Lesson 12: Volume of prisms

Goals

- Determine the volume of a prism by counting how many unit cubes it takes to build one layer and then multiplying by the number of layers.
- Generalise (orally) the relationship between the volume of a prism, the area of its base, and its height.
- Identify whether a given shape is a prism, and if so, identify its base and height.

Learning Targets

- I can explain why the volume of a prism can be found by multiplying the area of the base and the height of the prism.

Lesson Narrative

Students have previously calculated the volume of cuboids. In this lesson, students learn that they can calculate the volume of any prism by multiplying the area of the base times the height of the prism. Students make sense of this formula by picturing the prism decomposed into identical layers 1 unit tall. These layers are composed of a number of cubic units equal to the number of square units in the area of the base. The height of the prism tells how many of these layers there are. Therefore, multiplying the number of cubic units in one layer times the number of layers gives the total number of cubic units in the prism, regardless of the shape of the base.

Given some three-dimensional shapes that are prisms and some that are not, students decide whether they can apply the formula $V = Bh$ to calculate the volume. If so, they identify the base and measure the height, before calculating the volume. Students also apply the formula $V = Bh$ to find the height of a prism given its volume and the area of its base.

Addressing

- Solve real-world and mathematical problems involving area, volume and surface area of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and prisms.

Instructional Routines

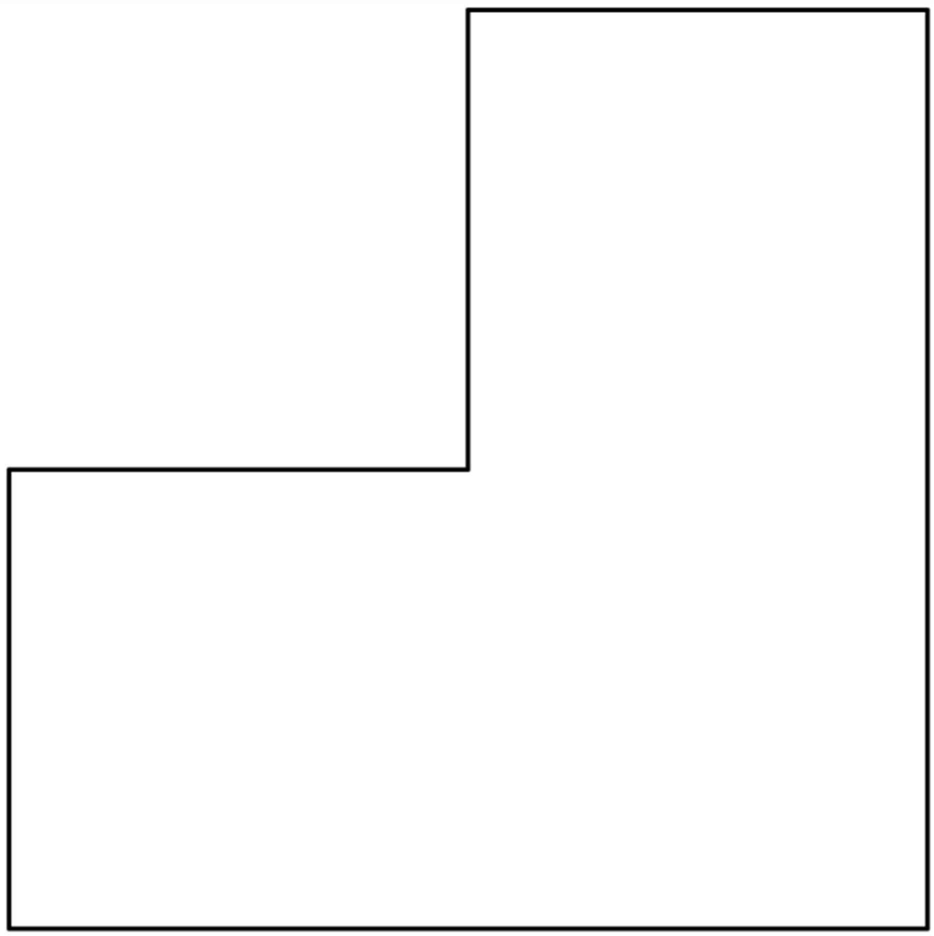
- Stronger and Clearer Each Time
- Clarify, Critique, Correct
- Discussion Supports

Required Materials

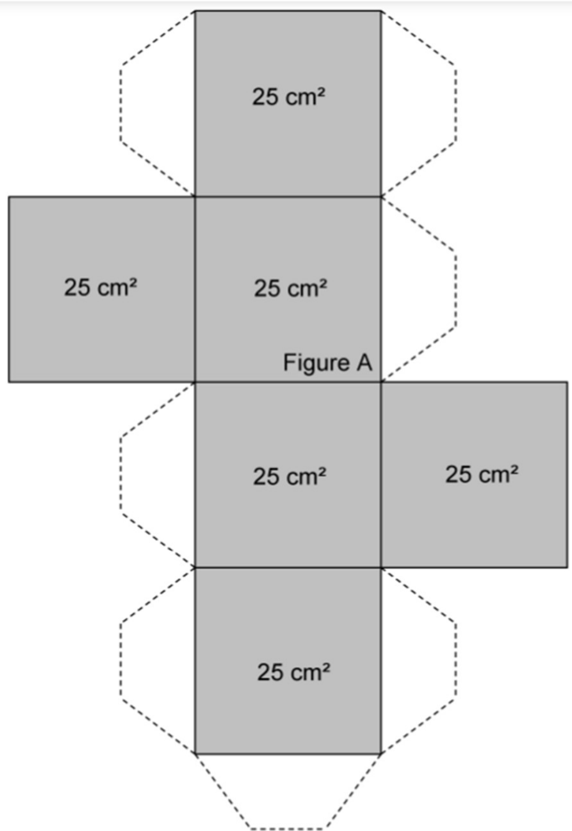
Copies of blackline master
Finding volume with cubes

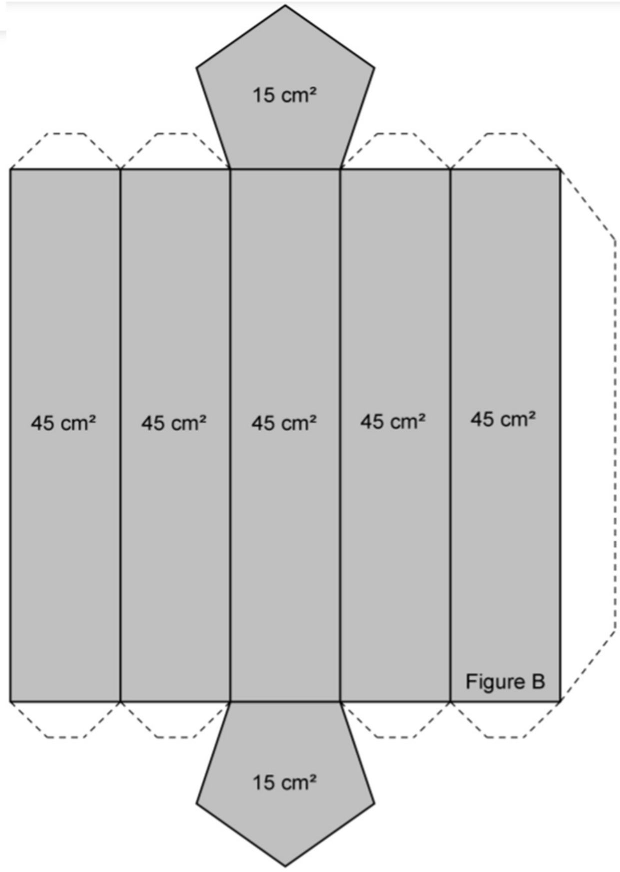
Check that the longer sides measure 12 cm and the shorter ones 6 cm. It may be better to task students with drawing the shape accurately on squared paper, rather than photo-copying them from the blackline master.

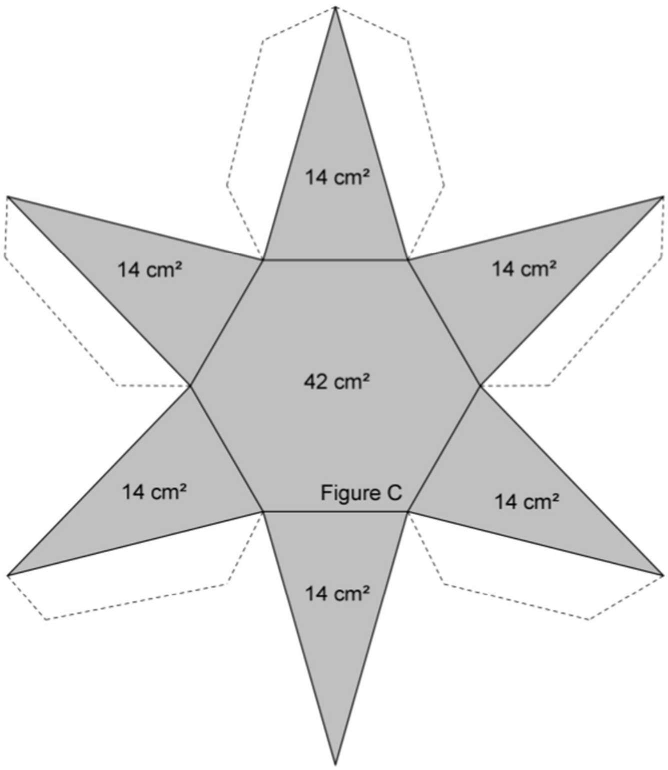
Use 2 cm cubes.

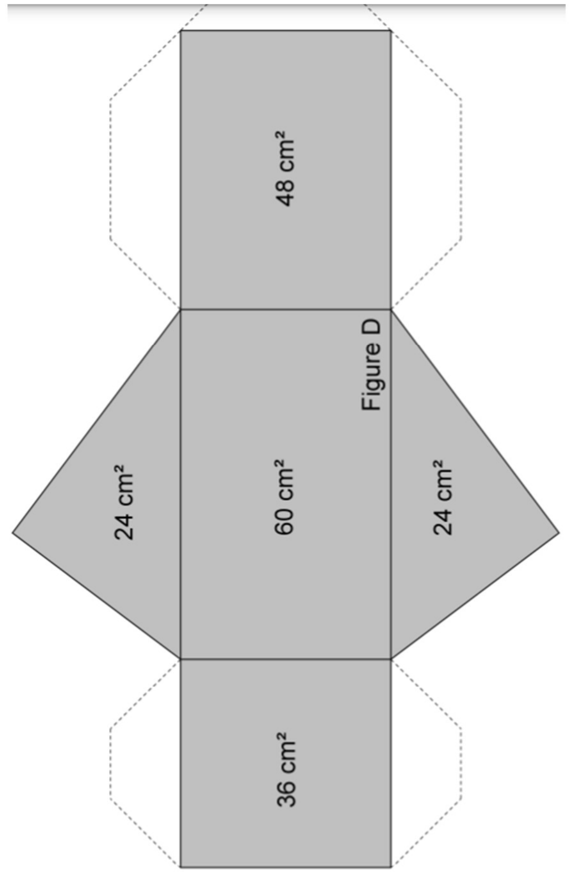


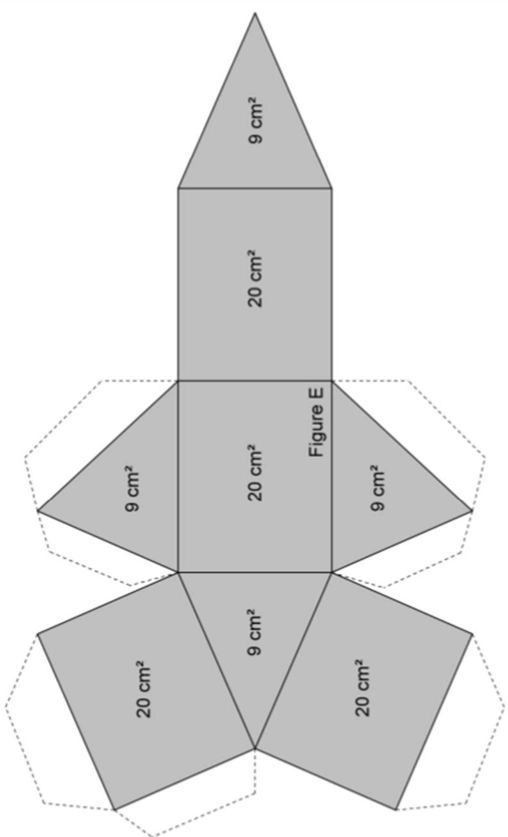
Can you find the volume?

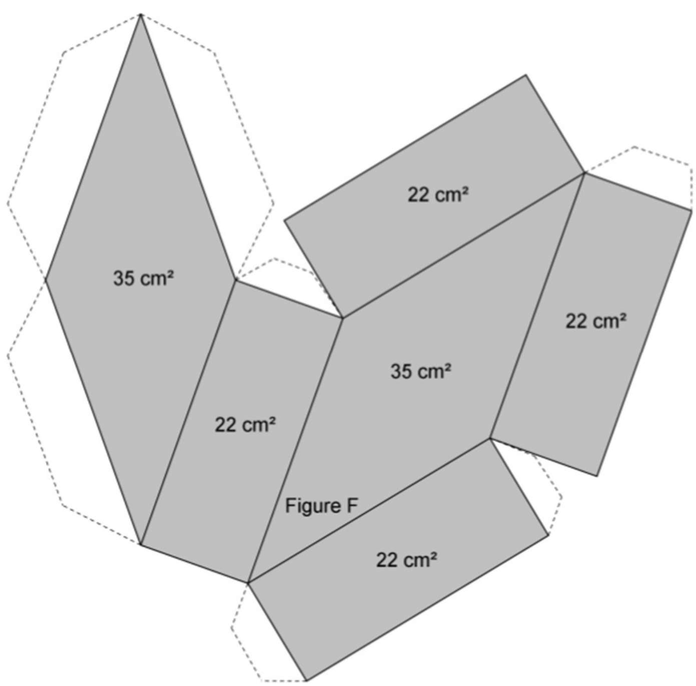

25 cm²
25 cm²
25 cm²
Figure A
25 cm²
25 cm²
25 cm²


15 cm²
45 cm²
45 cm²
45 cm²
45 cm²
45 cm²
Figure B
15 cm²


14 cm²
14 cm²
14 cm²
42 cm²
14 cm²
Figure C
14 cm²
14 cm²


48 cm²
Figure D
24 cm²
60 cm²
24 cm²
36 cm²


9 cm²
20 cm²
Figure E
9 cm²
20 cm²
9 cm²
9 cm²
20 cm²
20 cm²


22 cm²
35 cm²
22 cm²
35 cm²
22 cm²
Figure F
22 cm²

Pre-assembled polyhedra
Rulers marked with centimetres
Multi-link cubes

Required Preparation

You will need the Finding Volume with Cubes blackline master for this lesson. Prepare 1 copy for every 3 students.

Print, cut, and assemble the nets from the Can You Find the Volume? blackline master. Card is recommended. Make sure to print the blackline master at a suitable scale so the dimensions are accurate. Prepare 1 polyhedron for every 2 students (1 copy of the entire file for every 12-18 students).

It may work better for students to reason about the volume from the nets rather than try to photo-copy the nets at the right scale.

Make sure students have access to multi-link cubes and rulers marked in centimetres.

Student Learning Goals

Let's look at volumes of prisms.

12.1 Three Prisms with the Same Volume

Warm Up: 5 minutes

The purpose of this warm-up is to encourage students to think about possible heights of prisms with the same height and volume based on the area of a base. This is a review of previous work students have done with volume in which they found the volume of a cuboid by multiplying the area of a base and height. The ideas in this warm-up are revisited later in this lesson, so it is important students can clearly explain how they ordered their prisms based on them having the same volume and how they found the height of the prism with base C.

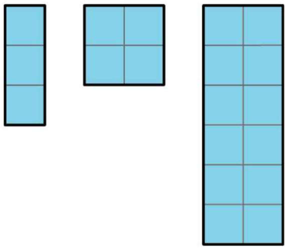
Launch

Arrange students in groups of 2. Give students 1 minute of quiet work time followed by time to discuss their explanations with a partner. Follow with a whole-class discussion.

Student Task Statement

Rectangles A, B, and C represent bases of three prisms.

A B C



1. If each prism has the same height, which one will have the greatest **volume**, and which will have the least? Explain your reasoning.

2. If each prism has the same volume, which one will have the tallest height, and which will have the shortest? Explain your reasoning.

Student Response

1. Prism C will have the greatest volume and prism A will have the least. Since the volume is the area of the base multiplied by the height the base with the greatest area will have the greatest volume, even if all the heights are the same.

2. Prism A will have the tallest height and prism C will have the shortest. Since the volume is the area of the base multiplied by the height, the base with the smallest area will have the tallest height and the base with the greatest area will have the shortest height.

Activity Synthesis

Select students to share the prism they found to have the greatest and least volume and the tallest and shortest height. Record and display their responses for all to see. Poll the class if they agree or disagree. If students all agree, ask a few students to share their reasoning. If they do not agree, ask students to share their reasoning until they reach an agreement.

If there is time, display this question for all to see: "If each prism has the same volume and the prism associated with base B has a height of 6 units, what is the height of the prism associated with base C?"

Have students share the volume of the prism with base C and their reasoning. Record and display the responses for all to see.

12.2 Finding Volume with Cubes

10 minutes (there is a digital version of this activity)

Previously, students worked with the volume of cuboids. In this activity, students extend their understanding to see that even when the base is not a rectangle, they can still calculate the volume of a prism by multiplying the area of the base times the height of the prism.

Students use multi-link cubes to build a prism with a base that matches the shape from the blackline master. Each group needs 30 multi-link cubes for the first question and a total of 60 multi-link cubes for the second question. If there are not enough multi-link cubes, two groups of 3 students may combine together after answering the first question to form one group of 6 students.

Instructional Routines

- Stronger and Clearer Each Time

Launch

Arrange students in groups of 3. Distribute copies of the blackline master, one half-page to each group, and 30–60 multi-link cubes to each group. Give students 2–3 minutes of quiet work time followed by a whole-class discussion.

For students using digital materials: depending on the needs of your class, either demonstrate how to build shapes using the applet, or instruct students to read and follow the instructions for working the applet.

Student Task Statement

Your teacher will give you a paper with a shape on it and some multi-link cubes.

1. Using the face of a multi-link cube as your area unit, what is the area of the shape? Explain or show your reasoning.
2. Use multi-link cubes to build the shape from the paper. Add another layer of cubes on top of the shape you have built. Describe this three-dimensional object.
3. What is the volume of your object? Explain your reasoning.
4. Right now, your object has a height of 2. What would the volume be:
   a. if it had a height of 5?
   b. if it had a height of 8.5?

Student Response

1. 27 units^2. Possible strategy: $3 \times 3 + 3 \times 6 = 27$.
2. Answers vary. Sample response: It is a prism with a base in the shape of an "L."
3. 54 units^3, because $2 \times 27 = 54$.
4.
   a. 135 units^3, because it would take 5 layers and $5 \times 27 = 135$.
   b. 229.5 units^3, because $8.5 \times 27 = 229.5$.

Activity Synthesis

Select students to share their reasoning.

Consider asking some of the following questions:

- "How do you know this shape is a prism?" (Cross sections parallel to the base are identical copies.)
- "What is the area of the base of this shape?" (It is the number of cubes in one layer of the prism.)
- "How do you calculate the total number of cubes to make the prism?" (Multiply the number of cubes in one layer by the number of layers.)
- "What is the volume of this prism?" (The volume is the same as calculating the number of cubes to make the prism.)
- "If you find the area of the base, how do you use that information to calculate the volume of the prism?" (Multiply the area of the base by the height of the prism.)
- "How would the volume of the prism change if we changed the shape of the base but still used 27 cubes to build it?" (The volume would not change.)

If not mentioned by students, explain that calculating the total number of cubes to make the prism is the same as calculating the volume of the prism. We can find the area of the base of the prism and multiply that by the number of layers in the prism which is the same as the height of the prism. The height of the prism is measured in units, the area of the base is measured in units^2 and the volume of the prism is measured in units^3.

Writing, Listening, Conversing: Stronger and Clearer Each Time. Use this routine to help students improve their writing, by providing them with multiple opportunities to clarify their explanations through conversation. Invite students to draft an initial response to the question: "How do you know if a three-dimensional shape is a prism?" Give students time to meet with 2–3 partners, to share and get feedback on their responses. Provide students with prompts for feedback that will help their partner strengthen their ideas and clarify their language (e.g., "What are some properties of prisms?", "Can you give an example and non-example?", etc.). Students can borrow ideas and language from each partner to refine and clarify their explanation. This will also help students to build and describe the three-dimensional shapes used in this lesson and the unit.
Design Principle(s): Optimise output (for explanation); Maximise meta-awareness

12.3 Can You Find the Volume?

15 minutes (there is a digital version of this activity)

The purpose of this activity is for students to get hands-on experience with polyhedra, recognising whether a shape is a prism and if so, determining which face is the base of the prism. Once students determine which face is the base, they use a ruler marked in centimetres to measure the height of the prism. The area of each face is labelled on the shape so that students do not get bogged down with calculating the base area and can focus on using the area to find the volume.

Instead of creating enough sets of polyhedra for every group to have one of every shape at the same time, consider having the students pass the shapes from one group to the next or rotate around to different stations so that fewer sets of shapes have to be constructed.

Instructional Routines

- Clarify, Critique, Correct

Launch

Arrange students in groups of 3–4. Distribute the three-dimensional shapes that were already assembled from the nets in the blackline master and rulers marked in centimetres. Give students 1–2 minutes of quiet work time with the polyhedra given to their group, have groups exchange objects so that each group gets to examine each shape. Follow with a whole-class discussion.

Students using the digital version have an applet with the six polyhedra in 3D. Students can rotate the view using the tool marked by two intersecting, curved arrows. Note that each polyhedron has only one label per unique face. If no other measurements are shown, the faces are congruent. Students can use the distance tool, marked with the "cm," to find the height or length of any segment. Troubleshooting tip: the cursor must be on the 3D Graphics window for the full toolbar to appear.

Anticipated Misconceptions

Some students may say that shape A is not a prism because it is a cube. Ask them whether the cross sections would be identical if you made various cuts parallel to one side. Explain that a cube is a special type of square prism where the height of the prism matches the side lengths of the base. Consider making a comparison to the fact that a square is a special type of rectangle.

Student Task Statement

Your teacher will give you a set of three-dimensional shapes.

1. For each shape, determine whether the shape is a prism.
2. For each prism:
    a. Find the area of the base of the prism.
    b. Find the height of the prism.
    c. Calculate the volume of the prism.

	Is it a prism?	area of prism base (cm²)	height (cm)	volume (cm³)
shape A				
shape B				
shape C				
shape D				
shape E				
shape F				

Student Response

1. Yes, it is a square prism. Any cross section parallel to any of the faces will be the same size square, so it is a prism.
2. Yes, it is a pentagonal prism. Any cross section parallel to the pentagon base will be the same size pentagon, so it is a prism.
3. No, it is a hexagonal pyramid. Cross sections parallel to the hexagon will be smaller hexagons.
4. Yes, it is a triangular prism. Any cross section parallel to the triangle base will be the same size triangle, so it is a prism.
5. No. Cross sections are not the same all the way through the object.
6. Yes, it is a rhombus-shaped prism. Any cross section parallel to the rhombus base will be the same size rhombus, so it is a prism.

	Is it a prism?	area of prism base (cm²)	height (cm)	volume (cm³)
shape A	Yes	25	5	125
shape B	Yes	15	15	225
shape C	No			
shape D	Yes	24	6	144
shape E	No			
shape F	Yes	35	3	105

Are You Ready for More?

Imagine a large, solid cube made out of 64 white multi-link cubes. Someone spray paints all 6 faces of the large cube blue. After the paint dries, they disassemble the large cube into a pile of 64 multi-link cubes.

1. How many of those 64 multi-link cubes have exactly 2 faces that are blue?
2. What are the other possible numbers of blue faces the cubes can have? How many of each are there?
3. Try this problem again with some larger-sized cubes that use more than 64 multi-link cubes to build. What patterns do you notice?

Student Response

1. 2 faces: 24 cubes
2. 3 faces: 8 cubes, 1 face: 24 cubes, 0 faces: 8 cubes.
3. Answers vary. Sample response: There are always 8 cubes with 3 blue faces, there is a cube on the inside with side length 2 units less than the large cube that doesn't get painted at all. (Students can make similar observations and perhaps find formulae for the number of cubes with 1 or 2 blue faces.)

Activity Synthesis

Poll the class for answers to the first column of the table. Make sure the class agrees about answers before proceeding. Select students to provide their answers to each part of the table and an explanation. Display answers on the table for all to see. Ask students:

- "What is different about the structure of non-prisms in comparison to prisms?" (The prisms have multiple layers of the same base, where the non-prism does not have that.)
- "Why can't you use 'area of the base times the height' to calculate the volume of the shapes that were not prisms?" (Because the non-prism isn't made up of multiple layers of the same base.)

Writing, Speaking: Clarify, Critique, Correct. Use this routine to support student reasoning about whether a shape is a prism or not. Before students share their answers, display an incorrect answer showing conceptual (or common) errors. For example, "Shape A is not a prism because it is a cube." or "Shape B is a prism because its base is a rectangle." Ask students to work with a partner to identify the errors and critique the reasoning shown. Provide questions for discussion such as, "What is unclear?" or "Are there any errors?" Give students 2–3 minutes of quiet time to revise the original statement, by drawing on the conversations with their partners. Improved statements should draw on properties of prisms, such as: cross sections are parallel to the base and are identical copies. This will facilitate students' understanding of prisms and their ability to evaluate, and improve on, the written mathematical arguments of others.
Design Principle(s): Optimise output (for explanation); Maximise meta-awareness

12.4 What's the Prism's Height?

Optional: 10 minutes

The purpose of this activity is for students to work backwards from the volume to the height of a prism. Students see that for two prisms to have the same volume, the one with the smaller base has the taller height and the one with the larger base has the shorter height. The grid helps students find the area of the base so they can focus their attention on what it means to have a prism made out of stacks of layers of the same base.

Instructional Routines

- Discussion Supports

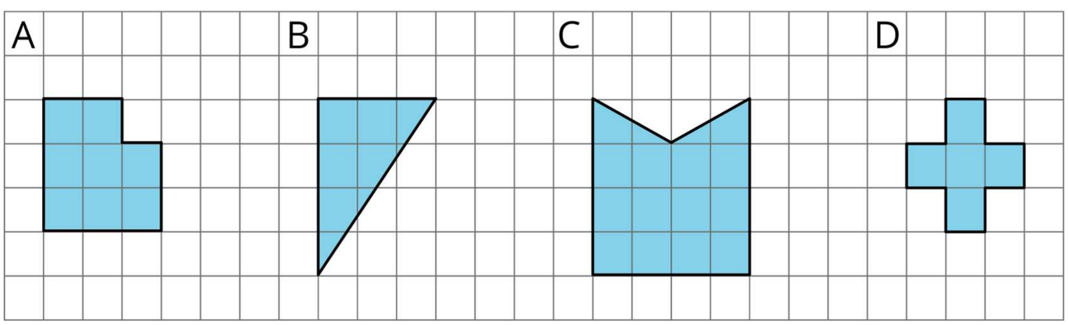
Launch

Arrange students in groups of 2. Give students 5 minutes of quiet work time followed by time to discuss their thinking with a partner. Follow with a whole-class discussion.

Representation: Develop Language and Symbols. Use virtual or concrete manipulatives to connect symbols to concrete objects or values. Provide students with geometric solids or multi-link cubes and a printed copy of the student task statement to draw on or annotate.
Supports accessibility for: Conceptual processing

Student Task Statement

There are 4 different prisms that all have the same volume. Here is what the base of each prism looks like.



1. Order the prisms from shortest to tallest. Explain your reasoning.
2. If the volume of each prism is 60 units^3, what would be the height of each prism?
3. For a volume other than 60 units^3, what could be the height of each prism?
4. Discuss your thinking with your partner. If you disagree, work to reach an agreement.

Student Response

1. D, B, A, C, since a larger base area means the height of the prism must be shorter to maintain the same volume.

2. If the volume of all 4 prisms is 60 units^3, prism A is 7.5 units tall, prism B is 10 units tall, prism C is 5 units tall, and prism D is 12 units tall.

3. Answers vary. Sample responses:

   - The volume of all 4 prisms is 48 units^3 if prism A is 6 units tall, B is 8 units tall, C is 4 units tall, and D is 2.4 units tall.

   - The volume of all 4 prisms is 120 units^3 if prism A is 15 units tall, B is 20 units tall, C is 10 units tall, and D is 6 units tall.

Activity Synthesis

Select students to share their responses and reasoning. If not brought up in student's explanation, explain that for the last problem, there is more than one possible correct answer. The smallest possible volume that involves all whole number side lengths is 120 units^3, but there is nothing in the problem that requires all the heights to be whole numbers.

To highlight connections to calculating volume, ask:

- "How do you calculate the volume of a prism?" (Display the equation for all to see: $V = B \times h$.)
- "Since $V = B \times h$, how could we find the area of the base if we knew the volume and height of the prism?" (Display the equation for all to see: $B = V \div h$.)
- "If we keep the volume the same, what happens to the height when we increase the area of the base?" (It decreases.)
- "If we keep the height the same, what happens to the volume when we increase the area of the base?" (It increases.)

Speaking: Discussion Supports. Use this routine to support whole-class discussion. For each response that is shared, ask students to restate and/or revoice what they heard using precise mathematical language. Consider providing students time to restate what they hear to a partner, before selecting one or two students to share with the class. Ask the original speaker if their peer was accurately able to restate their thinking. Call students' attention to any words or phrases that helped to clarify the original statement. This will provide more students with an opportunity to produce language as they interpret the reasoning of others.
Design Principle(s): Support sense-making

Lesson Synthesis

- "What information do we need to calculate the volume of a prism?" (Area of the base and the height)
- "Explain how you could use layers to find the volume of a prism." (If you look at the first layer of a prism, you can find how many cubes are in that layer by finding the area

of the base. Once you find the number of cubes on the first layer, you multiply that by the number of layers it takes to stack up to the height of the prism.)

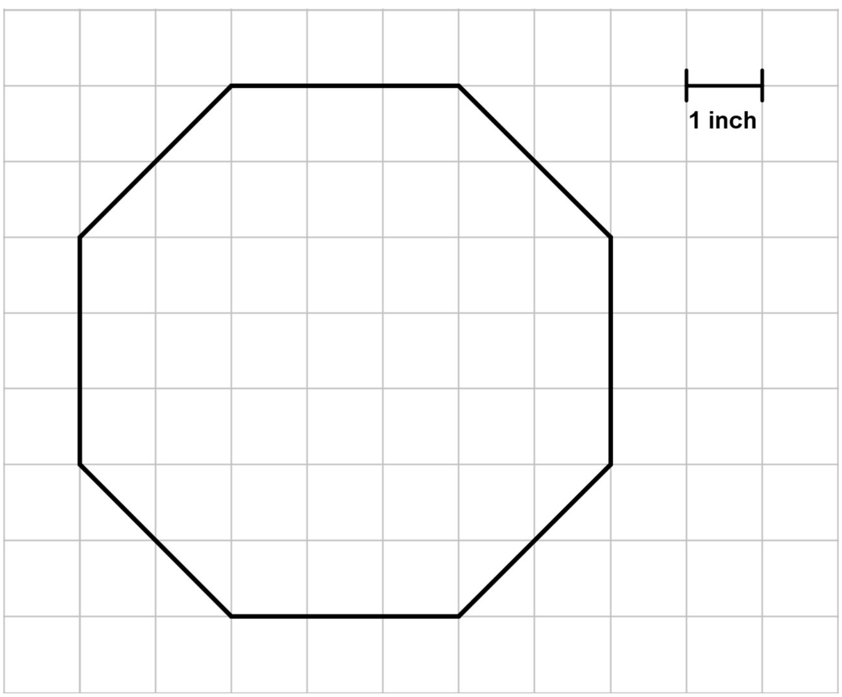
- "Two prisms have the same base area and height, but different base shapes. Which prism has a greater volume? Explain." (The two prisms have the same volume. The shape of the base does not matter if it is a prism, only the base area matters.)
- "Two clay prisms use the same amount of clay to make them, but the first has a larger height than the second. Which prism has a larger base area?" (The second prism will have a larger base area since a shorter height means a larger base area if the volume is held constant. Imagine squashing the first one down in a nice way to make a shorter, fatter version.)

12.5 Octagonal Box

Cool Down: 5 minutes

Student Task Statement

A box is shaped like an octagonal prism. Here is what the base of the prism looks like.



For each question, make sure to include the unit with your answer and explain or show your reasoning.

1. If the height of the box is 7 inches, what is the volume of the box?
2. If the volume of the box is 123 in^3, what is the height of the box?

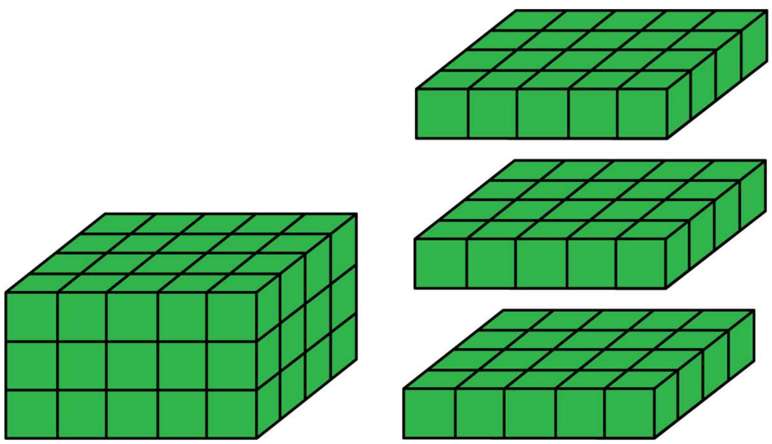
Student Response

1. 287 in^3, because the base has an area of 41 in^2 and $41 \times 7 = 287$.

2. 3 in, because $41 \times 3 = 123$.

Student Lesson Summary

Any cross section of a prism that is parallel to the base will be identical to the base. This means we can slice prisms up to help find their volume. For example, if we have a cuboid that is 3 units tall and has a base that is 4 units by 5 units, we can think of this as 3 layers, where each layer has 4×5 cubic units.

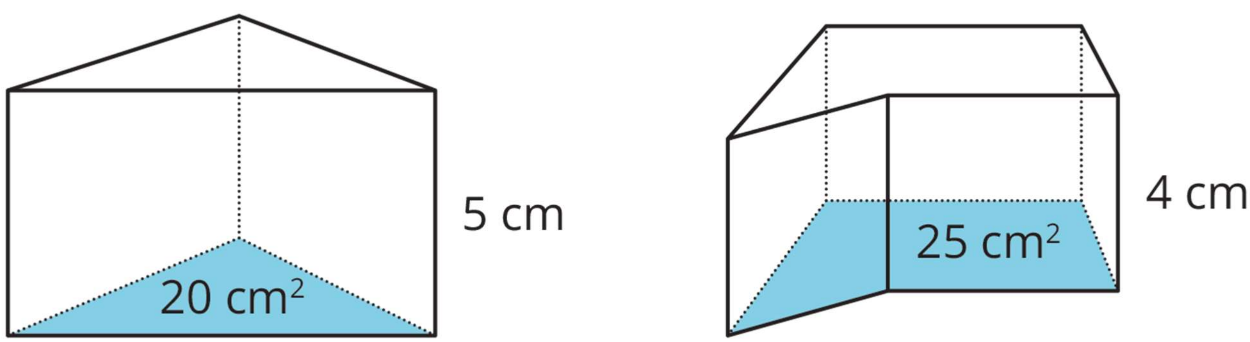


That means the volume of the original cuboid is $3(4 \times 5)$ cubic units.

This works with any prism! If we have a prism with height 3 cm that has a base of area 20 cm^2, then the volume is $3 \times 20 \text{ cm}^3$ regardless of the shape of the base. In general, the volume of a prism with height h and area B is

$V = B \times h$

For example, these two prisms both have a volume of 100 cm^3.



Glossary

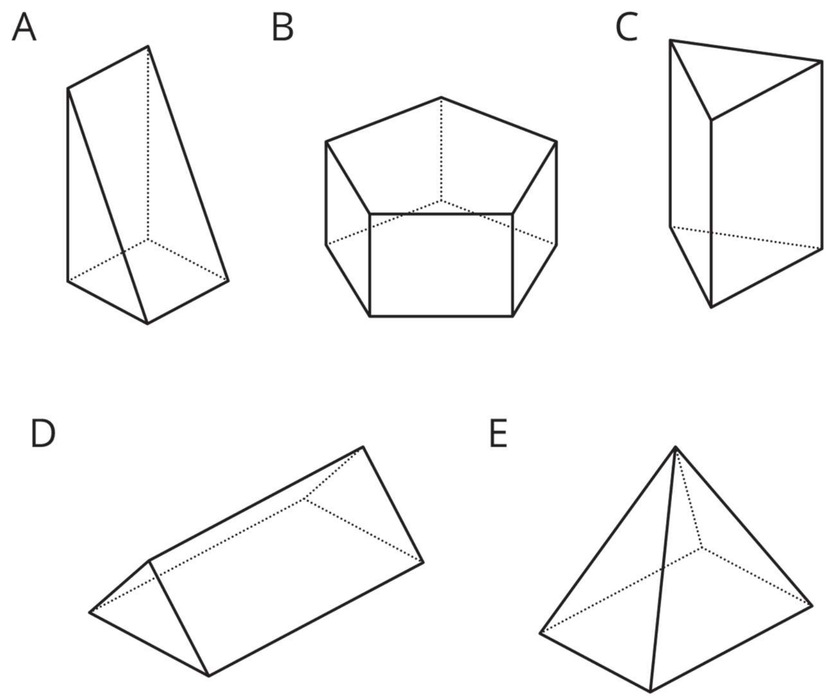
- volume

Lesson 12 Practice Problems

1. **Problem 1 Statement**

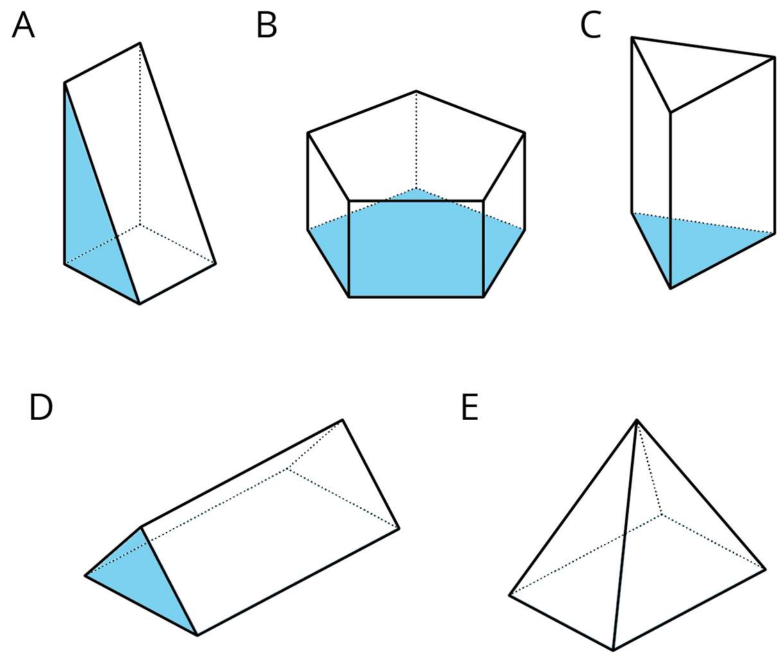
   a. Select **all** the prisms.

   b. For each prism, shade one of its bases.



Solution

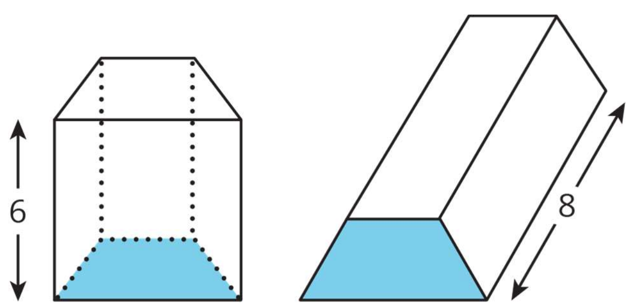
a. A, B C, D



b.

2. **Problem 2 Statement**

The volume of both of these trapezium-based prisms is 24 cubic units. Their heights are 6 and 8 units, as labelled. What is the area of a trapezium-shaped base of each prism?



Solution

The prism with a height of 6 units has a base with area 4 square units, because $24 \div 6 = 4$. The prism with a height of 8 units has a base with area 3 square units, because $24 \div 8 = 3$.

3. **Problem 3 Statement**

Two angles are complementary. One measures 19 degrees. What is the size of the other?

Solution

71 degrees

4. **Problem 4 Statement**

Two angles are supplementary. One is twice as large as the other. Find the two angles.

Solution

60° and 120°

5. **Problem 5 Statement**

Match each expression in the first list with an equivalent expression from the second list.

A. $7(x+2)-x+3$

B. $6x+3+4x+5$

C. $\frac{-2}{5}x-7+\frac{3}{5}x-3$

D. $8x-5+4-9$

E. $24x+36$

1. $\frac{1}{5}x-10$

2. $6x+17$

3. $2(5x+4)$

4. $12(2x+3)$

5. $8x+(-5)+4+(-9)$

Solution

- A: 2
- B: 3
- C: 1
- D: 5
- E: 4

6. **Problem 6 Statement**

Clare paid 50% more for her notebook than Priya paid for hers. Priya paid x for her notebook and Clare paid y pounds for hers. Write an equation that represents the relationship between y and x.

Solution

$y = 1.5x$ (or equivalent)