Lesson 5: The size of the scale factor

Goals

- Describe (orally and in writing) how scale factors of 1, less than 1, and greater than 1 affect the size of scaled copies.
- Explain and show (orally and in writing) how to recreate the original shape given a scaled copy and its scale factor.
- Recognise (orally and in writing) the relationship between a scale factor of a scaled copy to its original shape is the "reciprocal" of the scale factor of the original shape to its scaled copy.

Learning Targets

- I can describe the effect on a scaled copy when I use a scale factor that is greater than 1, less than 1, or equal to 1.
- I can explain how the scale factor that takes shape A to its copy shape B is related to the scale factor that takes shape B to shape A.

Lesson Narrative

In this lesson, students deepen their understanding of scale factors in two ways:

1. They classify scale factors by size (less than 1, exactly 1, and greater than 1) and notice how each class of factors affects the scaled copies, and
2. They see that the scale factor that takes an original shape to its copy and the one that takes the copy to the original are reciprocals. This means that the scaling process is reversible, and that if shape B is a scaled copy of shape A, then shape A is also a scaled copy of shape B.

Students also continue to apply scale factors and what they learned about corresponding distances and angles to draw scaled copies without a grid.

Two of the activities, Scaling a Puzzle, and Missing Shape, Factor, or Copy, are optional. In Scaling a Puzzle, students scale the 6 pieces of a puzzle individually and then assemble them to make a scaled copy of the puzzle. The individual pieces are rectangular with line segments partitioning them into regions. Students need to think strategically about which measurements to take in order to scale the pieces accurately. In Missing Shape, Factor, or Copy, students gain fluency dealing with the different aspects of scaled copies, supplying the missing information in each case.

Building On

- Find whole-number quotients of whole numbers with up to four-digit dividends and two-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

- Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.
- Interpret multiplication as scaling (resizing), by:
- Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem. For example, create a story context for $\left(\frac{2}{3}\right) \div \left(\frac{3}{4}\right)$ and use a visual fraction model to show the quotient; use the relationship between multiplication and division to explain that $\left(\frac{2}{3}\right) \div \left(\frac{3}{4}\right) = \frac{8}{9}$ because $\frac{3}{4}$ of $\frac{8}{9}$ is $\frac{2}{3}$. (In general, $\left(\frac{a}{b}\right) \div \left(\frac{c}{d}\right) = \frac{ad}{bc}$.) How much chocolate will each person get if 3 people share $\frac{1}{2}$ lb of chocolate equally? How many $\frac{3}{4}$ cup servings are in $\frac{2}{3}$ of a cup of yogurt? How wide is a rectangular strip of land with length $\frac{3}{4}$ m and area $\frac{1}{2}$ square m?

Addressing

- Solve problems involving scale drawings of geometric shapes, including computing actual lengths and areas from a scale drawing and reproducing a scale drawing at a different scale.

Building Towards

- Recognise and represent proportional relationships between quantities.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Clarify, Critique, Correct
- Discussion Supports
- Number Talk
- Think Pair Share

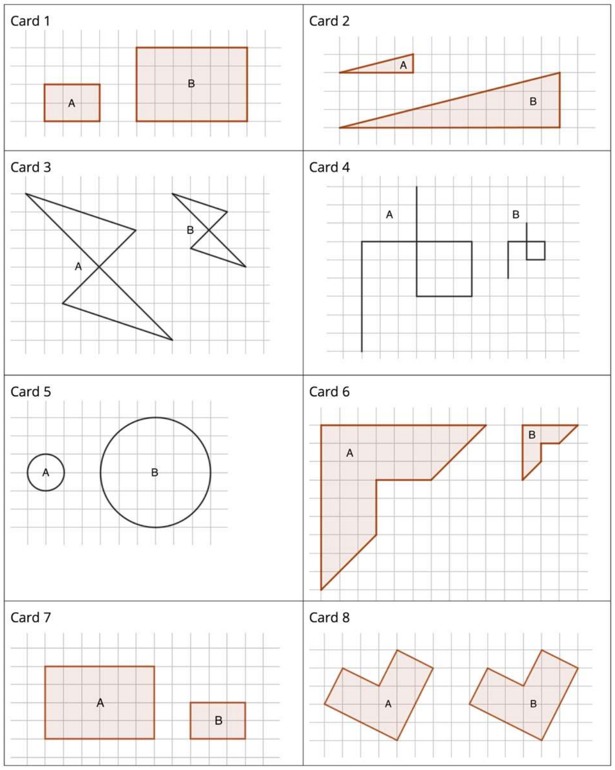
Required Materials

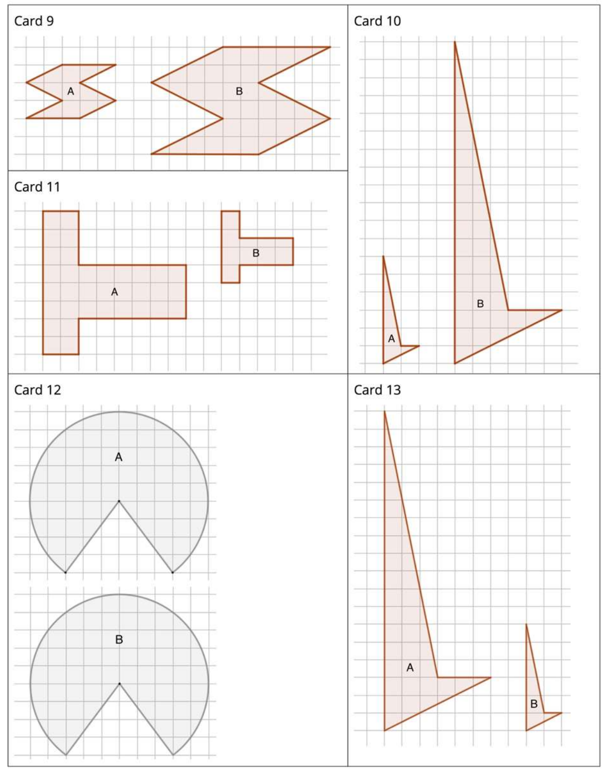
Geometry toolkits

tracing paper, graph paper, coloured pencils, scissors, and an index card to use as a straightedge or to mark right angles, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

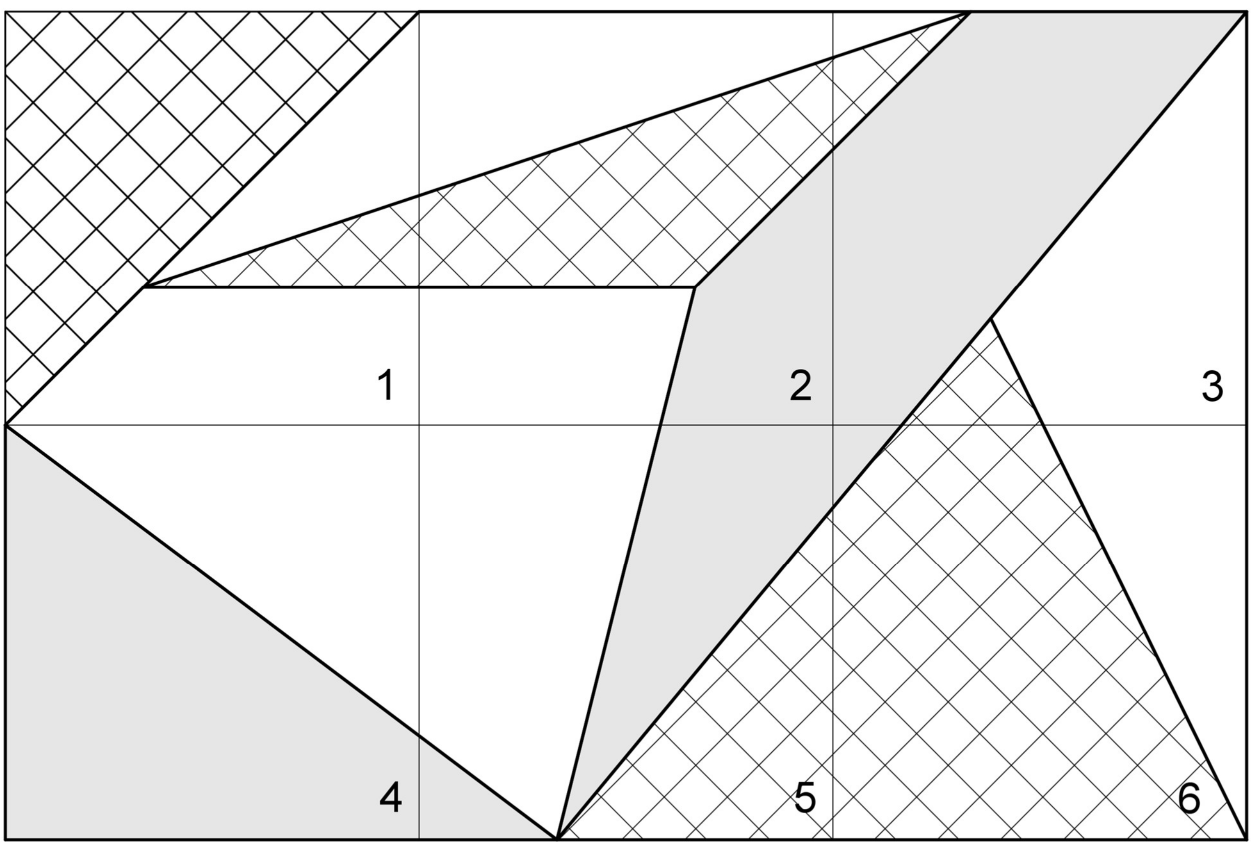
Pre-printed slips, cut from copies of the blackline master

Scaled copies Card Sort





Scaling a puzzle



Required Preparation

Print and cut sets of slips for the sorting activity from the Scaled Copies Card Sort blackline master. Make enough copies so that each group of 3–4 students has a set. If possible, copy each complete set on a different colour of paper, so that a stray slip can quickly be put back.

Print and cut puzzle pieces and blank squares for the Scaling a Puzzle activity from the Scaling a Puzzle blackline master. Make enough copies so that each group of 3 students has 1 original puzzle and 6 blank squares.

Make sure students have access to their geometry toolkits—especially rulers and protractors.

Student Learning Goals

Let's look at the effects of different scale factors.

5.1 Number Talk: Missing Factor

Warm Up: 10 minutes

This number talk encourages students to use structure and the relationship between multiplication and division to mentally solve problems involving fractions. It prompts students to think about how the size of factors impacts the size of the product. It reviews the idea of reciprocal factors in preparation for the work in the lesson.

Instructional Routines

- Discussion Supports
- Number Talk

Launch

Ask students what operation is meant when a number and a variable are placed right next to each other in an equation. (Multiplication)

Display one problem at a time. Give students 1 minute of quiet think time per problem and ask them to give a signal when they have an answer and a strategy. Follow with a whole-class discussion.

Representation: Internalise Comprehension. To support working memory, provide students with sticky notes or mini whiteboards.
Supports accessibility for: Memory; Organisation

Anticipated Misconceptions

Students might think that a product cannot be less than one of the factors, not realising that one of the factors can be a fraction. Use examples involving smaller and familiar numbers to remind them that it is possible. Ask, for example, "What times 10 is 5?"

Student Task Statement

Solve each equation mentally.

$16x = 176$

$16x = 8$

$16x = 1$

$\frac{1}{5}x = 1$

$\frac{2}{5}x = 1$

Student Response

- 11. Possible strategy: $16 \times 10 = 160$ and $16 \times 1 = 16$, so $16 \times 11 = 176$
- $\frac{1}{2}$ or equivalent. Possible strategy: 16 divided by 2 is 8, and dividing by 2 is the same as multiplying by $\frac{1}{2}$.
- $\frac{1}{16}$. Possible strategy: It takes 16 of $\frac{1}{16}$ to make 1.

- 5. Possible strategy: There are 5 copies of $\frac{1}{5}$ in 1, so $5 \times \frac{1}{5} = 1$.
- $\frac{5}{2}$ or equivalent. Possible strategy: 5 groups of $\frac{2}{5}$ make 2, so $\frac{5}{2}$ groups of $\frac{2}{5}$ make 1.

Activity Synthesis

Ask students to share their strategies for each problem. Record and display their explanations for all to see. To involve more students in the conversation, consider asking:

- Who can restate ___'s reasoning in a different way?
- Did anyone solve the problem the same way but would explain it differently?
- Did anyone solve the problem in a different way?
- Does anyone want to add on to _____'s strategy?
- Do you agree or disagree? Why?

Highlight that multiplying a factor by a fraction less than 1 results in a product that is less than one of the factors, and that two factors that multiply to be 1 are reciprocals.

Speaking: Discussion Supports.: Display sentence frames to support students when they explain their strategy. For example, "First, I _____ because . . ." or "I noticed _____ so I . . . ." Some students may benefit from the opportunity to rehearse what they will say with a partner before they share with the whole class.
Design Principle(s): Optimise output (for explanation)

5.2 Card Sort: Scaled Copies

15 minutes

Students have studied many examples of scaled copies and know that corresponding lengths in a shape and its scaled copy are related by the same scale factor. The purpose of this activity is for students to examine how the *size* of the scale factor is related to the original shape and the scaled copy. The activity serves several purposes:

1. To reinforce students' awareness of scale factors
2. To draw attention to how scaled copies behave when the scale factor is 1, less than 1, and greater than 1; and
3. To help students notice that reciprocal scale factors reverse the scaling.

You will need the Scaled Copies Card Sort blackline master for this activity.

Monitor for students who group the cards in terms of:

- Specific scale factors (e.g., 2, 3, $\frac{1}{2}$, etc.)
- Ranges of scale factors producing certain effects (e.g., factors producing larger, unchanged, or smaller copies)
- Reciprocal scale factors (e.g., one factor scales shape A to B, and its reciprocal reverses the scaling)

Select groups who use each of these approaches (and any others) and ask them to share during the discussion.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Clarify, Critique, Correct

Launch

Arrange students in groups of 3–4. Distribute one set of slips to each group. Give students 7–8 minutes of group work time, followed by whole-class discussion.

Action and Expression: Internalise Executive Functions. Begin with a small-group or whole-class demonstration and think aloud using card 1 to remind students how to determine the scale factor between shape A and shape B. Keep the worked-out calculations on display for students to reference as they work.
Supports accessibility for: Memory; Conceptual processing

Anticipated Misconceptions

Students may sort by the types of shapes rather than by how the second shape in each pair is scaled from the first. Remind students to sort based on how shape A is scaled to create shape B.

Students may think of the change in lengths between shapes A and B in terms of addition or subtraction, rather than multiplication or division. Remind students of an earlier lesson in which they explored the effect of subtracting the same length from each side of a polygon in order to scale it. What happened to the copy? (It did not end up being a polygon and was not a scaled copy of the original one.)

Students may be unclear as to how to describe how much larger or smaller a shape is, or may not recall the meaning of scale factor. Have them compare the lengths of each side of the shape. What is the common factor by which each side is multiplied?

Student Task Statement

Your teacher will give you a set of cards. On each card, shape A is the original and shape B is a scaled copy.

1. Sort the cards based on their scale factors. Be prepared to explain your reasoning.

2. Examine cards 10 and 13 more closely. What do you notice about the shapes and sizes of the shapes? What do you notice about the scale factors?

3. Examine cards 8 and 12 more closely. What do you notice about the shapes? What do you notice about the scale factors?

Student Response

1. Grouping categories vary. Sample categories:
   - Scale factor of 2: cards 1 and 9.
   - Scale factor of $\frac{1}{2}$: cards 3, 7, and 11.
   - Scale factor of 3: cards 2, 5, and 10.
   - Scale factor of $\frac{1}{3}$: cards 4, 6, and 13.
   - Scale factor of 1: cards 8 and 12.
2. Answers vary. Sample response: The shapes on both cards are the same, but on card 10, the scaled copy is larger and the scale factor is 3. On card 13, the scaled copy is smaller and the scale factor is $\frac{1}{3}$.
3. Answers vary. Sample response: The original and the copy are the same size on cards 8 and 12. The copy is identical to the original. The scale factor is 1.

Are You Ready for More?

Triangle B is a scaled copy of triangle A with scale factor $\frac{1}{2}$.

1. How many times bigger are the side lengths of triangle B when compared with triangle A?
2. Imagine you scale triangle B by a scale factor of $\frac{1}{2}$ to get triangle C. How many times bigger will the side lengths of triangle C be when compared with triangle A?
3. Triangle B has been scaled once. Triangle C has been scaled twice. Imagine you scale triangle A n times to get triangle N, always using a scale factor of $\frac{1}{2}$. How many times bigger will the side lengths of triangle N be when compared with triangle A?

Student Response

1. $\frac{1}{2}$
2. $\frac{1}{4}$
3. $\left(\frac{1}{2}\right)^n$

Activity Synthesis

Select groups to explain their sorting decisions following the sequence listed in the Activity Narrative. If no groups sorted in terms of ranges of scale factors (less than 1, exactly 1, and greater than 1) or reciprocal scaling, ask:

- What can we say about the scale factors that produce larger copies? Smaller copies? Same-size copies?
- Some cards had the same pair of shapes on them, just in a reversed order (i.e., pairs #1 and 7, #10 and 13). What do you notice about their scale factors?

Highlight the two main ideas of the lesson: 1) the effects of scale factors that are greater than 1, exactly 1, and less than 1; and 2) the reversibility of scaling. Point out that if shape B is a scaled copy of shape A, then A is also a scaled copy of B. In other words, A and B are scaled copies of one another, and their scale factors are reciprocals.

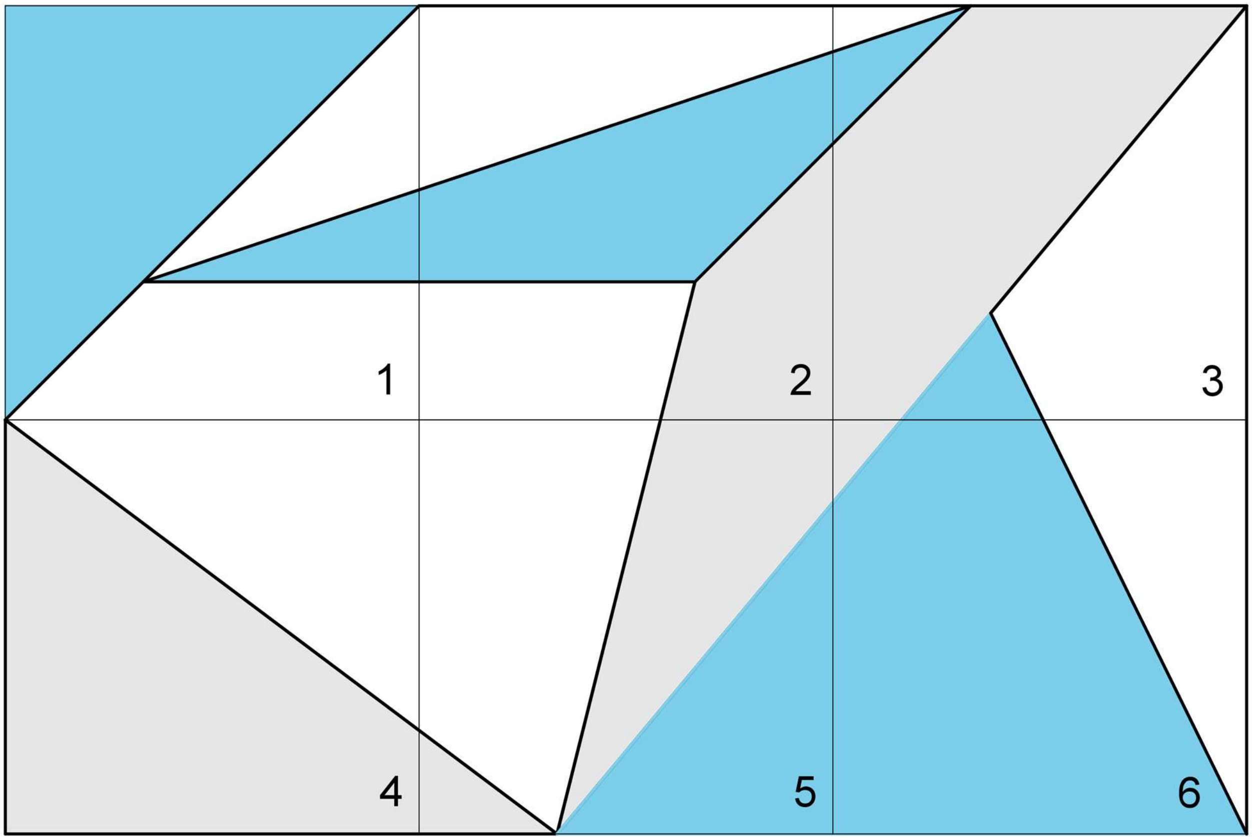
Suggest students add these observations to their answer for the last question.

Writing: Clarify, Critique, Correct. Present an incorrect statement that reflects a possible misunderstanding from the class for the last prompt. For example, “The scale factor of cards 8 and 12 is 0 because the shapes are the same and there was no change.” Prompt students to identify the error, and then write a correct version. In this discussion, highlight the use of disciplinary language by revoicing student ideas. This helps students evaluate, and improve on, the written mathematical arguments of others.
Design Principle(s): Maximise meta-awareness

5.3 Scaling A Puzzle

Optional: 15 minutes

This activity gives students a chance to apply what they know about scale factors, lengths, and angles and create scaled copies without the support of a grid. Students work in groups of 3 to complete a jigsaw puzzle, each group member scaling 2 non-adjacent pieces of a 6-piece puzzle with a scale factor of $\frac{1}{2}$. The group then assembles the scaled pieces and examines the accuracy of their scaled puzzle. Consider having students use a colour in place of the cross hatching.



As students work, notice how they measure distances and whether they consider angles. Depending on how students determine scaled distances, they may not need to transfer angles. Look out for students who measure only the lengths of drawn line segments rather than distances, e.g., between the corner of a square and where a line segment begins. Suggest that they consider other measurements that might help them locate the beginning and end of a line segment.

You will need the Scaling a Puzzle blackline master for this activity.

Launch

Arrange students in groups of 3. Give pre-cut puzzle squares 1 and 5 to one student in the group, squares 2 and 6 to a second student, and squares 3 and 4 to the third. After students have answered the first question, give each student 2 blank squares cut from the second section of the blackline master, whose sides are half of the side length of the puzzle squares. Provide access to geometry toolkits.

Engagement: Develop Effort and Persistence. Connect a new concept to one with which students have experienced success. Before students begin, ask them to predict which tools, from their geometry toolkits, they anticipate they will need, and to describe how they might use them. During the activity, make sure that students are using their rulers and protractors correctly.
Supports accessibility for: Visual-spatial processing; Conceptual processing; Fine-motor skills

Anticipated Misconceptions

Students may incorporate the scale factor when scaling line segments but neglect to do so when scaling distances between two points not connected by a line segment. Remind them that all distances are scaled by the same factor.

Students may not remember to verify that the angles in their copies must remain the same as the original. Ask them to notice the angles and recall what happens to angles when a shape is a scaled copy.

Student Task Statement

Your teacher will give you 2 pieces of a 6-piece puzzle.

1. If you drew scaled copies of your puzzle pieces using a scale factor of $\frac{1}{2}$, would they be larger or smaller than the original pieces? How do you know?

2. Create a scaled copy of each puzzle piece on a blank square, with a scale factor of $\frac{1}{2}$.

3. When everyone in your group is finished, put all 6 of the original puzzle pieces together like this:

1	2	3
4	5	6

   Next, put all 6 of your scaled copies together. Compare your scaled puzzle with the original puzzle. Which parts seem to be scaled correctly and which seem off? What might have caused those parts to be off?

4. Revise any of the scaled copies that may have been drawn incorrectly.

5. If you were to lose one of the pieces of the original puzzle, but still had the scaled copy, how could you recreate the lost piece?

Student Response

1. They would be smaller because $\frac{1}{2} < 1$.

2. Copies of puzzle pieces scaled by $\frac{1}{2}$.

3. Answers vary depending on how each student drew their pieces.

4. Revised copies of puzzle pieces.

5. Use the copy and a scale factor of 2 to recreate the original puzzle piece. Sample explanation: 2 is the reciprocal of $\frac{1}{2}$, so it would scale the copy back to the original.

Activity Synthesis

Much of the conversations about creating accurate scaled copies will have taken place among partners, but consider coming together as a class to reflect on the different ways students worked. Ask questions such as:

- How is this task more challenging than creating scaled copies of polygons on a grid?
- Besides distances or lengths, what helped you create an accurate copy?
- How did you know or decide which distances to measure?
- Before your drawings were assembled, how did you check if they were correct?

Student responses to these questions may differ: for example, for piece 6, the two lines can be drawn by measuring distances on the border of the puzzle piece; the angles work out correctly automatically. For piece 2, however, to get the three lines that meet in a point in the middle of the piece just right, students can either measure angles, or extend those line segments until they meet the border of the piece (and then measure distances).

5.4 Missing Shape, Factor, or Copy

Optional: 10 minutes

In this activity, students investigate different aspects of the shape, scaled copy, and scale factor trio. Given any two of these three, we can find the third. Students work with these different scenarios on a grid, dealing with scale factors greater than 1, less than 1, and equal to 1. In addition, they create a scaled copy of a non-polygonal shape off of a grid.

Both on and off of a grid, students need to decide what tools to use in order to measure angles and side lengths in order to produce the scaled copies.

Instructional Routines

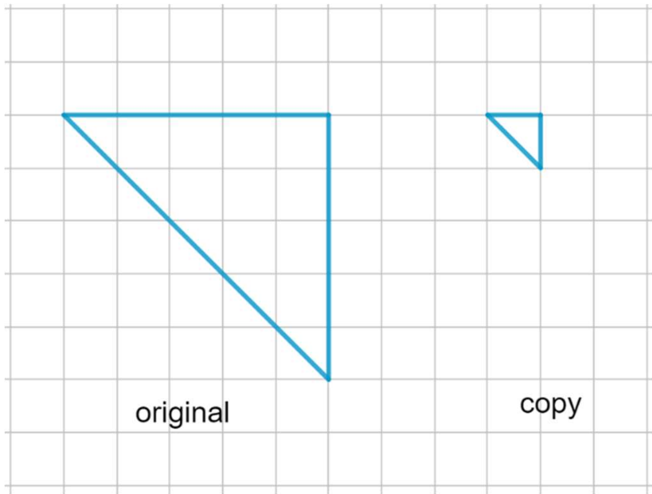
- Discussion Supports
- Think Pair Share

Launch

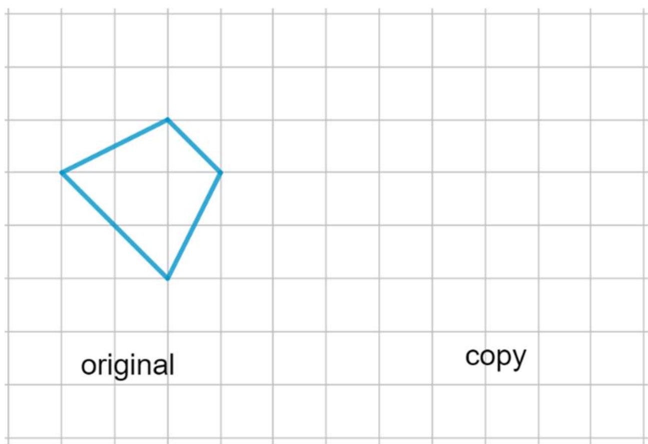
Arrange students in groups of 2. Provide access to geometry toolkits. Give students 3–4 minutes of quiet work time, followed by 2 minutes of group discussion and then whole-class discussion.

Student Task Statement

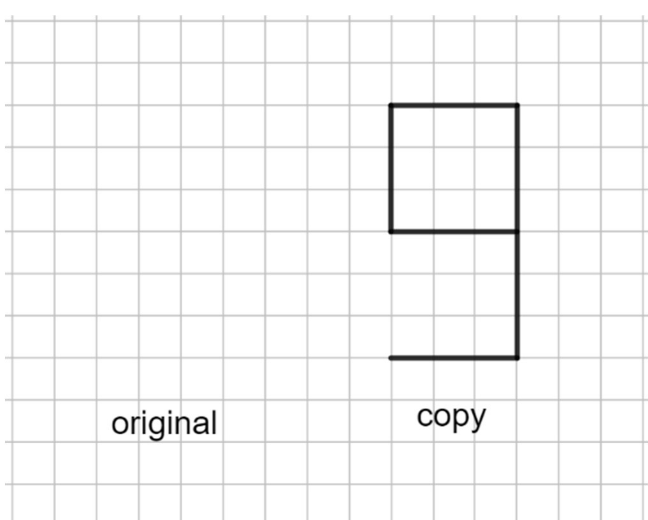
1. What is the scale factor from the original triangle to its copy? Explain or show your reasoning.



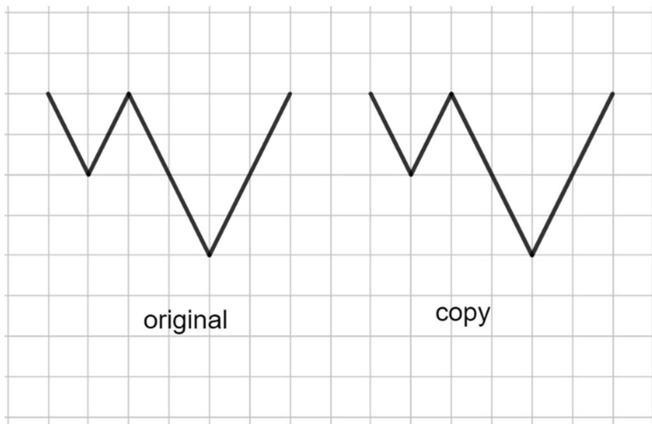
2. The scale factor from the original trapezium to its copy is 2. Draw the scaled copy.



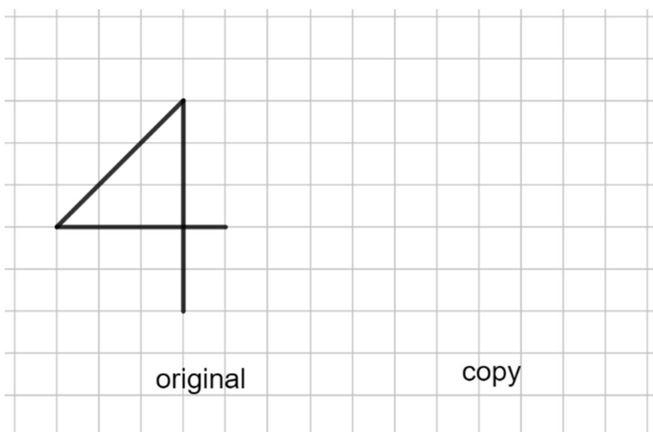
3. The scale factor from the original shape to its copy is $\frac{3}{2}$. Draw the original shape.



4. What is the scale factor from the original shape to the copy? Explain how you know.



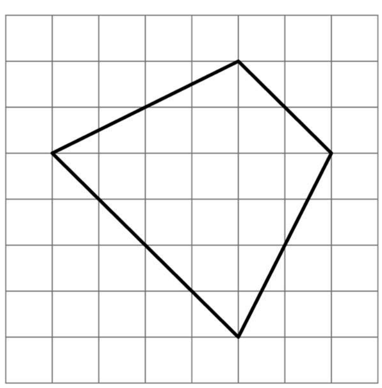
5. The scale factor from the original shape to its scaled copy is 3. Draw the scaled copy.



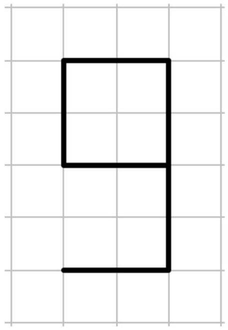
Student Response

1. Scale factor: $\frac{1}{5}$ since the vertical and horizontal sides on the original are 5 grid units in length and the corresponding sides on the copy are 1 grid unit in length.

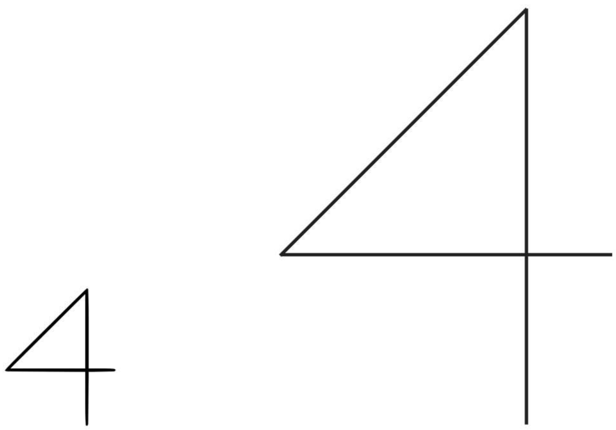
2. Scaled copy:



3. Original shape:



4. Scale factor: 1 since the shape and the copy are the same size.



original copy

5.

Activity Synthesis

The purpose of this discussion is to ensure that students understand the connections between the original shape, scaled copy, and scale factor—given any two of the three, they should understand a method for finding the third.

Ask students how they solved the first three questions. Which one was most challenging? Why? Individual responses will likely vary here, but the question with the missing original shape deserves special attention. This requires going backward since we have the scaled copy and the scale factor. This scenario also reinforces that the way to “undo” the scale factor of $\frac{3}{2}$ that has been applied to produce the copy is to apply a scale factor of $\frac{2}{3}$.

For the last problem, ask students how they used the line segment lengths and angles of the original shape. They may have sketched the copy without checking angles. Ask them why measuring angles is important.

Speaking, Listening: Discussion Supports. Use this routine to support whole-class discussion as students share their strategies and discuss which strategies they found to be easier or

more challenging. Press for details in students' explanations by requesting that students challenge an idea, elaborate on an idea, or give an example. Revoice student ideas to demonstrate mathematical language use by restating a statement as a question in order to clarify, apply appropriate language, and involve more students.
Design Principle(s): Support sense-making

Lesson Synthesis

- What happens to the copy when it is created with a scale factor greater than 1? Less than 1? Exactly 1?
- How can we reverse the scaling to get back to the original shape when we have a scaled copy?

When the scale factor is greater than 1, the scaled copy is larger than the original. When it is less than 1, the copy is smaller than the original. A scale factor of exactly 1 produces a same-size copy.

Scaling can be reversed by using reciprocal factors. If we scale shape A by a factor of 4 to obtain shape B, we can scale B back to A using a factor of $\frac{1}{4}$. This means that if B is a scaled copy of A, A is also a scaled copy of B; they are scaled copies of each other.

5.5 Scaling a Rectangle

Cool Down: 5 minutes

Student Task Statement

A rectangle that is 2 inches by 3 inches has been scaled by a factor of 7.

1. What are the side lengths of the scaled copy?
2. Suppose you want to scale the copy back to its original size. What scale factor should you use?

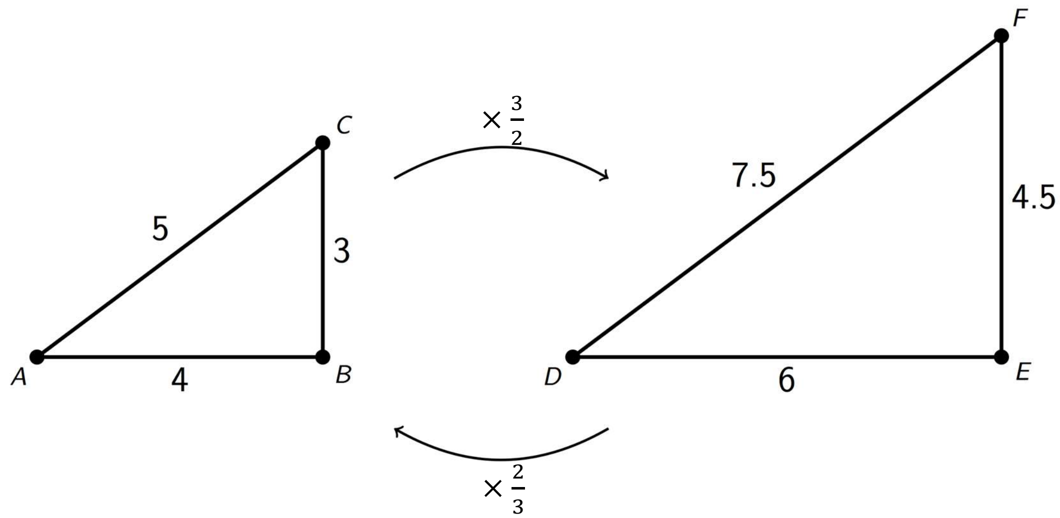
Student Response

1. 14 inches by 21 inches, because $2 \times 7 = 14$ and $3 \times 7 = 21$.
2. $\frac{1}{7}$, because it is the reciprocal of 7.

Student Lesson Summary

The size of the scale factor affects the size of the copy. When a shape is scaled by a scale factor greater than 1, the copy is larger than the original. When the scale factor is less than 1, the copy is smaller. When the scale factor is exactly 1, the copy is the same size as the original.

Triangle DEF is a larger scaled copy of triangle ABC, because the scale factor from ABC to DEF is $\frac{3}{2}$. Triangle ABC is a smaller scaled copy of triangle DEF, because the scale factor from DEF to ABC is $\frac{2}{3}$.



This means that triangles ABC and DEF are scaled copies of each other. It also shows that scaling can be reversed using **reciprocal** scale factors, such as $\frac{2}{3}$ and $\frac{3}{2}$.

In other words, if we scale shape A using a scale factor of 4 to create shape B, we can scale shape B using the reciprocal scale factor, $\frac{1}{4}$, to create shape A.

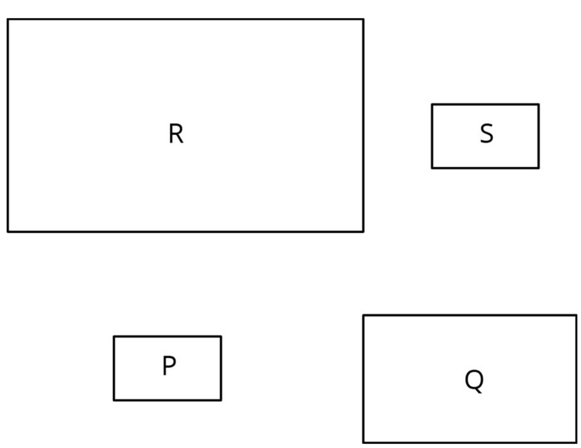
Glossary

- reciprocal

Lesson 5 Practice Problems

Problem 1 Statement

Rectangles P, Q, R, and S are scaled copies of one another. For each pair, decide if the scale factor from one to the other is greater than 1, equal to 1, or less than 1.



a. from P to Q

b. from P to R

c. from Q to S

d. from Q to R

e. from S to P

f. from R to P

g. from P to S

Solution

a. Greater than 1

b. Greater than 1

c. Less than 1

d. Greater than 1

e. Equal to 1

f. Less than 1

g. Equal to 1

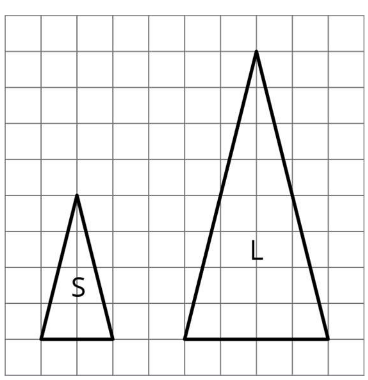
Problem 2 Statement

Triangle S and triangle L are scaled copies of one another.

a. What is the scale factor from S to L?

b. What is the scale factor from L to S?

c. Triangle M is also a scaled copy of S. The scale factor from S to M is $\frac{3}{2}$. What is the scale factor from M to S?



Solution

a. 2

b. $\frac{1}{2}$

c. $\frac{2}{3}$. The two scale factors are reciprocals of each other.

Problem 3 Statement

Are two squares with the same side lengths scaled copies of one another? Explain your reasoning.

Solution

Yes. There is a scale factor of 1 between them.

Problem 4 Statement

Quadrilateral A has side lengths 2, 3, 5, and 6. Quadrilateral B has side lengths 4, 5, 8, and 10. Could one of the quadrilaterals be a scaled copy of the other? Explain.

Solution

No. For the shortest sides to match up, the scale factor from A to B would have to be 2. But scaling the side of A with length 3 by a factor of 2 would give a side of length 6, which doesn't match any of the side lengths of B.

Problem 5 Statement

Select **all** the ratios that are equivalent to the ratio $12:3$.

a. $6:1$

b. $1:4$

c. $4:1$

d. $24:6$

e. $15:6$

f. $1\,200:300$

g. $112:13$

Solution ["C", "D", "F"]