Lesson 12: Applications of arithmetic with powers of 10

Goals

- Determine what information is needed to answer a question about large numbers, and explain (orally) how that information would help solve the problem.
- Use exponent rules and powers of 10 to solve problems in context, and explain (orally) the steps used to organise thinking.

Learning Targets

- I can apply what I learned about powers of 10 to answer questions about real-world situations.

Lesson Narrative

Students apply what they have learned about working with exponents (especially powers of ten) to solve rich problems in context. The style of questioning requires students to identify essential features of the problem and persevere to calculate and interpret the solutions in context. Students must attend to precision when considering appropriate units of measurement.

Addressing

- Use numbers expressed in the form of a single digit times an integer power of 10 to estimate very large or very small quantities, and to express how many times as much one is than the other. For example, estimate the population of the United States as 3×10^8 and the population of the world as 7×10^9, and determine that the world population is more than 20 times larger.
- Perform operations with numbers expressed in standard form, including problems where both decimal and standard form are used. Use standard form and choose units of appropriate size for measurements of very large or very small quantities (e.g., use millimetres per year for seafloor spreading). Interpret standard form that has been generated by technology.

Building Towards

- Perform operations with numbers expressed in standard form, including problems where both decimal and standard form are used. Use standard form and choose units of appropriate size for measurements of very large or very small quantities (e.g., use millimetres per year for seafloor spreading). Interpret standard form that has been generated by technology.

Instructional Routines

- Stronger and Clearer Each Time
- Discussion Supports

Student Learning Goals

Let's use powers of 10 to help us make calculations with large and small numbers.

12.1 What Information Do You Need?

Warm Up: 5 minutes

The purpose of this warm-up is for students to reason about a real-world situation and consider the essential information required to solve problems.

Launch

Arrange students in groups of 2. Give students 1 minute of quiet think time, followed by 1 minute to share their responses with a partner. Follow with a whole-class discussion.

Student Task Statement

What information would you need to answer these questions?

1. How many metre sticks does it take to equal the mass of the Moon?
2. If all of these metre sticks were lined up end to end, would they reach the Moon?

Student Response

1. The mass of the Moon and the mass of a metre stick.
2. Distance to the Moon.

Activity Synthesis

Ask students to share their responses for each question. Record and display the responses for all to see.

Consider asking questions like these to encourage students to reason further about each question:

- "Why do you need that piece of information?"
- "How would you use that piece of information in finding the solution?"
- "Where would you look to find that piece of information?"

If there is time, ask students for predictions for each of the questions. Record and display their responses for all to see.

12.2 Metre Sticks to the Moon

20 minutes

The large quantities involved in these questions lend themselves to arithmetic with powers of 10, giving students the opportunity to make use of standard form before it is formally introduced. This activity was designed so students could practice modelling skills such as identifying essential features of the problem and gathering the required information. Students use powers of 10 and the number line as tools to make it easier to calculate and interpret results.

Notice the ways in which students use relevant information to answer the questions. Identify students who can explain why they are calculating with one operation rather than another. Speed is not as important as carefully thinking through each problem.

Instructional Routines

- Discussion Supports

Launch

From the warm-up, students have decided what information they need to solve the problem. Invite students to ask for the information they need. Provide students with only the information they request. Display the information for students to see throughout the activity. If students find they need more information later, provide it to the whole class then.

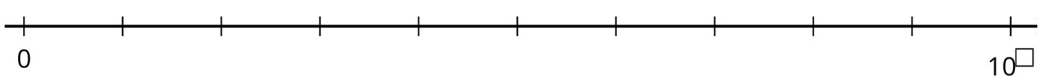
Here is information students might ask for in order to solve the problems:

- The mass of an average classroom metre stick is roughly 0.2 kg.
- The length of an average classroom metre stick is 1 metre.
- The mass of the Moon is approximately 7×10^{22} kg.
- The Moon is roughly $(3.8) \times 10^{8}$ metres away from Earth.
- The distances to various astronomical bodies the students might recognise, in light years, as points of reference for their last answer. (Consider researching other distances in advance or, if desired, encouraging interested students to do so.)

Arrange students in groups of 2–4 so they can discuss how to use the information to solve the problem. Give students 15 minutes of work time.

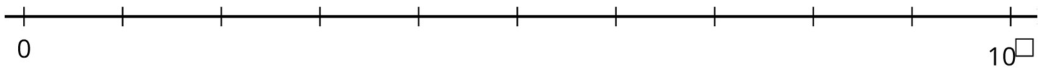
Student Task Statement

1. How many metre sticks does it take to equal the mass of the Moon? Explain or show your reasoning.
2. Label the number line and plot your answer for the number of metre sticks.



3. If you took all the metre sticks from the last question and lined them up end to end, will they reach the Moon? Will they reach beyond the Moon? If yes, how many times farther will they reach? Explain your reasoning.

4. One light year is approximately 10^{16} metres. How many light years away would the metre sticks reach? Label the number line and plot your answer.



Student Response

1. $(3.5) \times 10^{23}$ metre sticks, because the mass of the Moon divided by the mass of a metre stick is $\frac{7\times10^{22}}{2\times10^{-1}} = \frac{7}{2} \times 10^{22-(-1)} = (3.5) \times 10^{23}$

2. The right side of the number line should be labelled with 10^{24} with the tick marks labelled as multiples of 10^{23}. The number of metre sticks should be placed between the 3rd and 4th tick marks.

3. About 10^{15}, or a thousand trillion times as far as the Moon, because 4×10^{23} is 10^{15} times as much as 4×10^{8}.

4. $(3.5) \times 10^{7}$, or 35 million light years away. This should be plotted on a number line with 10^{8} as the rightmost tick mark, labelled with multiples of 10^{7}. The number of light years should be placed between the 3rd and 4th tick marks.

Are You Ready for More?

Here is a problem that will take multiple steps to solve. You may not know all the facts you need to solve the problem. That is okay. Take a guess at reasonable answers to anything you don't know. Your final answer will be an estimate.

If everyone alive on Earth right now stood very close together, how much area would they take up?

Student Response

Answers vary (and are likely to vary wildly). Sample response: There are between 7 billion and 8 billion people on Earth right now, so let's say 8 billion. Most teenagers and adults can fit into a rectangle of about 1 metre by half a metre when standing, so half a square metre. Smaller children would take up less space—maybe about half the space of an adult, so say children take up a quarter of a square metre. Let's guess that about a quarter of the people on Earth are small children. The space the adults will take up should be 6 billion times half a square metre, which is 3 billion square metres. The children will take up 2 billion times a quarter of a square metre, which is half a billion square metres. In total, the people will take up about 3.5 billion square metres.

Students may want to convert this answer to kilometres (or feet to miles, etc). To do this requires the tricky realisation that, even though there are 1 000 metres in a kilometre, there are 1 000^2 square metres in a square kilometre. Knowing this, we can divide $(3.5) \times 10^9$ by 1×10^6 to get $(3.5) \times 10^3$, or 35 000 square kilometres.

Activity Synthesis

Select previously identified students to share how they organised their relevant information and how they planned to use the information to answer the questions. The important idea students should walk away with is that powers of 10 are a great tool to tackle challenging, real-world problems that involve very large numbers.

It might be illuminating to put 35 million light years into some context. It is over a thousand trillion times as far as the distance to the Moon, or about the size of a supercluster of galaxies. The Sun is less than 1.6×10^{-5} light year away from Earth.

Engagement: Develop Effort and Persistence. Encourage and support opportunities for peer interactions. Prior to the whole-class discussion, invite students to share their work with a partner. Display sentence frames to support student conversation such as: “First, I _____ because . . .”, “I noticed _____ so I . . .”, “Why did you . . .?”, “I agree/disagree because . . ..”
Supports accessibility for: Language; Social-emotional skills Speaking: Discussion Supports. Give students additional time to make sure everyone in their group can explain their response to the first question. Then, vary who is called on to represent the ideas of each group. This routine will prepare students for the role of group representative and to support each other to take on that role.
Design Principle(s): Optimise output (for explanation); Cultivate conversation

12.3 That’s a Tall Stack of Cash

Optional: 20 minutes

This activity also illustrates the utility of using powers of 10 to work with and interpret very large quantities. Students practice modelling skills, such as identifying essential features of a problem and gathering the required information. Students use numbers and exponents flexibly and interpret their results in context.

As students work, look for students who use powers of 10 and the number line as tools to make it easier to calculate and interpret their results.

Instructional Routines

- Stronger and Clearer Each Time

Launch

Ask the class to predict which is taller, the Burj Khalifa or a stack of the money it cost to build the Burj Khalifa. Push them further by asking them to predict how high they think the stack would go. Record some of these predictions. Students will ask for the information they need to solve these problems:

- A 1-metre stack of 100-pound notes is about 1 000 000 pounds. The Burj Khalifa is 830 metres tall and cost 1.5 billion pounds.

- The Burj Khalifa weighs 450 000 000 kg. A penny weighs $(3.5) \times 10^{-3}$ kg. There are 100 pence in a pound.

Arrange students in groups of 2–4. Give students 10–15 minutes to work. As students work to finish the fourth problem (plotting the heights on a number line), tell the class that their next step is to read the fifth problem and think about what additional information they would need to know to solve the problem. When many students have finished problem 4, pause to allow the class to ask these questions before proceeding.

Representation: Internalise Comprehension. Begin the activity with concrete or familiar contexts. Review an image or video of the Burj Khalifa to activate prior knowledge of the context of the problem.
Supports accessibility for: Conceptual processing; Memory Writing, Speaking: Stronger and Clearer Each Time. Use this routine to give students a structured opportunity to revise and refine their response to the question: "Which is taller, the Burj Khalifa, or the stack of the money it cost to build the Burj Khalifa?" Ask each student to meet with 2–3 other partners in a row for feedback. Provide students with prompts for feedback that will help them to strengthen their ideas and clarify their language (e.g., "What did you do first?", "How do you know…?", "How did you compare the two values?", etc.). Students can borrow ideas and language from each partner to strengthen their final version.
Design Principle(s): Optimise output (for explanation)

Anticipated Misconceptions

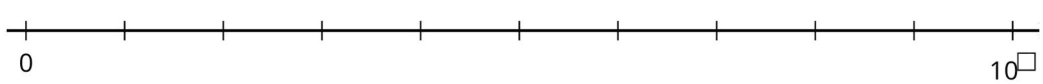
Students may overlook the fact that there are 100 pence in a pound. Remind these students of this fact and ask, "If you use pennies instead of pounds, would there be more coins or fewer coins? How many times more? If 1.5 billion pounds is $(1.5) \times 10^9$, then how would you find the number of pennies?"

Student Task Statement

In 2016, the Burj Khalifa was the tallest building in the world. It was very expensive to build.

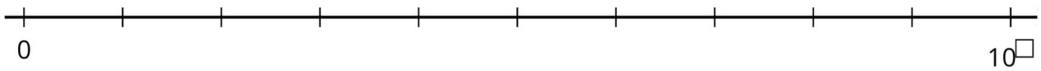
Consider the question: Which is taller, the Burj Khalifa or a stack of the money it cost to build the Burj Khalifa?

1. What information would you need to be able to solve the problem?
2. Record the information your teacher shares with the class.
3. Answer the question "Which is taller, the Burj Khalifa or a stack of the money it cost to build the Burj Khalifa?" and explain or show your reasoning.
4. Decide what power of 10 to use to label the rightmost tick mark of the number line, and plot the height of the stack of money and the height of the Burj Khalifa.



5. Which has more mass, the Burj Khalifa or the mass of the pennies it cost to build the Burj Khalifa? What information do you need to answer this?

6. Decide what power of 10 to use to label the rightmost tick mark of the number line, and plot the mass of the Burj Khalifa and the mass of the pennies it cost to build the Burj Khalifa.



Student Response

1. The height of the Burj Khalifa, the cost to build it, and the height of a stack of a million pounds.
2. The Burj Khalifa is 830 metres tall and cost 1.5 billion pounds to build. A 1 metre stack of £100 notes is about 1 000 000 pounds.
3. The stack of cash is $(1.5) \times 10^3$ or 1 500 metres tall, about twice as tall as the Burj Khalifa. This is because $\frac{(1.5)\times 10^9}{10^6} = (1.5) \times 10^3$
4. 10^4 for the rightmost tick mark and the first 9 integer multiples of 10^3 for tick marks leading up to it. The height of the stack of cash should be placed between the first and second tick marks. The height of the building should be placed between 0 and the first tick mark, but closer to the first tick mark than 0.
5. The Burj Khalifa has a mass of $(4.5) \times 10^8$ kg, and a penny has a mass of $(3.5) \times 10^{-3}$kg. Since there are 100 pennies in each pound, the number of pennies in 1.5 billion pounds is $(1.5) \times 10^9 \times 10^2 = (1.5) \times 10^{11}$ pennies. Since each penny has a mass of $(3.5) \times 10^{-3}$ kg, then the total mass of the pennies is $(3.5) \times 10^{-3} \times (1.5) \times 10^{11} = (5.25) \times 10^8$ kg, which is more massive than the Burj Khalifa.
6. 10^9 for the rightmost tick mark and the first 9 integer multiples of 10^8 for tick marks leading up to it. The mass of the pennies should be placed $\frac{1}{4}$ of the way between the 5th and 6th tick marks, and the mass of the Burj Khalifa should be placed between the 4th and 5th tick marks.

Activity Synthesis

If time allows, return to some of the recorded predictions. Acknowledge predictions that were accurate and discuss how powers of 10 made this problem much more approachable.

Lesson Synthesis

To prompt students to reflect on the modelling process and on using exponents to solve problems, consider asking some of these questions:

- "To solve the problems in this lesson you had to determine what information was needed. Did you find that to be fairly straightforward or challenging? What made it straightforward or challenging?"
- "Describe your thinking as you planned a solution path for the problems. For example, did you ask for information first and then decide what to do with it, or did you decide what needs to be done first before asking for certain information?"
- "Once you had the information you needed, what were some difficulties you encountered? How did you work through them?"
- "How did exponent rules and powers of 10 make the calculations easier?" (Powers of 10 make the numbers easier to express and interpret. The rules of exponents were handy for comparing how many times as large or as small one number is as another number.)
- "Would an estimate be an acceptable answer for problems like these? Why or why not? When might we need more precise solutions?" (It depends on the questions and how the answers would be used. For example, if we were planning a space exploration, we would likely need a high level of precision to ensure that we hit our targets. But if the answers are for comparison or general information, estimates are likely adequate.)

12.4 Reflecting on Using Powers of 10

Cool Down: 5 minutes

Student Task Statement

What is a mistake you would expect to see others make when doing problems like the ones in this lesson? Give an example of what such a mistake looks like.

Student Response

Answers vary. Sample response: I would expect to see others make mistakes using the exponent rules if they are not careful. For example, when multiplying two powers of 10 together, the exponents are added together. Someone might try to multiply the exponents instead. In the "Tall Stack of Cash" activity, someone might try to calculate how much an 830-metre stack of money would be and write $(8.3) \times 10^2$ metres times 10^6 pounds per metre and get $(8.3) \times 10^{12}$ pounds.

Student Lesson Summary

Powers of 10 can be helpful for making calculations with large or small numbers. For example, in 2014, the United States had

318 586 495

people who used the equivalent of

2 203 799 778 107

kilograms of oil in energy. The amount of energy per person is the total energy divided by the total number of people. We can use powers of 10 to estimate the total energy as 2×10^{12} and the population as 3×10^8 So the amount of energy per person in the U.S. is roughly $(2 \times 10^{12}) \div (3 \times 10^8)$ That is the equivalent of $\frac{2}{3} \times 10^4$ kilograms of oil in energy. That's a lot of energy—the equivalent of almost 7 000 kilograms of oil per person!

In general, when we want to perform arithmetic with very large or small quantities, estimating with powers of 10 and using exponent rules can help simplify the process. If we wanted to find the exact quotient of 2 203 799 778 107 by 318 586 495, then using powers of 10 would not simplify the calculation.

Lesson 12 Practice Problems

1. **Problem 1 Statement**

   Which is larger: the number of metres across the Milky Way, or the number of cells in all humans? Explain or show your reasoning.

   Some useful information:

   – The Milky Way is about 100 000 light years across.

   – There are about 37 trillion cells in a human body.

   – One light year is about 10^{16} metres.

   – The world population is about 7 billion.

   Solution

   There are more human cells than there are metres across the Milky Way. Since 100 000 is 10^5, it is about $10^5 \times 10^{16}$ or 10^{21} metres across the Milky Way. Notice that 37 trillion is $(3.7) \times 10^{13}$ and 7 billion is 7×10^9, so the total number of cells of all humans is $(3.7) \times 10^{13} \times 7 \times 10^9$. This gives $(25.9) \times 10^{22}$ human cells. This is about 260 times larger than 10^{21}, the approximate number of metres across the Milky Way. Using more precise values for population and the number of metres in a light year will yield slightly different results.

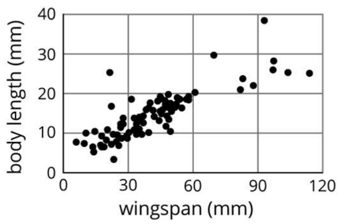
2. **Problem 2 Statement**

   Ecologists measure the body length and wingspan of 127 butterfly specimens caught in a single field.

   a. Draw a line that you think is a good fit for the data.

   b. Write an equation for the line.

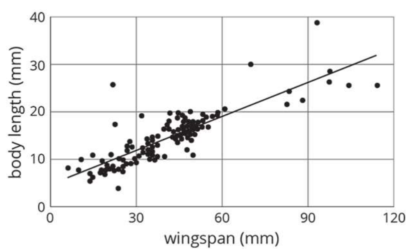
c. What does the gradient of the line tell you about the wingspans and lengths of these butterflies?



Solution

Answers vary. Sample response:

a.



b. $y = \frac{1}{4}x + 5$

c. For every 4 millimetres the length of the wingspan increases, the body length increases 1 millimetre.

3. **Problem 3 Statement**

Diego was solving an equation, but when he checked his answer, he saw his solution was incorrect. He knows he made a mistake, but he can't find it. Where is Diego's mistake and what is the solution to the equation?

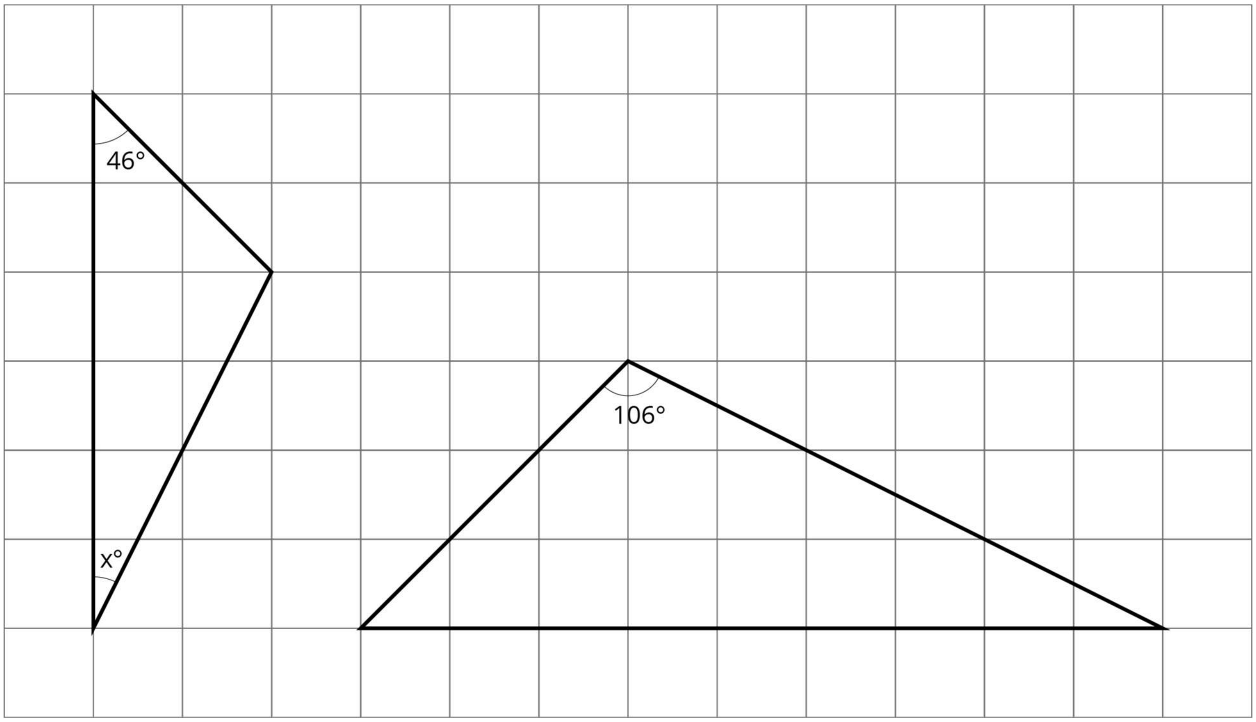
$$
\begin{aligned}
-4(7 - 2x) &= 3(x + 4) \\
-28 - 8x &= 3x + 12 \\
-28 &= 11x + 12 \\
-40 &= 11x \\
-\frac{40}{11} &= x
\end{aligned}
$$

Solution

Diego's mistake occurred in the transition from the first line to the second line. The distributive property with $-4(7 - 2x)$ should give $-28 + 8x$. The correct solution is $x = 8$.

4. **Problem 4 Statement**

The two triangles are similar. Find the value of x.



Solution

$x = 28$ (The obtuse angle in both triangles measures 106° because they are similar. The sum of the three angles in a triangle is 180°.)