Lesson 8: Similar triangles

Goals

- Generalise a process for identifying similar triangles and justify (orally) that finding two pairs of congruent angles is sufficient to show similarity.
- Justify (orally) that two triangles are similar by finding a sequence of transformations that takes one triangle to the other or checking that two pairs of corresponding angles are congruent.

Learning Targets

- I know how to decide if two triangles are similar just by looking at their angles.

Lesson Narrative

In the previous lesson, students found that, in order to check if two quadrilaterals are similar, it is important, in general, to check that corresponding angles are congruent and that corresponding side lengths are proportional. This lesson focuses on triangles. Triangles are special since it is possible to determine whether or not they are similar by looking *only* at the angles. If two triangles share three corresponding angle measurements, then they are similar. In fact, since the sum of the angles in a triangle is 180 degrees, two angles determine the third. Hence for triangles, all that is needed to deduce similarity is having *two* corresponding angles with equal size.

Students deduce the criteria for similarity in terms of angles by experimenting with triangles built out of pasta. As a result, they will need to make sense of measurements and account for possible inaccuracies.

Students will use the similarity criterion in future lessons to understand the concept of the gradient of a line. Later on, they will learn that three proportional sides (but not two) is also enough to deduce that two triangles are similar.

Building On

- Apply and extend previous understandings of operations with fractions to add, subtract, multiply, and divide rational numbers.
- Decide whether two quantities are in a proportional relationship, e.g., by testing for equivalent ratios in a table or graphing on a coordinate grid and observing whether the graph is a straight line through the origin.
- Understand that a two-dimensional shape is similar to another if the second can be obtained from the first by a sequence of rotations, reflections, translations, and enlargements; given two similar two-dimensional shapes, describe a sequence that exhibits the similarity between them.

Addressing

- Understand congruence and similarity using physical models, transparencies, or geometry software.
- Use informal arguments to establish facts about the angle sum and exterior angles of triangles, about the angles created when parallel lines are cut by a transversal, and the angle-angle criterion for similarity of triangles. For example, arrange three copies of the same triangle so that the sum of the three angles appears to form a line, and give an argument in terms of transversals why this is so.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Group Presentations
- Collect and Display
- Co-Craft Questions

Required Materials

Blank paper

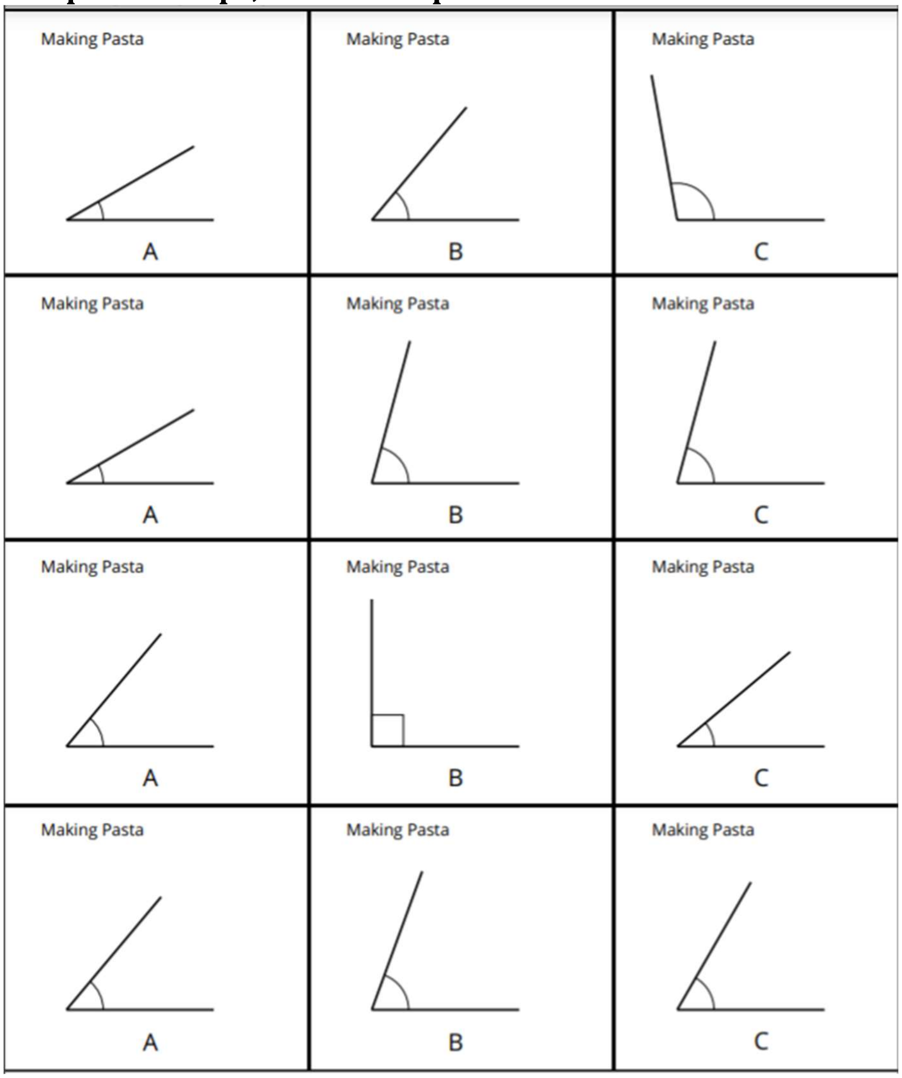
Dried linguine pasta

We specified linguine since it is flatter and less likely to roll around than spaghetti.

Geometry toolkits

tracing paper, graph paper, coloured pencils, scissors, and an index card to use as a straightedge or to mark right angles, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Pre-printed slips, cut from copies of the blackline master



Sticky tape

Required Preparation

Make 1 copy of the blackline master for every 4 students. Cut these into strips horizontally. Each student will receive one strip, which is a set of three angles labelled A, B, and C.

For the dried pasta that will be used to create the sides of the triangles, we recommend fettuccine or linguine so it doesn't roll off the table and is easy to break as needed.

Student Learning Goals

Let's look at similar triangles.

8.1 Equivalent Expressions

Warm Up: 5 minutes

This warm-up prompts students to use what they know about operations to create related expressions. While many warm-ups in the unit encourage students to work mentally and verbally, students will write their responses to this prompt. Since many different responses are possible, the task is accessible to all students and provides an opportunity to hear how each student reasons about the operations.

Launch

Arrange students in groups of 2. Display problem. Give students 1 minute of quiet think time and ask them to give a signal when they have three expressions. Follow with a 1 minute partner discussion and then a whole-class discussion.

Anticipated Misconceptions

If students only rely on the associative and commutative properties, suggest that they try to include at least one expression containing brackets.

Student Task Statement

Create three different expressions that are each equal to 20. Each expression should include only these three numbers: 4, -2, and 10.

Student Response

Answers vary. Possible responses:

- $-(4 \times 10) \div -2$
- $-(10 \div -2) \times 4$
- $-(4 \div -2) \times 10$

Activity Synthesis

Ask selected students to share their expressions. Record and display their responses for all to see. Ask students if or how the factors in the problem impacted their strategy. To involve more students in the conversation, consider asking:

- “Did anyone have a similar expression?”
- “Did anyone have a different expression related to this one?”
- “Do you agree or disagree? Why?”

8.2 Making Pasta Angles and Triangles

30 minutes

In this activity, students create triangles with given angles and compare them to classmates’ triangles to see that they are not necessarily congruent, but they are similar.

Each angle in a triangle imposes a constraint and students are examining in this activity how many constraints there are in similar triangles. There are only two. This makes sense since we can place one side of the triangle and then the triangle is determined by the length of that side and the two angles made with that side. (Note: the length of that side can be scaled with an appropriate enlargement to give any particular similar triangle with these angles.)

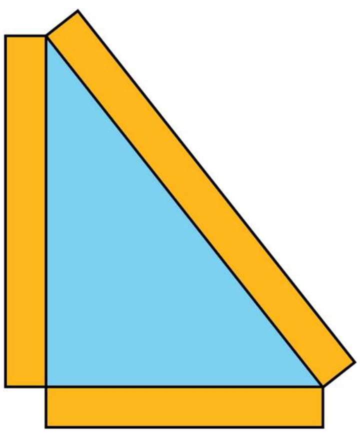
Watch for how students deal with measurement error. Remind students who are looking for proportional side lengths in similar triangles that they may not be exactly proportional. Also, students may find that their rounded angles do not add up to 180°. Remind them that the measurements are only approximate.

Instructional Routines

- Group Presentations
- Collect and Display

Launch

Tell students that they are going to build triangles using pasta and some of their given angles. Then they will find classmates who used the same angle(s) and compare their triangles. Tell them that they will need to move around the classroom to identify partners who use the same angle(s). For large classrooms, consider asking students to find three matching partners instead of two. Demonstrate how to measure angles in a pasta triangle.



In this picture, the approximate angle sizes are 90°, 50°, and 40°. Tell students that they can trace their angle(s) on the sheet of paper they are going to use to build their pasta triangle.

Provide students with 1 strip of 3 angles (A, B, and C) pre-cut from the blackline master. Each student should also have access to sticky tape, and extra paper to tape their triangles to. Finally, each student will need a ruler (which should be available in their geometry toolkits).

Pause students after they have made and compared triangles with one angle (first problem) and three angles (second problem). Make sure that students have found some non-similar triangles on the first problem and similar triangles for the second. Consider doing a gallery walk to see how the first sets of triangles differ and how the second sets of triangles are alike. Then have students work on the third problem.

Representation: Develop Language and Symbols. Use virtual or concrete manipulatives to connect symbols to concrete objects or values. Demonstrate how to build a triangle using pasta and how to use a protractor and ruler to measure the angles and lengths of each side. Be sure to emphasise appropriate rounding on the measurements.
Supports accessibility for: Visual-spatial processing; Conceptual processing Conversing, Reading: Collect and Display. As students find others who have the same angles in their triangles, circulate and listen to students as they decide whether their triangles are similar. Write down the words and phrases students use to justify why the triangles are or are not similar. Listen for students who state that two shared angles is enough to guarantee that two triangles are similar, but one shared angle is not enough. As students review the language collected in the visual display, encourage students to clarify the meaning of a word or phrase. For example, a phrase such as "the triangles are not similar because the angles do not match" can be clarified by restating it as "the triangles are not similar because their corresponding angles are not congruent." This routine will provide feedback to students in a way that supports sense-making while simultaneously increasing meta-awareness of language.
Design Principle(s): Support sense-making; Maximise meta-awareness

Anticipated Misconceptions

Students may need a reminder that the sum of angles in any triangle is 180 degrees.

Student Task Statement

Your teacher will give you some dried pasta and a set of angles.

1. Create a triangle using three pieces of pasta and angle A. Your triangle *must* include the angle you were given, but you are otherwise free to make any triangle you like. Tape your pasta triangle to a sheet of paper so it won't move.

    a. After you have created your triangle, measure each side length with a ruler and record the length on the paper next to the side. Then measure the angles to the nearest 5 degrees using a protractor and record these measurements on your paper.

    b. Find two others in the room who have the same angle A and compare your triangles. What is the same? What is different? Are the triangles congruent? Similar?

    c. How did you decide if they were or were not congruent or similar?

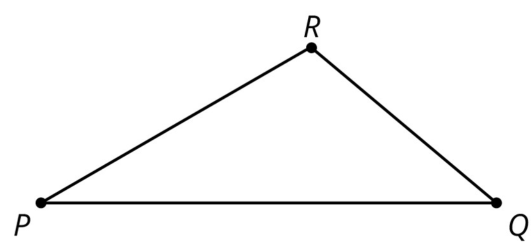
2. Now use more pasta and angles A, B, and C to create another triangle. Tape this pasta triangle on a separate sheet of paper.

a. After you have created your triangle, measure each side length with a ruler and record the length on the paper next to the side. Then measure the angles to the nearest 5 degrees using a protractor and record these measurements on your paper.

b. Find two others in the room who used your same angles and compare your triangles. What is the same? What is different? Are the triangles congruent? Similar?

c. How did you decide if they were or were not congruent or similar?

3. Here is triangle PQR. Break a new piece of pasta, different in length than line segment PQ.



– Tape the piece of pasta so that it lays on top of line PQ with one end of the pasta at P (if it does not fit on the page, break it further). Label the other end of the piece of pasta S.

– Tape a full piece of pasta, with one end at S, making an angle congruent to $\angle PQR$.

– Tape a full piece of pasta on top of line PR with one end of the pasta at P. Call the point where the two full pieces of pasta meet T.

a. Is your new pasta triangle PST similar to $\triangle PQR$? Explain your reasoning.

b. If your broken piece of pasta were a different length, would the pasta triangle still be similar to $\triangle PQR$? Explain your reasoning.

Student Response

1. Measurements and answers vary. The initial one-angle triangles will often only have the one angle in common. They are not necessarily similar. One way to check that they are not similar is to see if some of the angles have different sizes.

2. Answers vary. Sample response: The three-angle triangles are similar but the measured side lengths may not be exactly proportional because of possible measurement error.

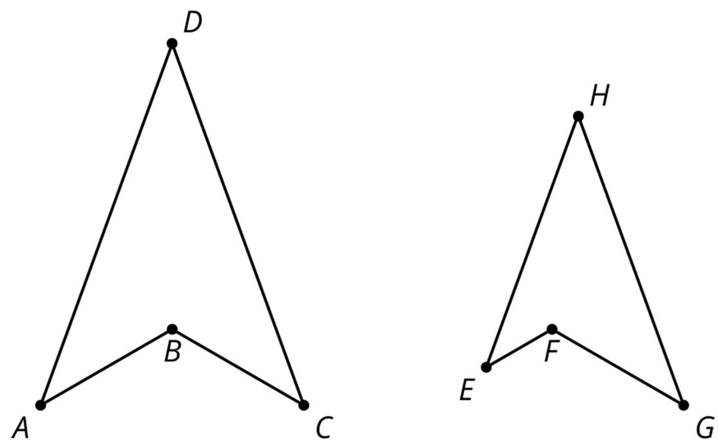
3. Answers vary. The two-angle triangles are similar. The interior angles of a triangle sum to 180. So if two pair of angles are congruent, the third pair are also congruent. Again, the side lengths may not be exactly proportional due to measurement error.

Are You Ready for More?

Quadrilaterals $ABCD$ and $EFGH$ have four angles measuring $240°$, $40°$, $40°$, and $40°$. Do $ABCD$ and $EFGH$ have to be similar?

Student Response

No. We can start with a 240 degree angle ABC and then place D so that BAD and BCD are 40 degree angles.



Then angle ADC will also be 40 degrees. We can make one of these quadrilaterals no matter where A and C are placed. The diagram shows another example of this construction: by placing E closer to the 240 degree angle EFG than A was to angle ABC, we do not change the angles, but we get a non-similar quadrilateral.

Activity Synthesis

Ask students if they needed all three angle measurements to build a triangle in the second problem. Some students may notice that it was not necessary: once they knew two angles, that was enough to build the pasta triangle, the third angle automatically being the right size. Other students may find that having the third angle was helpful to get better accuracy for all three angle sizes. Ask students how this relates to their findings on the last question: the last question shows that two shared angle sizes would be enough to guarantee that two triangles are similar. This makes sense because the sum of the angles in a triangle is 180 degrees so triangles that share two pairs of congruent angles actually share three pairs of congruent angles.

Some important discussion questions include

- "Did your angles in the triangle always add up to 180 degrees?" (Answers may vary for the first triangles because the angles were rounded to the nearest 5 degrees and the rounding can influence the sum of the angles.)
- "How did you decide whether or not the sides of your triangle were proportional to other triangles?" (Answers vary, the key being that measurement error means that quotients computed from measured side lengths may not be *exactly* equal even if the triangles are similar.)
- "How did you check whether or not your triangle was similar to another?" (Answers should include aligning an angle and then enlarging when the triangles were similar, and observing that the angles were different or the sides were not proportional when they were not similar.)

A big conclusion from this activity is that if triangles share two pair of congruent angles then they are similar. For the quadrilaterals studied in the previous activity, there was another instance where a pair of congruent angles was enough to decide that quadrilaterals are similar, namely rhombuses. But the triangle result is more surprising because there are three side lengths and *no* restrictions are made on these.

8.3 Similar Shapes in a Regular Pentagon

Optional: 10 minutes

This activity presents a complex shape with many triangles and asks students to find triangles similar to a given triangle. From the previous activity, students know that finding two pair of congruent angles is sufficient to show similarity. Students may also measure all three angles and check that they are congruent. In addition to using angle measures, students can describe transformations that take one triangle to another.

Monitor for students who use these methods to show that their chosen triangles are similar to triangle DJI:

- finding a sequence of translations, rotations, reflections, and enlargements
- checking that two (or three) corresponding angles are congruent

Select students who use these methods and invite them to present during the discussion.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Co-Craft Questions

Launch

Tell students that they may use anything in their geometry toolkits as well as a copy of triangle DJI to find triangles similar to it.

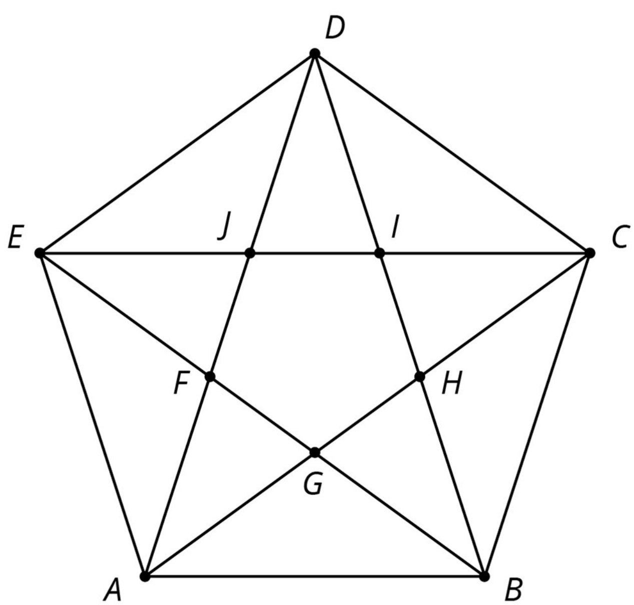
Representation: Internalise Comprehension. Demonstrate and encourage students to use colour coding and annotations to highlight connections between representations in a problem. For example, use the same colour to illustrate which triangles are similar to DJI. Annotate side and angle measurements in the appropriate places on the image.
Supports accessibility for: Visual-spatial processing Conversing, Writing: Co-Craft Questions. Before presenting the questions in this activity, display the diagram and ask students to write possible mathematical questions about the diagram. Ask students to compare the questions they generated with a partner before sharing questions with the whole class. Listen for and amplify questions about which triangles are similar to each other. Then reveal and ask students to work on the actual questions of the task. This routine will help develop students' meta-awareness of language as they generate questions about identifying similar triangles.
Design Principle(s): Maximise meta-awareness

Anticipated Misconceptions

If students eyeball the triangles in predicting similar triangles, make sure they justify their decisions by including specific measurements.

Student Task Statement

1. This diagram has several triangles that are similar to triangle DJI.



   a. Three different scale factors were used to make triangles similar to DJI. In the diagram, find at least one triangle of each size that is similar to DJI.

b. Explain how you know each of these three triangles is similar to DJI.

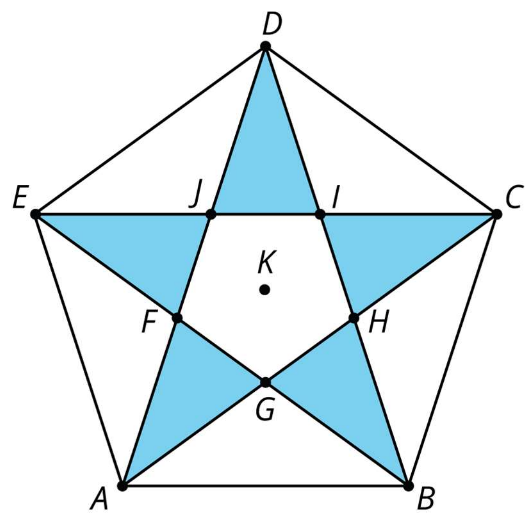
2. Find a triangle in the diagram that is not similar to DJI.

Student Response

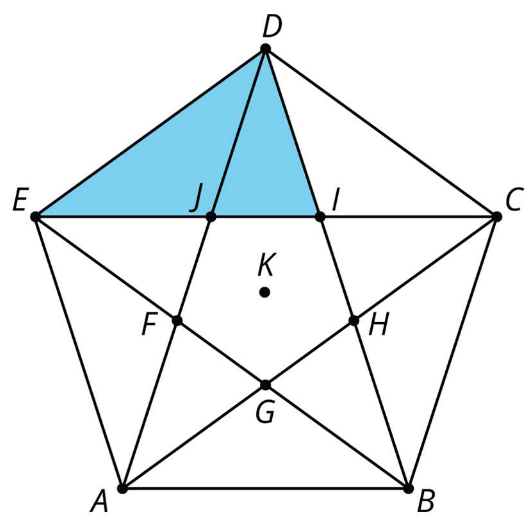
1. Answers vary.

    a. Options are (one from each)

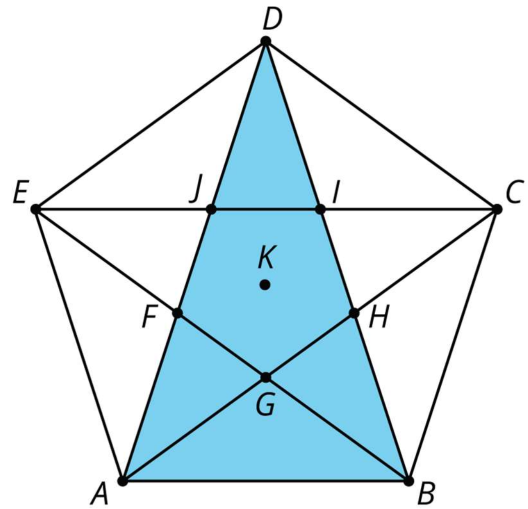
        i. Congruent to DJI (scale factor 1): DJI, EJF, AFG, BGH, CHI



        ii. Middle sized (scale factor about 1.5 from DJI): $EID, DJC, EGA, AEJ, BAF, ABH, CGB, BIC, DHC, DEF$



        iii. Large sized (scale factor about 2.5 from DJI): DAB, BCE, ACD, BDE, CEA



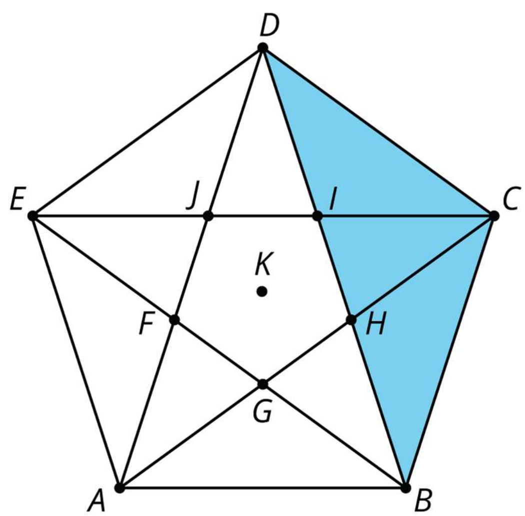
b. Answers vary. Some options:

   i. Check two of the angles to be the same as DJI.

   ii. Find transformations that take DJI to the other triangles.

   iii. Measure the angles to be congruent and the side lengths to be proportional.

2. Answers vary. Options are:
$DEJ, EFA, ABG, BCH, CDI, BDC, CDE, DEA, ABE, ABC, DFB, EGC, DAH, BIE, ACJ$.

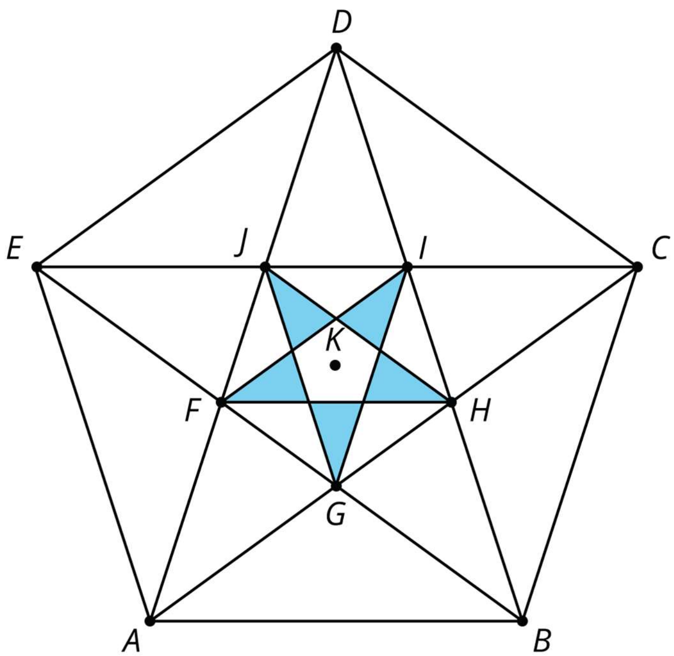


Are You Ready for More?

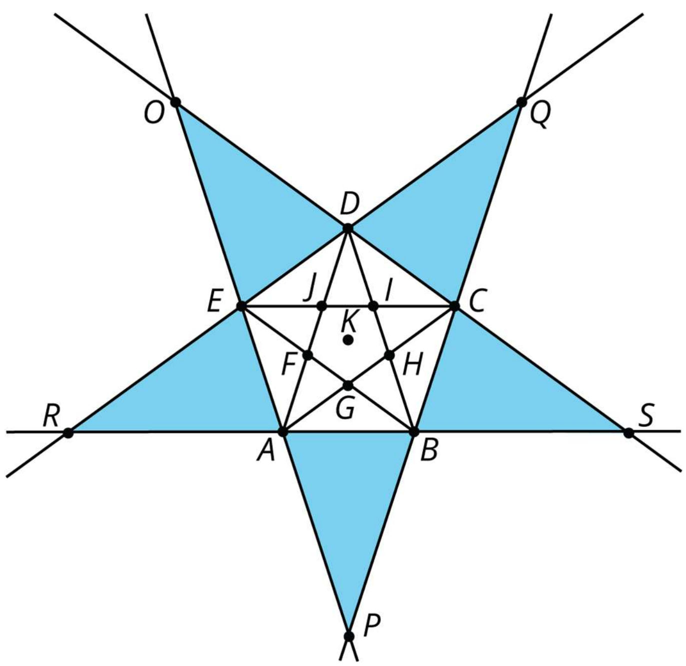
Figure out how to draw some more lines in the pentagon diagram to make more triangles similar to DJI.

Student Response

Some possibilities: Draw in the star inscribed in the inner pentagon $FGHIJ$, or just one of these line segments.



Extend the sides of pentagon $ABCDE$ until the lines intersect.



Activity Synthesis

Invite selected students to share some of the triangles they found and explain how they determined the triangles to be similar. Sequence them so students who used translations, rotations, reflections and enlargements go first, followed by those who use angle sizes. Concluding that two triangles are similar using the angle criterion from the previous task is a quick, efficient argument. On the other hand, finding explicit transformations and enlargements taking one triangle to another provides a more tactile, concrete experience. Both are important methods and students can choose based on their own personal comfort and the nature of the problem at hand.

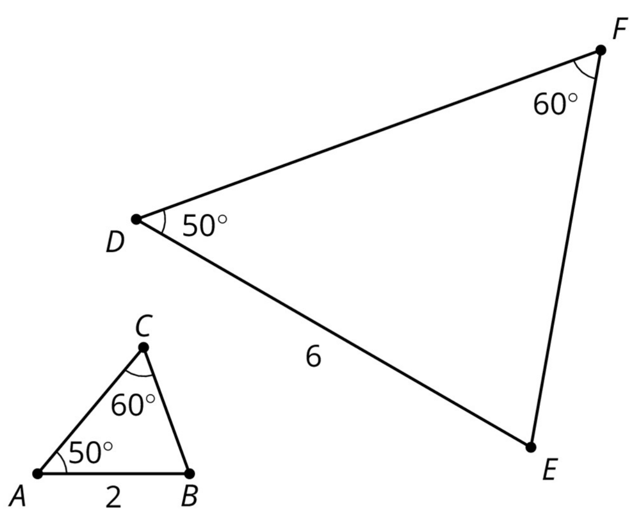
Make a list of different ways to show that two triangles are similar, including:

- using transformations (cutting out the triangle and placing it appropriately)
- finding two congruent corresponding angles in the triangles
- finding three congruent corresponding angles in the triangles

Among those triangles that are not similar to DJI, ask if some of these may be similar to each other.

Lesson Synthesis

This lesson focuses on how the angles influence the shape of a triangle. If a triangle has a 50 degree angle, what does that tell me about its shape? Could it be isosceles? (yes, two 50 degree angles and an 80 degree angle). Could it fail to be isosceles? (yes, one 50 degree angle, one 20 degree angle, and one 110 degree angle). Are all triangles with a 50 degree angle similar? (no, because the other two pairs of angles could be different).



If triangles have two pairs of congruent angles, then they are similar. For example, here are two triangles which each have a 50° angle and a 60° angle. If we enlarge ABC with centre A

and scale factor 3, then AB has the same length as DE. We can apply translations, rotations and reflections so that angle A matches up with angle D and angle B matches up with angle E. The vertices C and F also match up and ABC is similar to DEF.

8.4 Applying Angle-Angle Similarity

Cool Down: 5 minutes

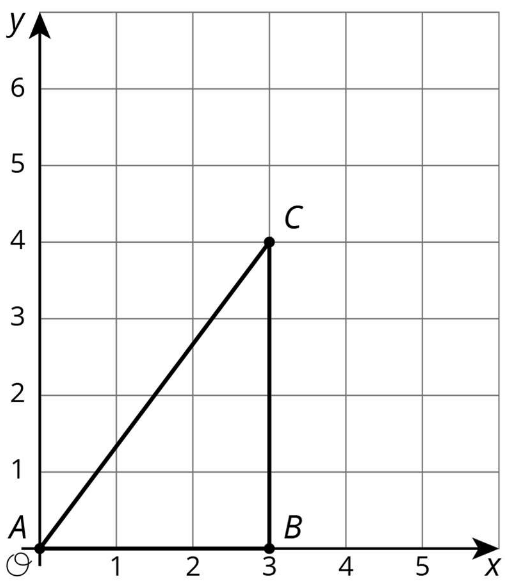
Students explain why two triangles are similar. They have at least two good options available. They can measure angles in the triangles and use what they learned in this lesson or they can describe similarity transformations (in this case an enlargement centred at (0,0)) that take one triangle to the other.

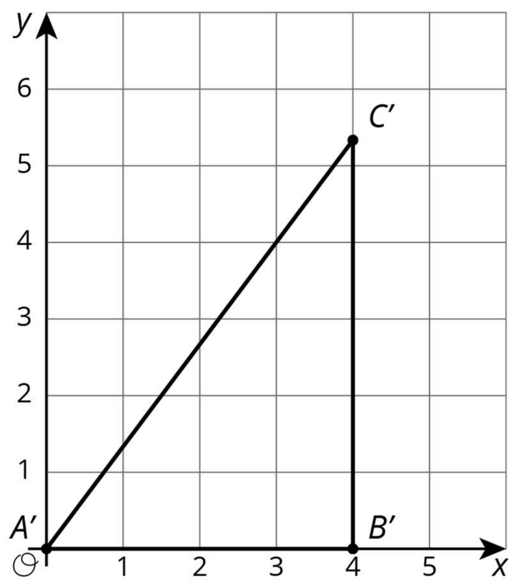
Launch

Students should have access to Geometry toolkit, especially a protractor

Student Task Statement

Here are two triangles.





1. Show that the triangles are similar.

2. What is the scale factor from triangle ABC to triangle $A'B'C'$?

Student Response

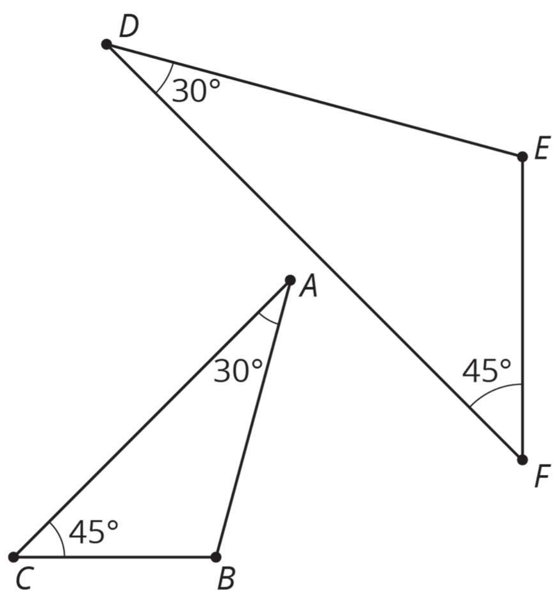
1. All three corresponding angles are congruent. Angle ABC and angle $A'B'C'$ are both right angles. Angle CAB and angle $C'A'B'$ are congruent (they both measure about 37 degrees). Angle ACB and angle $A'C'B'$ are congruent as well (they both measure about 53 degrees).

2. The scale factor is $1\frac{1}{3}$ since the length of line segment AB is 3 and the length of corresponding line segment $A'B'$ is 4.

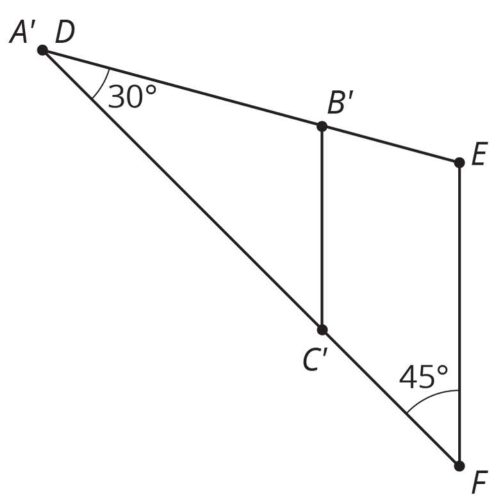
Student Lesson Summary

We learned earlier that two polygons are similar when there is a sequence of translations, rotations, reflections, and enlargements taking one polygon to the other. When the polygons are triangles, we only need to check that that both triangles have two corresponding angles to show they are similar—can you tell why?

Here is an example. Triangle ABC and triangle DEF each have a 30 degree angle and a 45 degree angle.



We can translate A to D and then rotate so that the two 30 degree angles are aligned, giving this picture:



Now an enlargement with centre D and appropriate scale factor will move C' to F. This enlargement also moves B' to E, showing that triangles ABC and DEF are similar.

Lesson 8 Practice Problems

1. **Problem 1 Statement**

   In each pair, some of the angles of two triangles in degrees are given. Use the information to decide if the triangles are similar or not. Explain how you know.

   - Triangle A: 53, 71, ___; Triangle B: 53, 71, ___
   - Triangle C: 90, 37, ___; Triangle D: 90, 53, ___
   - Triangle E: 63, 45, ____; Triangle F: 14, 71, ____
   - Triangle G: 121, ___, ___; Triangle H: 70, ___, ___

   Solution

   - Similar: They have two pairs of angles with equal measurement.
   - Similar: Since the angles in a triangle add up to 180°, the missing angle in Triangle C must be 53°. The two triangles therefore have two pairs of angles with equal measurement, so they are similar.
   - Not similar: Similar triangles have equal angle measurements, and there is no way to fill in the blanks so that this is true for these two triangles.
   - Not similar: Similar triangles have equal angle measurements, but no triangle can have angles which measure 121 and 70 degrees as these add up to more than 180.

2. **Problem 2 Statement**

   a. Draw two equilateral triangles that are not congruent.

   b. Measure the side lengths and angles of your triangles. Are the two triangles similar?

   c. Do you think two equilateral triangles will be similar *always*, *sometimes*, or *never*? Explain your reasoning.

   Solution

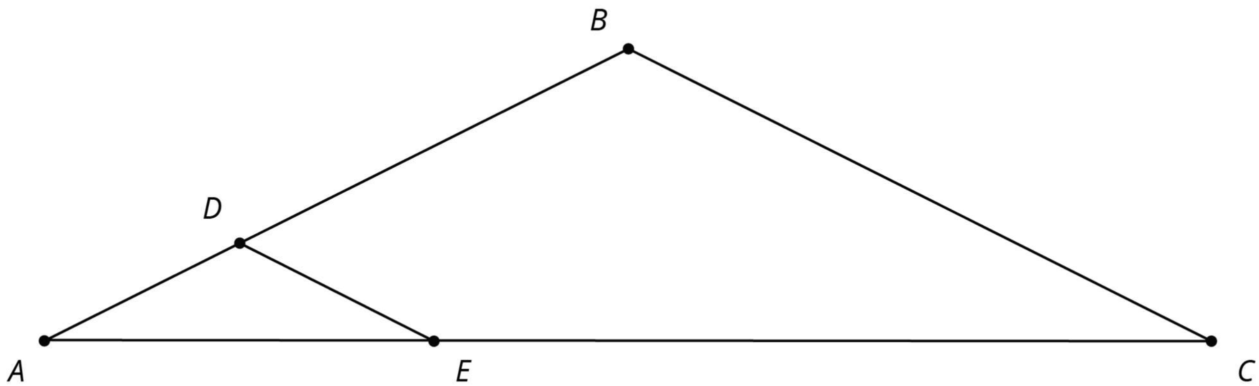
   a. Answers vary.

   b. The side lengths in each triangle should be equal, and the angles should all be 60°. The triangles are similar, because the angles are equal.

   c. Always. All equilateral triangles have the same angles, so they are all similar.

3. **Problem 3 Statement**

In the shape, line BC is parallel to line DE.



Explain why $\triangle ABC$ is similar to $\triangle ADE$.

Solution

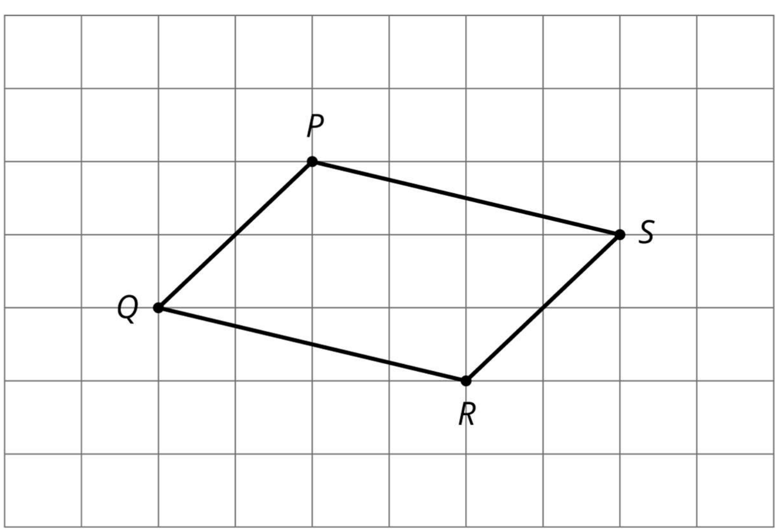
Answers vary.

Sample Solution 1: Triangles ABC and ADE share angle A. Line AC is a transversal for parallel lines BC and DE. Therefore, angles ADE and ABC are congruent. Since they share two congruent angles, triangles ABC and ADE are similar.

Sample Solution 2: An enlargement with centre A and appropriate scale factor will take triangle ABC to triangle ADE. The scale factor looks like it is about $\frac{1}{3}$.

4. **Problem 4 Statement**

The quadrilateral $PQRS$ in the diagram is a parallelogram. Let $P'Q'R'S'$ be the image of $PQRS$ after applying an enlargement centred at a point O (not shown) with scale factor 3.



Which of the following is true?

a. $P'Q' = PQ$

b. $P'Q' = 3PQ$

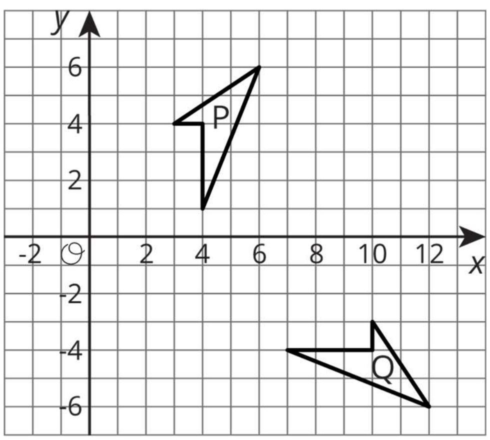
c. $PQ = 3P'Q'$

d. Cannot be determined from the information given

Solution B

5. **Problem 5 Statement**

Describe a sequence of transformations for which quadrilateral P is the image of quadrilateral Q.



Solution

Answers vary. Sample response: Translate Q 3 units left and 5 units up. Now, they share a point. Rotate using this point as the centre, 90 degrees anti-clockwise.

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