton-upon-Severn, Worcester WR8 0RE

Appendix B: Coordinates

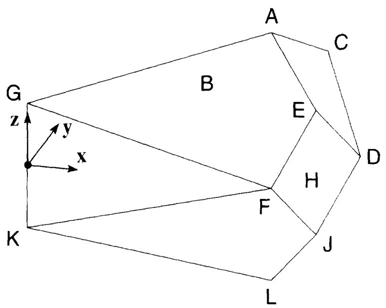
Bisymmetric hendecahedron

	x	y	z
A:	0	0	2
B:	2	1	1
C:	0	−1	1
D:	−2	1	1
E:	0	2	0
F:	1	−1	0
G:	−1	−1	0
H:	2	1	−1
J:	0	−1	−1
K:	−2	1	−1
L:	0	0	−2



For an elongated version add $\frac{1}{2}$ to the z ordinates of points A, B, C, D, E, F and G and subtract $\frac{1}{2}$ from the z ordinates of points E, F, G, H, J, K and L, thus creating three additional vertices.

Sphenoid hendecahedron

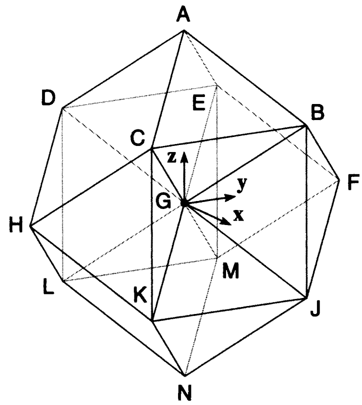


	x	y	z
A:	$\frac{1}{4}(9-\sqrt{3})$	$\frac{1}{4}(1+\sqrt{3})$	1
B:	1	$\sqrt{3}$	0
C:	2	$\sqrt{3}$	$\frac{1}{2}$
D:	$\frac{5}{2}$	$\frac{1}{2}\sqrt{3}$	0
E:	$\frac{9}{4}$	$\frac{1}{4}\sqrt{3}$	$\frac{1}{2}$
F:	2	0	0
G:	0	0	$\frac{1}{2}$
H:	2	$\sqrt{3}$	$-\frac{1}{2}$
J:	$\frac{9}{4}$	$\frac{1}{4}\sqrt{3}$	$-\frac{1}{2}$
K:	0	0	$-\frac{1}{2}$
L:	$\frac{1}{4}(9-\sqrt{3})$	$\frac{1}{4}(1+\sqrt{3})$	−1

For an elongated version add $\frac{1}{2}$ to the z ordinates of points A, B, C, D, E, F and G and subtract $\frac{1}{2}$ from the z ordinates of points B, D, F, H, J, K and L, thus creating three additional vertices.

Dodecahemioctahedron

	x	y	z
A:	0	0	2
B:	1	1	1
C:	1	−1	1
D:	−1	−1	1
E:	−1	1	1
F:	0	2	0
G:	0	0	0
H:	0	−2	0
J:	1	1	−1
K:	1	−1	−1
L:	−1	−1	−1
M:	−1	1	−1
N:	0	0	−2



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Juries do not apply mathematical formulae

Evidence of the Bayes Theorem or any similar statistical method of analysis in a criminal trial plunged the jury into inappropriate and unnecessary realms of theory and complexity deflecting them from their proper task.

. . .

At trial, the defence were permitted to lead evidence of the Bayes Theorem in connection with the statistical evaluation of the DNA profile.

. . .

The Bayes Theorem might be an appropriate and useful tool for statisticians, but it was not appropriate for use in jury trials or as a means to assist the jury in their task. In the first place, the theorem's methodology required that items of evidence be assessed separately according to their bearing on the guilt of the accused, before being combined in the overall formula.

That in their Lordships' view was too rigid an approach to evidence of the nature which a jury characteristically had to assess.

More fundamentally, the attempt to determine guilt or innocence on the basis of a mathematical formula, applied to each separate piece of evidence, was simply inappropriate to the jury's task. Jurors evaluated evidence and reached conclusions not by means of a formula, mathematical or otherwise, but by the joint application of their individual common sense and knowledge of the world to the evidence before them.

It was common for juries to evaluate scientific evidence but their Lordships had never heard it suggested that a jury should consider the relationship between such scientific evidence and other evidence by reference to probability formulae.

From *The Times* 9 May 1996, Nick Lord, Tonbridge School.