# Lesson 4: Tables, equations, and graphs of functions

## Goals

- Determine whether a graph represents a function, and explain (orally) the reasoning.
- Identify the graph of an equation that represents a function, and explain (orally and in writing) the reasoning.
- Interpret (orally and in writing) points on a graph, including a graph of a function and a graph that does not represent a function.

## Learning Targets

- I can identify graphs that do, and do not, represent functions.
- I can use a graph of a function to find the output for a given input and to find the input(s) for a given output.

## Lesson Narrative

In this lesson, students work with graphs and tables that represent functions in addition to the equations and descriptions used previously. They learn the conventions of graphing the independent variable (input) on the horizontal axis and the dependent variable (output) on the vertical axis and that each coordinate point represents an input-output pair of the function.

By matching contexts and graphs and reading information about functions from graphs and tables, students become familiar with the different representations and draw connections between them.

### Addressing

Understand that a function is a rule that assigns to each input exactly one output. The graph of a function is the set of ordered pairs consisting of an input and the corresponding output. Function notation is not required.

- Interpret the equation $y = mx + c$ as defining a linear function, whose graph is a straight line; give examples of functions that are not linear. For example, the function $A = s^2$ giving the area of a square as a function of its side length is not linear because its graph contains the points (1,1), (2,4) and (3,9), which are not on a straight line.

### Instructional Routines

- Stronger and Clearer Each Time
- Co-Craft Questions
- Notice and Wonder

### Student Learning Goals

Let's connect equations and graphs of functions.

## 4.1 Notice and Wonder: Doubling Back

**Warm Up: 5 minutes**

The purpose of this warm-up is to familiarise students with one of the central graphical representations they will be working with in the lesson. As students notice and wonder, they have the opportunity to reason abstractly and quantitatively if they consider the situation the graph represents.

### Instructional Routines

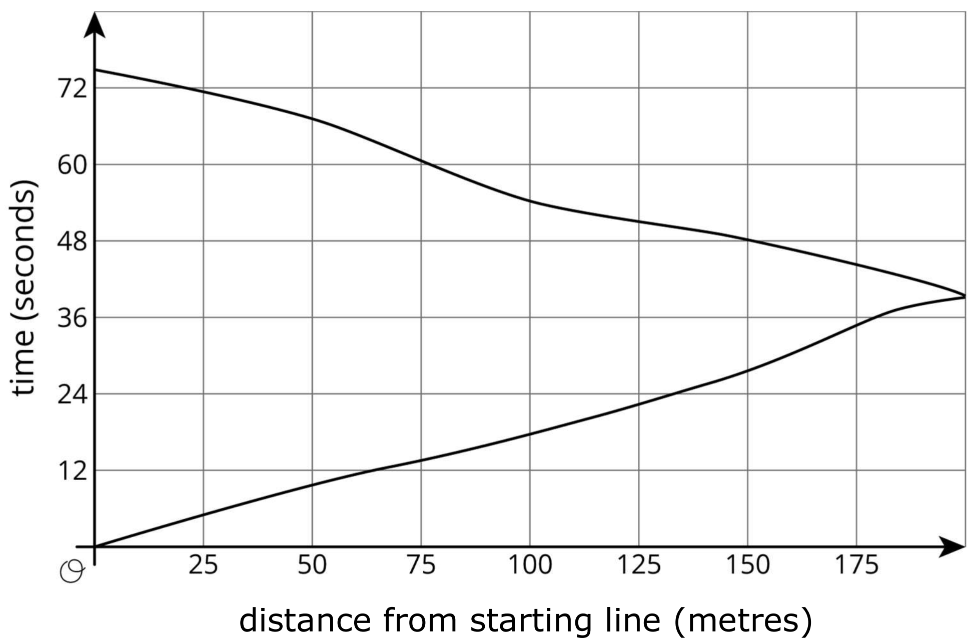
- Notice and Wonder

### Launch

Tell students they will look at a graph, and their job is to think of at least one thing they notice and at least one thing they wonder about the picture. Display the graph for all to see and give 1 minute of quiet think time. Ask students to give a signal when they have noticed or wondered about something.

### Student Task Statement

What do you notice? What do you wonder?



## Student Response

Students may notice things such as:

- The graph looks like a sideways V.
- The graph doubles back on itself.
- The graph is made of pieces that are almost, but not quite, straight.
- Distances go between 0 and 200 metres.
- Times go from 0 to a little more than 72.
- The horizontal axis tells us the distance from the starting line in metres.
- The vertical axis tells us the time in seconds.
- The person gets farther from the starting line and then comes back.
- The person was 75 metres from the starting line at about 14 seconds and 60 seconds.
- The person got back to the starting line in about 75 seconds.
- The furthest distance the person got from the start line was 200 metres.

Students may wonder things such as:

- Is this about a person, or more than one person?
- Why does the graph double back?
- Who is the graph about?
- Why are they coming back?
- Are they running or walking?
- Did they go out and come back in the exact same time?
- What is the title of this graph?

## Activity Synthesis

Ask students to share things they noticed and wondered about the graph. Record and display these ideas for all to see. For each of the things students notice and wonder, ask them to reference the graph in their explanation. If no one notices that at every distance from the starting line, there are two associated times (except at 200 metres), bring that to their attention.

If there is time, ask students to make some estimations or guesses for the wonders that refer to the information in the graph. For example, if someone wonders what the title of the graph may be, ask them to create a title that would make sense for this context.

## 4.2 Equations and Graphs of Functions

**15 minutes**

The purpose of this activity is for students to connect different function representations and learn the conventions used to label a graph of a function. Students first match function contexts and equations to graphs. They next label the axes and calculate input-output pairs for each function. The focus of the discussion should be on what quantities students used to label the axes and recognising the placement of the independent or dependent variables on the axes.

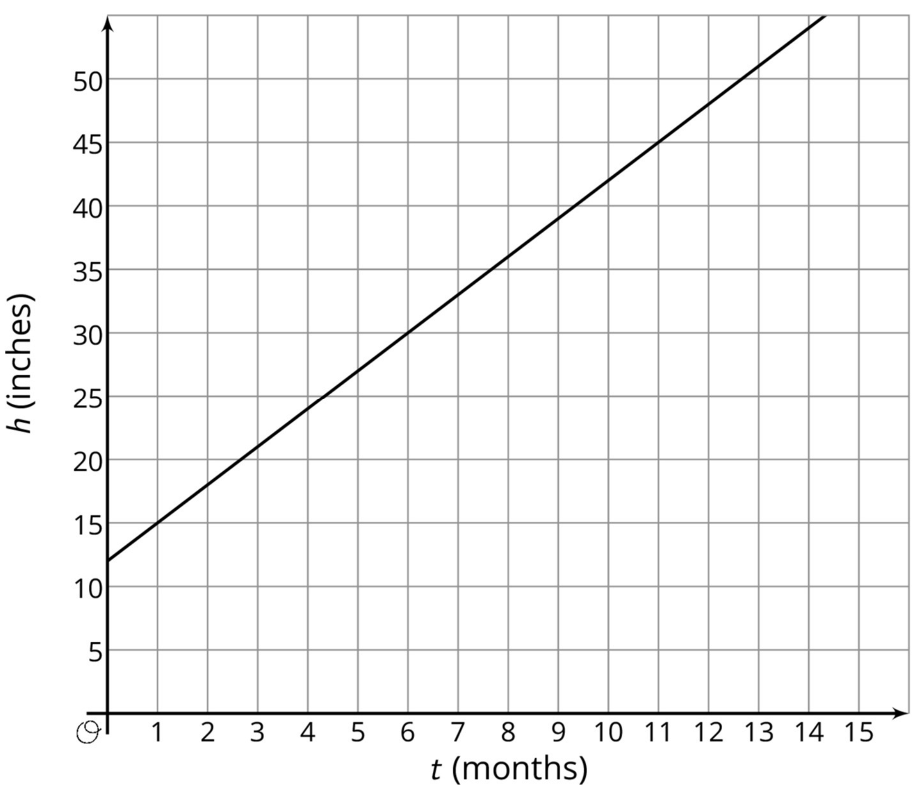
Monitor for students who recognise that there is one graph that is not linear and match that graph with the equation that is not linear.

### Instructional Routines

- Stronger and Clearer Each Time

### Launch

Arrange students in groups of 2. Display the graph for all to see. Ask students to consider what the graph might represent.



After brief quiet think time, select 1–2 students to share their ideas. (For example, something starts at 12 inches and grows 15 inches for every 5 months that pass.)

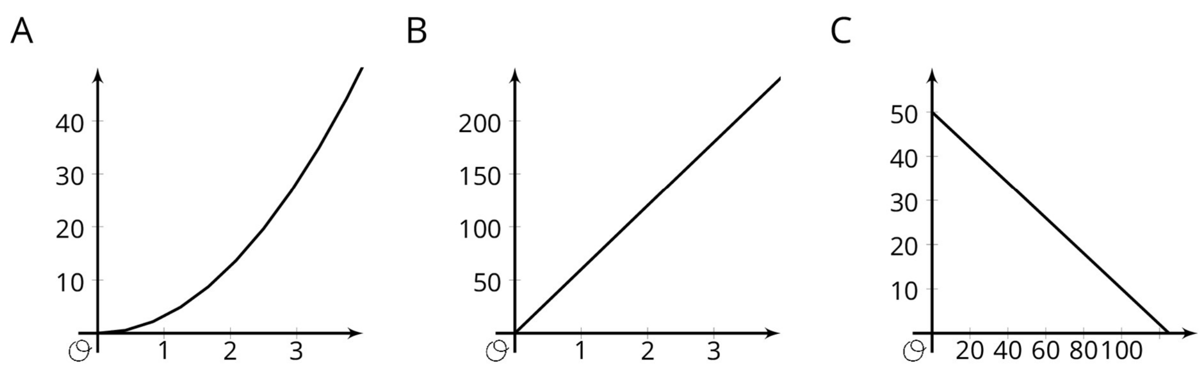
Remind students that axes labels help us determine what quantities are represented and should always be included. Let them know that in this activity the graphs of three functions have been started, but the labels are missing and part of their work is to figure out what those labels are meant to be.

Give students 3–5 minutes of quiet work time and then time to share responses with their partner. Encourage students to compare their explanations for the last three problems and resolve any differences. Follow with a whole-class discussion.

*Representation: Internalise Comprehension.* Demonstrate and encourage students to use colour coding and annotations to highlight connections between representations in a problem. For example ask students to use the same colour to denote inputs and another colour for outputs. Students can use the colours when calculating values, labelling axes, and plotting points in the graphs.
*Supports accessibility for: Visual-spatial processing Writing, Speaking: Stronger and Clearer Each Time.* Use this routine to give students a structured opportunity to revise and refine their explanation of how they matched the equations and graphs in the first problem. Ask each student to meet with 2–3 other partners in a row for feedback. Provide students with prompts for feedback that will help teams strengthen their ideas and clarify their language (e.g., "What did you do first?", "How did you verify your match?", etc.). Students can borrow ideas and language from each partner to strengthen their final version.
*Design Principle(s): Optimise output (for explanation)*

### Student Task Statement

The graphs of three functions are shown.




1. Match one of these equations to each of the graphs.

    a. $d = 60t$, where $d$ is the distance in miles that you would travel in $t$ hours if you drove at 60 miles per hour.

b. $q = 50 - 0.4d$, where $q$ is the number of quarters, and $d$ is the number of dimes, in a pile of coins worth \$12.50. A quarter is worth 25 cents and a dime is worth 10 cents.

c. $A = \pi r^2$, where $A$ is the area in square centimetres of a circle with radius $r$ centimetres.

2. Label each of the axes with the independent and dependent variables and the quantities they represent.

3. For each function: What is the output when the input is 1? What does this tell you about the situation? Label the corresponding point on the graph.

4. Find two more input-output pairs. What do they tell you about the situation? Label the corresponding points on the graph.

### Student Response

1.

a. B. This graph represents a proportional relationship with a constant positive slope.

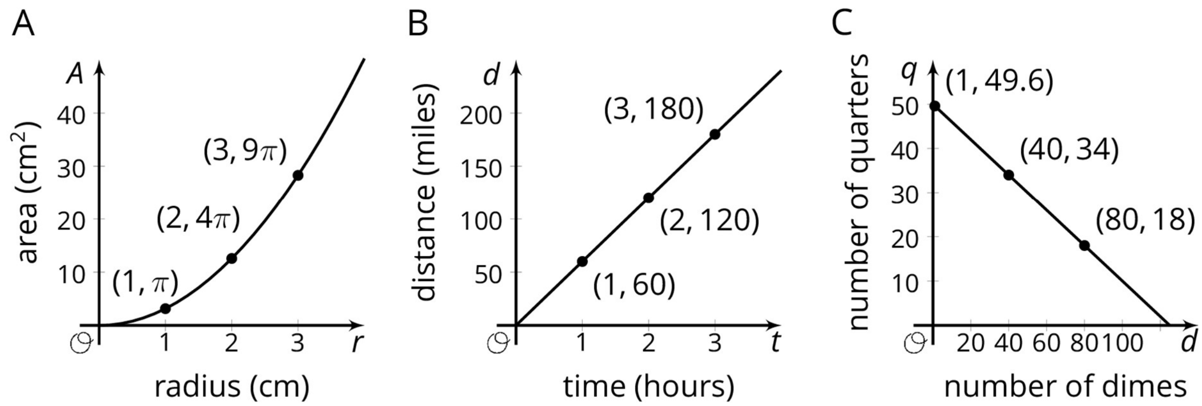
b. C. This is the only one of the graphs, and the only equation, with a negative gradient and vertical intercept of 50.

c. A. This is the only non-linear relationship.

2. See answer to part 4.

3. In graph A, we have point $(1, \pi)$, representing the fact that a circle of radius 1 cm has area $\pi$ $cm^2$.
In graph B, we have point (1,60), representing the fact that after travelling for 1 hour at 60 miles per hour, you would travel 60 miles.
In graph C, we have point (1,49.6). This does not have a concrete interpretation in terms of the context, as it says that if you had only one dime, you would have 49.6 quarters.
See answer to part 4 for the graphs.

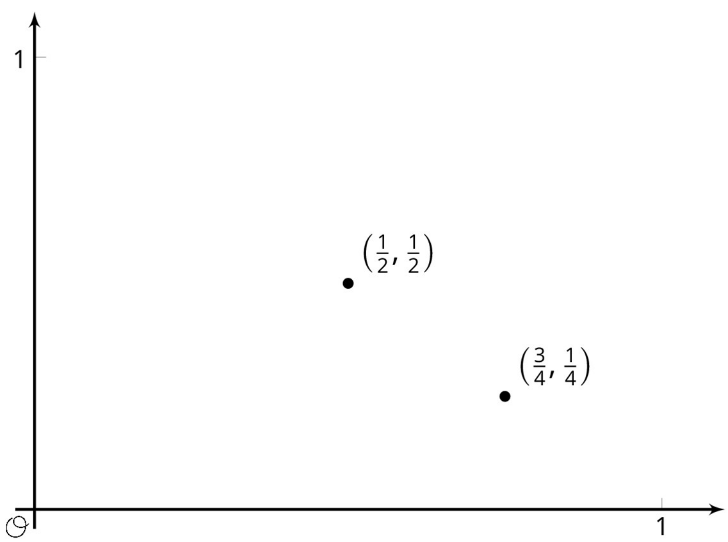
4. In graph A, we mark the points $(2,4\pi)$ and $(3,9\pi)$, representing the fact that circles of radius 2 cm and 3 cm have respective areas $4\pi$ $cm^2$ and $9\pi$ $cm^2$.
In graph B, we mark the points (2,120) and (3,180), representing the fact that after travelling for 2 and 3 hours at 60 miles per hour, you would respectively travel 120 and 180 miles.
In graph C, we mark the points (40,34) and (80,18), representing the fact that if there were 40 or 80 dimes in the pile, there would have to be 34 and 18 quarters, respectively.




## Are You Ready for More?

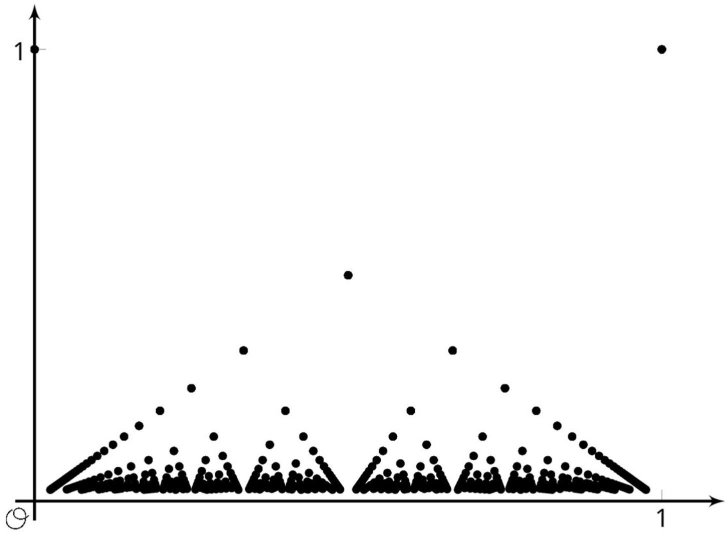
A function inputs fractions $\frac{a}{b}$ between 0 and 1 where $a$ and $b$ have no common factors, and outputs the fraction $\frac{1}{b}$. For example, given the input $\frac{3}{4}$ the function outputs $\frac{1}{4}$, and to the input $\frac{1}{2}$ the function outputs $\frac{1}{2}$. These two input-output pairs are shown on the graph.

Plot at least 10 more points on the graph of this function. Are most points on the graph above or below a height of 0.3? Of height 0.01?




## Student Response

This is a very complicated graph! Here is a computer-generated plot of several hundred inputs with denominators $b < 50$. Only very few inputs have height above 0.3. The only ones above are $\frac{0}{1}, \frac{0}{2}, \frac{0}{3}, \frac{1}{2}, \frac{1}{3}$ and $\frac{2}{3}$. Every other fraction between 0 and 1 has a denominator of $b \geq 4$, so $\frac{1}{b} \leq \frac{1}{4}$. Less obvious is that the same is true for height 0.01. Having an output less than 0.01 is the same as having $b > 100$. Since more fractions have $b > 100$ than $b < 100$, there are more points on the graph with height under 0.01 than over.




For more information, do some research on the Thomae function.

### Activity Synthesis

The purpose of this discussion is for students to understand the conventions of constructing a graph of a function and where input and outputs are found on a graph. Select students previously identified to share how they figured out $A = \pi r^2$ matched the non-linear graph.

Ask students:

- "Where are the independent variables labelled on the graphs?" (The horizontal axis)
- "Where are the dependent variables labelled on the graphs?" (The vertical axis)

Tell students that by convention, the independent variable is on the horizontal axis and the dependent variable is on the vertical axis. This means that when we write coordinate pairs, they are in the form of (input, output). For some functions, like the one with quarters and dimes, we can choose which variable is the independent and which is the dependent, which means the graph could be constructed either way based on our decisions.

Conclude the discussion by asking students to share their explanations for the point (1,49.6) for graph C. (There is no such thing as 0.6 of a quarter.) Remind students that sometimes we have to restrict inputs to only those that make sense. Since it's not possible to have 49.6 quarters, an input of 1 dime does not make sense. Similarly, 2, 3, or 4 dimes result in numbers of quarters that do not make sense. 0 dimes or 5 dimes, however, do produce outputs that make sense. Sometimes it is easier to sketch a graph of the line even when graphing discrete points would be more accurate for the context. Keeping the context of a function in mind is important when making sense of the input-output pairs associated with the function.

## 4.3 Running around a Track

**15 minutes**

The purpose of this activity is for students to interpret coordinates on graphs of functions and non-functions as well as understand that context does not dictate the independent and dependent variables.

In the first problem time is a function of distance, and the graph and table help determine how long it takes for Kiran to run a specific distance. In the second problem, however, time is not a function of Priya's distance from the starting line. This results in a graph where each input does not give exactly one output.

### Instructional Routines

- Co-Craft Questions

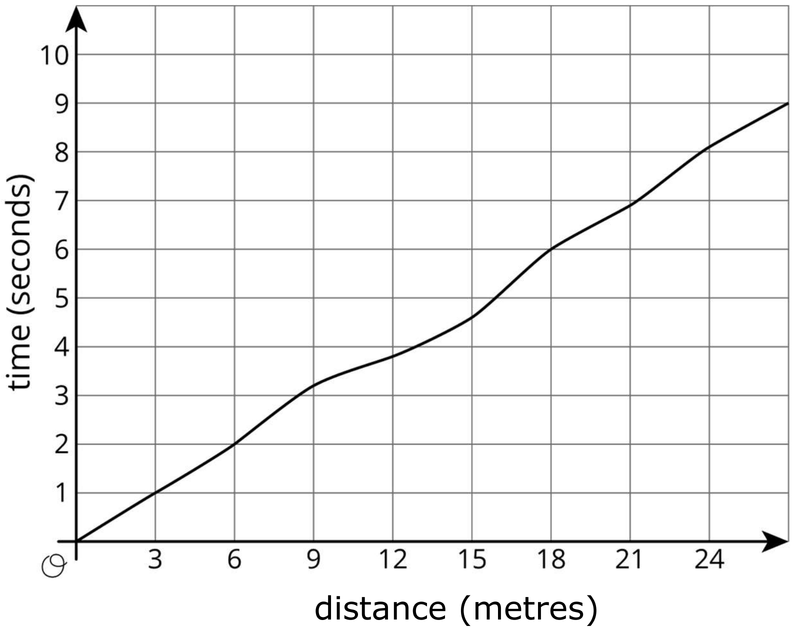
### Launch

Give students 3–5 minutes of quiet work time followed by whole-class discussion.

*Writing, Conversing: Co-Craft Questions.* Display only the graph and context (i.e., "Kiran was running around the track. The graph shows the time, $t$, he took to run various distances, $d$."). Ask pairs of students to write possible questions that could be answered by the graph. Invite pairs to share their questions with the class. Look for questions that ask students to interpret quantities represented in the graph. Next, reveal the questions of the activity. This helps students produce the language of mathematical questions and talk about the relationships between the two quantities in this task (e.g., distance and time).
*Design Principle(s): Maximise meta-awareness; Support sense-making*

### Student Task Statement

1. Kiran was running around the track. The graph shows the time, $t$, he took to run various distances, $d$. The table shows his time in seconds after every three metres.




| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| *d* | 0 | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 |
| *t* | 0 | 1.0 | 2.0 | 3.2 | 3.8 | 4.6 | 6.0 | 6.9 | 8.09 | 9.0 |

a. How long did it take Kiran to run 6 metres?

b. How far had he gone after 6 seconds?

c. Estimate when he had run 19.5 metres.

d. Estimate how far he ran in 4 seconds.

e. Is Kiran's time a function of the distance he has run? Explain how you know.

2. Priya is running once around the track. The graph shows her time given how far she is from her starting point.


a. What was her farthest distance from her starting point?

b. Estimate how long it took her to run around the track.

c. Estimate when she was 100 metres from her starting point.

d. Estimate how far she was from the starting line after 60 seconds.

e. Is Priya's time a function of her distance from her starting point? Explain how you know.

### Student Response

1

a. 2 seconds, since from the table, when $d = 6$, we have $t = 2$.

b. 18 metres, since from the table, when $t = 6$, we have $d = 18$.

c. Answers vary, but 6.45 seconds is a reasonable estimate. It took Kiran 6 seconds to run 18 metres, and 6.9 seconds to run 21 metres. Since 19.5 is halfway between 18 and 21, it is reasonable to estimate halfway between 6 seconds and 6.9 seconds. This estimate is further supported by the graph.

d. Answers vary, but 12.75 metres is a reasonable estimate. He runs 12 metres in 3.8 seconds and 15 metres in 4.6 seconds. Since 4 is a quarter of the way from 3.8 to 4.6, a reasonable estimate for distance would be a quarter of the way from 12 to 15, which is 12.75. This estimate is further supported by the graph.

e. Kiran's time is a function of the distance he has travelled. By reading the graph, you can use the distance he has travelled to find the time it took him to travel it.

2.

a. 200 metres, reflected by the rightmost point on the provided graph.

b. 75 seconds. From the top-left point on the graph, we can see that the first time after departing that Priya returns to the start line after her initial departure occurs at about 75 seconds.

c. 18 seconds and 54 seconds. There are two points on the graph representing a distance of 100 metres from the starting line, at heights 18 and 54.

d. 75 metres. There is only one point on the graph corresponding to a time of 60 seconds, and it occurs at distance 75 metres.

e. No. The time is not determined by the distance from the starting line, as the example of 100 metres above shows. There are two different times corresponding to the distance of 100 metres.

### Activity Synthesis

The purpose of this discussion is for students to understand that independent and dependent variables are not determined by the context (and specifically that time is not always a function of distance). Select students to share their strategies for calculating the answers for the first set of problems. For each problem, ask students whether the graph or table was more useful. Further the discussion by asking:

- "When are tables useful for answering questions?" (The exact answers that are listed in table while with the graph we can only approximate.)
- "When are graphs more useful for answering questions?" (A graph shows more input-output pairs than a table can list easily.)
- "Why does it make sense to have time be a function of distance in this problem?" (The farther Kiran runs, the longer it will take so it makes sense to represent time as a function of distance.)
- "Does time always have to be a function of distance?" (No, this graph could be made the other way with time on the horizontal and distance on the vertical and then it would show distance as a function of time, which makes sense since the longer Kiran runs, the further he will travel.)

For the second graph, ask students to indicate if they think it represents a function or not. If there are students who say yes and no, invite students from each side to say why they think it is or is not a function and try to persuade the rest of the class to their side. If all students are not persuaded that the graph is not a function, remind students that functions can only have one output for each input, and ask students to look back at their answer to the question "Estimate when she was 100 metres from her starting point." Since that question has two responses, the graph cannot be a function.

It time allows, ask students "If the axes of the second graph were switched, that is, time was the independent variable and distance from the starting line was the dependent variable, then would the graph be a representation of a function?" (Yes, because each input (time) would have only one output (distance).)

*Engagement: Develop Effort and Persistence.* Break the class into small discussion groups and then invite a representative from each group to report back to the whole class. All group members should be prepared to share if invited.
*Supports accessibility for: Language; Social-emotional skills; Attention*

## Lesson Synthesis

Each representation of a function presents information about the input and output of the function in different ways. Tell students to imagine we have a function with independent variable $x$ and dependent variable $y$.

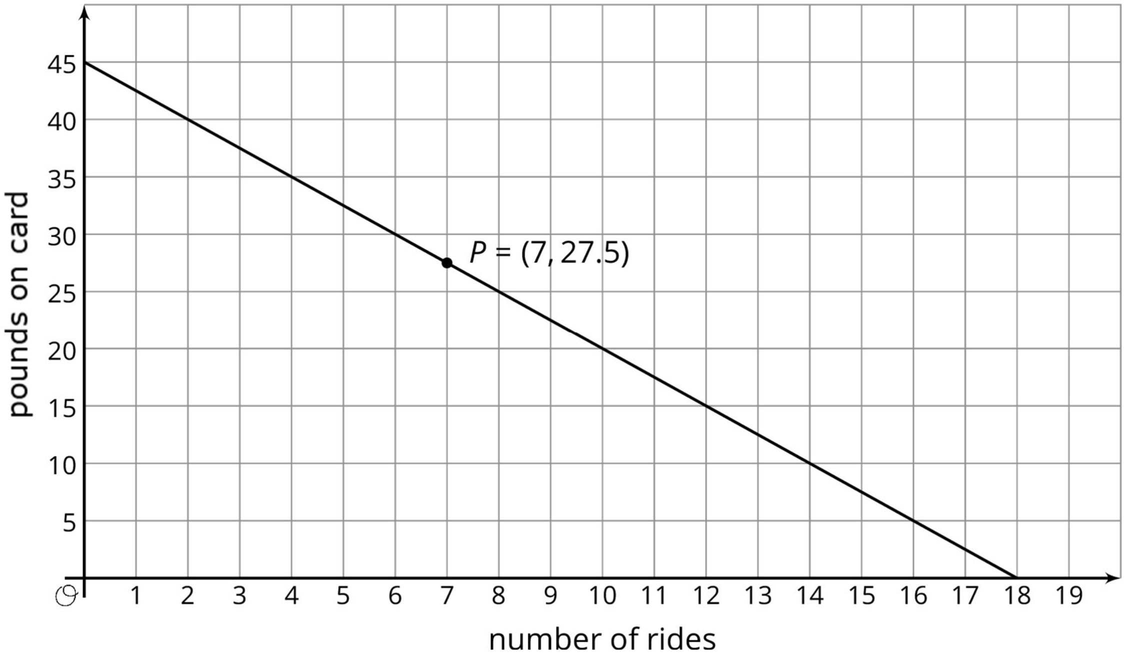
- "How do we find input-output pairs from a graph of the function?" (Any coordinate on the graph gives an input-output pair where the input is the $x$-value and the output is the $y$-value.)
- "What is something you won't see on the graph of the function?" (The graph will never "double-back" or have two $y$-values for the same $x$-value, because each input will have only one output.)
- "If the graph of the function contains the point (18,6), what else do we know about the function?" (If we input 18 into the function we will get 6 as an output. An equation for the function could by $y = \frac{1}{3}x$, but we would need to know more points on the graph to be sure.)

# 4.4 Subway Fare Card

**Cool Down: 5 minutes**

### Student Task Statement

Here is the graph of a function showing the amount of money remaining on a subway fare card as a function of the number of rides taken.




1. What is the output of the function when the input is 10? On the graph, plot this point and label its coordinates.

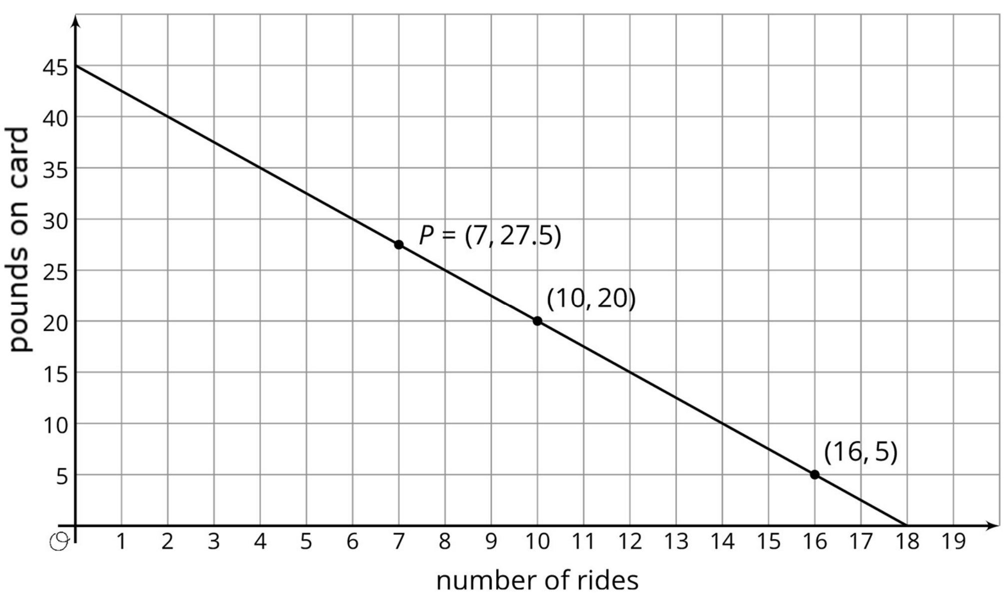
2. What is the input to the function when the output is 5? On the graph, plot this point and label its coordinates.

3. What does point *P* tell you about the situation?

### Student Response

1. 20. See graph in part 2.

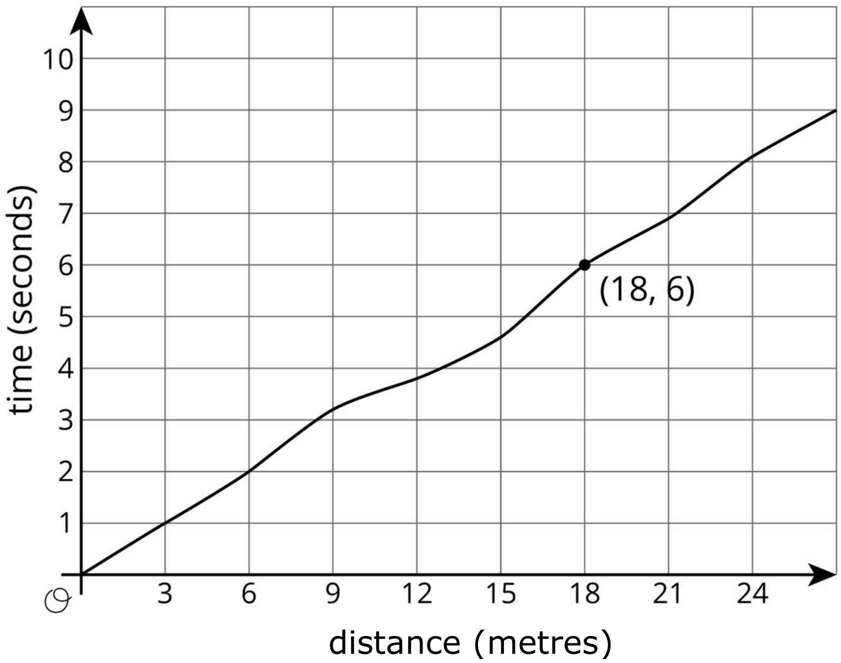
2. 16



3. After taking 7 rides, there will be £27.50 remaining on the card.

## Student Lesson Summary

Here is the graph showing Noah's run.



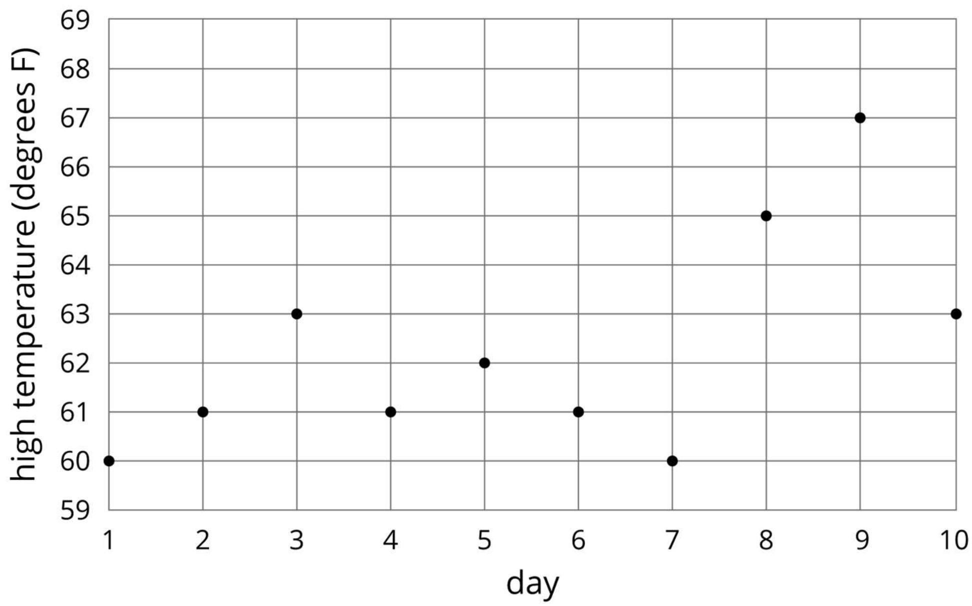

The time in seconds since he started running is a function of the distance he has run. The point (18,6) on the graph tells you that the time it takes him to run 18 metres is 6 seconds. The input is 18 and the output is 6.

The graph of a function is all the coordinate pairs, (input, output), plotted in the coordinate plane. By convention, we always put the input first, which means that the inputs are represented on the horizontal axis and the outputs, on the vertical axis.

## Lesson 4 Practice Problems

1. **Problem 1 Statement**

The graph and the table show the highest temperatures in a city over a 10-day period.




| day | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| temperature (degrees F) | 60 | 61 | 63 | 61 | 62 | 61 | 60 | 65 | 67 | 63 |

a. What was the highest temperature on Day 7?

b. On which days was the highest temperature 61 degrees?

c. Is the highest temperature a function of the day? Explain how you know.

d. Is the day a function of the highest temperature? Explain how you know.

**Solution**

a. 60 degrees F

b. Days 2, 4, 6

c. The highest temperature is a function of the day. There are no different outputs for the same input. That is, there is no day with two different highest temperatures.

d. Day could not be a function of temperature as there are multiple days that have the same highest temperature. There would be the different outputs for the same input.

2. **Problem 2 Statement**

The amount Lin's sister earns at her part-time job is proportional to the number of hours she works. She earns £9.60 per hour.

a. Write an equation in the form $y = kx$ to describe this situation, where $x$ represents the hours she works and $y$ represents the pounds she earns.

b. Is $y$ a function of $x$? Explain how you know.

c. Write an equation describing $x$ as a function of $y$.

**Solution**

a. $y = 9.6x$, where $x$ is number of hours worked and $y$ is amount earned in pounds

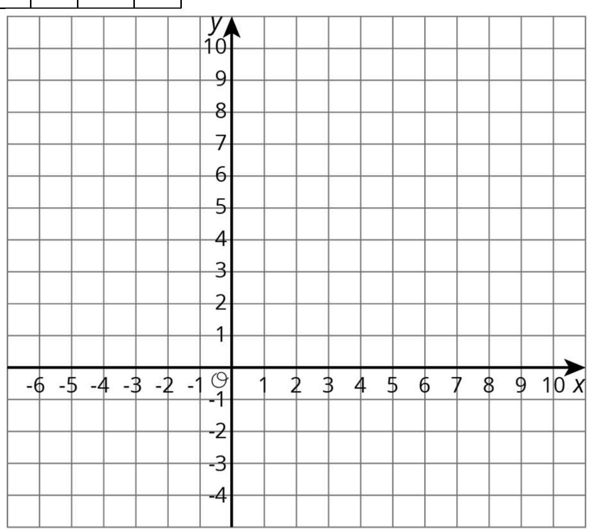
b. $y$ is a function of $x$ because there is only one output for each input.

c. $x = \frac{1}{9.6}y$ or $\left(x = \frac{5}{48}y\right)$

3. **Problem 3 Statement**

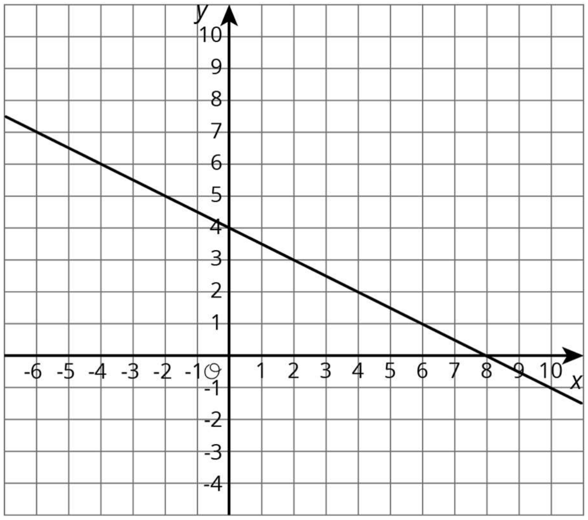
Use the equation $2m + 4s = 16$ to complete the table, then graph the line using $s$ as the dependent variable.

| | | | | |
|---|---|---|---|---|
| $m$ | 0 | | -2 | |
| $s$ | | 3 | | 0 |




**Solution**

| | | | | |
|---|---|---|---|---|
| $m$ | 0 | 2 | -2 | 8 |
| $s$ | 4 | 3 | 5 | 0 |




4. **Problem 4 Statement**

Solve the system of equations: $\begin{cases} y = 7x + 10 \\ y = -4x - 23 \end{cases}$

**Solution**

(-3,-11)

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