Lesson 2: Introduction to functions

Goals

- Comprehend the structure of a function as having one and only one output for each allowable input.
- Describe (orally and in writing) a context using function language, e.g., “the [output] is a function of the [input]” or “the [output] depends on the [input]”.
- Identify (orally) rules that produce exactly one output for each allowable input and rules that do not.

Learning Targets

- I know that a function is a rule with exactly one output for each allowable input.
- I know that if a rule has exactly one output for each allowable input, then the output depends on the input.

Lesson Narrative

In this second introductory lesson, students learn the term **function** for a rule that produces a single output for a given input. They also start to connect function language to language they learned in earlier years about independent and dependent variables.

We can say, "the output is a function of the input," and we also say, "the output depends on the input." In the optional activity, students see how it is possible to use different words to describe the same function as long as all input-output pairs are the same. This helps solidify the notion of a function as something different from the method of calculating its values.

Addressing

Understand that a function is a rule that assigns to each input exactly one output. The graph of a function is the set of ordered pairs consisting of an input and the corresponding output. Function notation is not required in KS3.

Building Towards

Understand that a function is a rule that assigns to each input exactly one output. The graph of a function is the set of ordered pairs consisting of an input and the corresponding output.

Instructional Routines

- Stronger and Clearer Each Time
- Collect and Display
- Discussion Supports
- Think Pair Share

Student Learning Goals

Let's learn what a function is.

2.1 Square Me

Warm Up: 5 minutes

The purpose of this warm-up is to remind students that two different numbers can have the same square. This is an example of two inputs having the same output for a given rule—in this case "square the number." Later activities in the lesson explore rules that have multiple outputs for the same input.

Launch

Give students 1–2 minutes of quiet work time followed by a whole-class discussion.

Student Task Statement

Here are some numbers in a list:

1, -3, $-\frac{1}{2}$, 3, 2, $\frac{1}{4}$, 0.5

1. How many different numbers are in the list?
2. Make a new list containing the squares of all these numbers.
3. How many different numbers are in the new list?
4. Explain why the two lists do not have the same number of different numbers.

Student Response

1. 7
2. 1, 9, $\frac{1}{4}$, 9, 4, $\frac{1}{16}$, 0.25
3. 5
4. Answers vary. Sample response: Some numbers in the list are different but have the same square. This can happen because a negative times a negative is a positive. For example, -3 squared is 9. 3 squared is also 9.

Activity Synthesis

The focus of this discussion should be on the final question, which, even though the language isn't used in the problem, helps prepare students for thinking about the collection of values that make up the input and output of rules. Here, the input is a list of 7 unique values while the output has only 5 unique values.

Invite students to share their responses to the second problem and display the list of numbers for all to see along with the original list. Next, invite different students to share their explanations from the forth problem. Emphasise the idea that when we square a negative number, we get a positive number. This means two different numbers can have the same square, or, using the language of inputs and outputs, two different inputs can have the same output for a rule.

If time allows, ask "Can you think of other rules where different inputs can have the same output?" After 30 seconds of quiet think time, select students to share their rules. They may recall the previous lesson where they encountered the rule "write 7," which has only one output for all inputs, and the rule "extract the digit in the tenths place," which has only 10 unique outputs for all inputs.

2.2 You Know This, Do You Know That?

15 minutes

In this activity students are presented with a series of questions like, “A person is 60 inches tall. Do you know their height in feet?” For some of the questions the answer is "yes" (because you can convert from inches to feet by dividing by 12). In other cases the answer is "no" (for example, “A person is 14 years old. Do you know their height?”). The purpose is to develop students’ understanding of the structure of a function as something that has one and only one output for each allowable input. In cases where the answer is yes, students draw an input-output diagram with the rule in the box. In cases where the answer is no, they give examples of an input with two or more outputs. In the Activity Synthesis, the word function is introduced to students for the first time.

Identify students who use different ways to describe the rules and different notation for the input and outputs of the final two problems.

Instructional Routines

- Discussion Supports

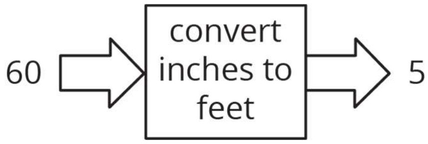
Launch

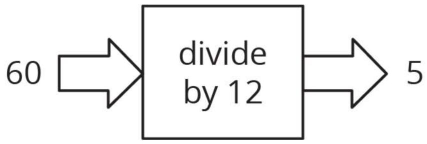
Display the example statement (but not the input-output diagram) for all to see.

Example: A person is 60 inches tall. Do you know their height in feet?

Give students 30 seconds of quiet think time, and ask them to be prepared to justify their response. Select students to share their answers, recording and displaying different justifications for all to see.

Display the following input-output diagrams for all to see. Ask students if the rules in the diagrams match the justifications they just heard:





or

Tell students that they will draw input-output diagrams like these as part of the task. Answer any questions students might have around the input-output diagrams. Be sure students understand that if they answer yes to the question they will need to draw the input-output diagram and if they answer no they need to give an example of why the question does not have one answer.

Give students 8–10 minutes of quiet work time for the problems followed by a whole-class discussion.

Representation: Access for Perception. Read all statements aloud. Students who both listen to and read the information will benefit from extra processing time.
Supports accessibility for: Language

Student Task Statement

Say yes or no for each question. If yes, draw an input-output diagram. If no, give examples of two different outputs that are possible for the same input.

1. A person is 5.5 feet tall. Do you know their height in inches?

2. A number is 5. Do you know its square?

3. The square of a number is 16. Do you know the number?

4. A square has a perimeter of 12 cm. Do you know its area?

5. A rectangle has an area of 16 cm^2. Do you know its length?

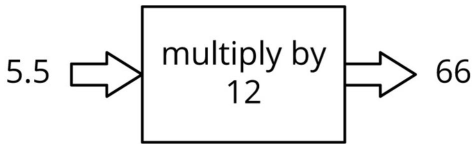
6. You are given a number. Do you know the number that is $\frac{1}{5}$ as big?

7. You are given a number. Do you know its reciprocal?

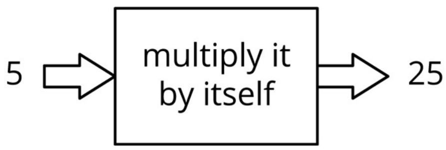
Student Response

Answers vary. Sample responses:

1. Yes, multiply a person's height in feet by 12 to get their height in inches. Since $5.5 \times 12 = 66$, this person is 66 inches tall.

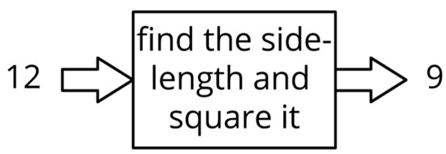


2. Yes, the square of 5 is 25.



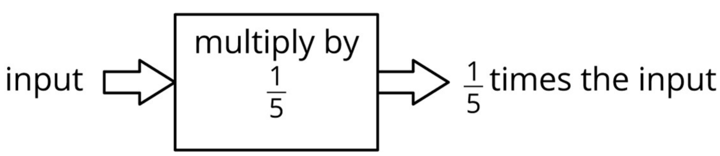
3. No, there are two different numbers whose square is 16, namely 4 and -4.

4. Yes, a square with perimeter 12 cm must have a side length of 3 cm, and so an area of 9 cm^2.

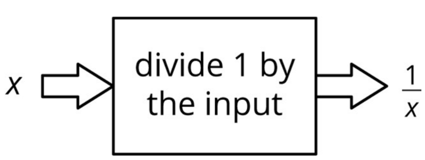


5. No, a rectangle with length 8 cm and width 2 cm has area 16 cm^2, as does a rectangle with length 16 cm and width 1 cm.

6. Yes, for any number, we can simply divide that number by 5.



7. Yes, as long as the number isn't 0 (since 0 doesn't have a reciprocal).



Activity Synthesis

The goal of this discussion is for students to understand that functions are rules that have one distinct output for each input. For several problems, select previously identified students to share their rule descriptions. For example, some students might write the rule for finding the area of a square from its perimeter as "find the side length then square it" and others might write, "divide by four and then square it." Compare the different rules and ask students if they agree (or disagree) that the statements represent the same rule.

For the final two problems where the input is not a specific value, select previously identified students to share what language they used. For example, in the final problem some students may use a letter to stand for the input while others may just write "input" or "a number." Ask, "How can using a letter sometimes make it easier to represent the output?" (Writing "$\frac{1}{5}n$" is shorter than writing "$\frac{1}{5}$ of a number.")

Tell students that each time they answered one of the questions with a yes, the sentence defined a **function**, and that one way to represent a function is by writing a rule to define the relationship between the input and the output. Functions are special types of rules where each input has only one possible output. Because of this, functions are useful since once we know an input, we can find the single output that goes with it. Contrast this with something like a rolling a number cube where the input "roll" has many possible outputs. For the questions students responded to with no, these are not functions because there is no single output for each input.

To highlight how rules that are not functions do not determine outputs in a unique way, end the discussion by asking:

- "Was the warm-up, where you have to square numbers, an example of a function?" (Yes, each input has only one output, even though some inputs have the same output.)
- "Is the reverse, that is knowing what number was squared to get a specific number, a function?" (No, if a number squared is 16, we don't know if the number was 4 or -4.)

Speaking: Discussion Supports. Use this routine to support whole-class discussion. Call on students to use mathematical language to restate and/or revoice the rule descriptions that are presented. Consider providing students time to restate what they hear to a partner, before selecting one or two students to share with the class. Ask the original speaker if their peer accurately restated their rule. Call students' attention to any words or phrases that helped to clarify the original statement. This will provide more students with an opportunity to produce language to describe functions.
Design Principle(s): Support sense-making; Maximise meta-awareness

2.3 Using Function Language

15 minutes

In this activity students revisit the questions in the previous activity and start using the language of functions to describe the way one quantity depends on another. For the "yes: questions students write a statement like, “[the output] depends on [the input]” and “[the output] is a function of [the input].” For the "no" questions, they write a statement like, “[the output] does not depend on [the input].” Students will use this language throughout the rest of the unit and course when describing functions.

Depending on the time available and students' needs, you may wish to assign only a subset of the questions, such as just the odds.

Instructional Routines

- Collect and Display
- Think Pair Share

Launch

Display the example statement from the previous activity ("A person is 60 inches tall. Do you know their height in feet?") for all to see. Tell students that since the answer to this question is yes, we can write a statement like, "height in feet depends on the height in inches" or "height in feet is a function of height in inches."

Arrange students in groups of 2. Give students 5–8 minutes of quiet work time and then additional time to share their responses with their partner. If they have a different response than their partner, encourage them to explain their reasoning and try to reach agreement. Follow with a whole-class discussion.

Student Task Statement

Here are the questions from the previous activity. For the ones you said yes to, write a statement like, “The height a rubber ball bounces to depends on the height it was dropped from” or “Bounce height is a **function** of drop height.” For all of the ones you said no to, write a statement like, “The day of the week does not determine the temperature that day” or “The temperature that day is not a function of the day of the week.”

1. A person is 5.5 feet tall. Do you know their height in inches?
2. A number is 5. Do you know its square?
3. The square of a number is 16. Do you know the number?
4. A square has a perimeter of 12 cm. Do you know its area?
5. A rectangle has an area of 16 cm^2. Do you know its length?

6. You are given a number. Do you know the number that is $\frac{1}{5}$ as big?

7. You are given a number. Do you know its reciprocal?

Student Response

1. Yes, height in feet depends on the height in inches.
2. Yes, the square of a number depends on the number.
3. No, knowing the square of a number does not determine the number.
4. Yes, the area of a square is determined by the perimeter of the square.
5. No, knowing the area of a rectangle does not determine its length.
6. Yes, every number determines the number which is $\frac{1}{5}$ as large.
7. Yes, every (non-zero) number determines its reciprocal.

Activity Synthesis

The goal of this discussion is for students to use the language like "[the output] depends on [the input]" and "[the output] is a function of [the input]" while recognising that a function means each input gives exactly one output.

Begin the discussion by asking students if any of them had a different response from their partner that they were not able to reach agreement on. If any groups say yes, ask both partners to share their responses. Next, select groups to briefly share their responses for the other questions and address any questions. For example, students may have a correct answer but be unsure since they used different wording than the person who shared their answer verbally with the class.

If time permits, give groups 1–2 minutes to invent a new question like the ones in the task that is *not* a function. Select 2–3 groups to share their question and ask a different group to explain why it is not a function using language like, "the input does not determine the output because. . . ."

Speaking, Writing: Collect and Display. While pairs are working, circulate and listen to student talk as they use the language like "[the output] depends on [the input]" and "[the output] is a function of [the input]." Capture student language that reflect a variety of ways to describe that a function means each input gives exactly one output. Display the language collected visually for the whole class to use as a reference during further discussions throughout the lesson and unit. Invite students to suggest revisions, updates, and connections to the display as they develop new mathematical ideas and new ways of communicating ideas. This will help students increase awareness of use of maths language as they progress through the unit.
Design Principle(s): Support sense-making; Maximize meta-awareness

2.4 Same Function, Different Rule?

Optional: 5 minutes

The activity calls back to a previous lesson where students filled out tables of values from input-output diagrams. Here, students determine if a rule is describing the same function but with different words, giving them an opportunity to look for and make use of the structure of a function.

Students are given 3 different input-output diagrams and need to determine which rules could describe the same function. A key point in this activity is that context plays an important role. For example, if the first rule is limited to positive inputs and the second rule is about sides of squares (which also has only positive inputs), then the two input-output rules describe the same function.

Instructional Routines

- Stronger and Clearer Each Time

Launch

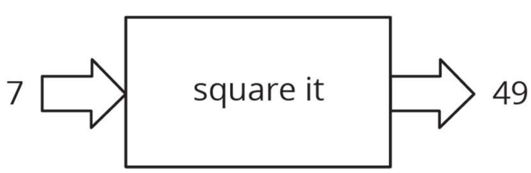
Give students 1–2 minutes of quiet work time followed by a whole-class discussion.

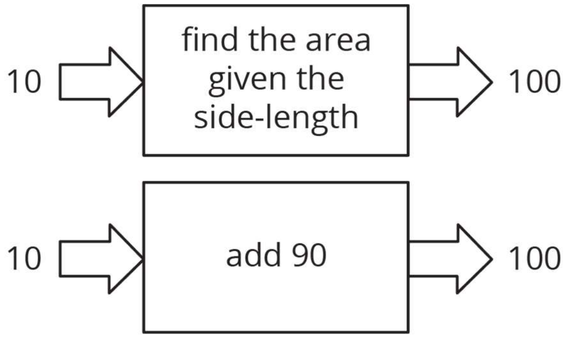
Action and Expression: Develop Expression and Communication. To help get students started, display sentence frames such as “Is it always true that…?” or “That could/couldn’t be the same function because…”
Supports accessibility for: Language; Organization Writing, Listening, Speaking: Stronger and Clearer Each Time. Use this routine to give students a structured opportunity to revise and refine their explanations for "Which input-output rules could describe the same function (if any)?" Give students time to meet with 2–3 partners, to share and get feedback on their responses. Provide prompts for feedback that will help students strengthen their ideas and clarify their language (e.g., “Can you say more about why?”, “Can you give an example?”, “How could you say that another way?”, etc.). Give students 1–2 minutes to revise their writing based on the feedback they received.

Design Principle(s): Optimize output (for explanation)

Student Task Statement

Which input-output rules could describe the same function (if any)? Be prepared to explain your reasoning.





Student Response

The first two rules define the same function if we are talking about the area of a square. Since the area of a square is the square of its side length, the second rule is the same as taking the input and squaring it to find the output. On the other hand, even though adding 90 to 10 gives the same result as squaring 10, this third rule does not define the same function as the first two. For example, adding 90 to 20 gives 110, which is not the same as squaring it, or finding the area of the square with side length 20.

Are You Ready for More?

The phrase "is a function of" gets used in non-mathematical speech as well as mathematical speech in sentences like, "The range of foods you like is a function of your upbringing." What is that sentence trying to convey? Is it the same use of the word "function" as the mathematical one?

Student Response

Answers vary.

Activity Synthesis

The goal of this discussion is for students to explain how two different rules can describe the same function and that two functions are the same if and only if all of their input-output pairs are the same.

Consider asking some of the following questions:

- "Do the latter two input-output rules describe the same function since they both take an input of 10 to an output of 100?" (No, every input-output pair needs to match in order for the two rules to describe the same function not just a few input-output pairs.)
- "Do any of the input-output rules describe the same function?" (Yes, if the first is restricted to positive inputs and the second is about areas of squares, then they share the same input-output pairs and describe the same function.)

Lesson Synthesis

The purpose of this lesson was to define functions as rules that assign exactly one output to each allowable input. We say things like “the output is a **function** of the input,” and “the output depends on the input” when talking about the relationship between inputs and outputs of functions.

To highlight the language and definition of functions from the lesson, ask:

- “How else could we describe the function 'double the input'?” (Multiply the input by 2 or, if the input is x, $2x$.)
- “Is the rule 'the radius of a circle with circumference C' a function? Why or why not?” (Yes, this rule is a function because the radius of circle depends on the circumference and each radius gives only one circumference.)
- “Why does the description 'A person's age is 14 years old. What is their height in inches?' not define a function?” (The age of a person does not determine what height they are. Different 14 year olds are different heights. The same 14 year old can be different heights depending on how long ago they turned 14.)

2.5 Wait Time

Cool Down: 5 minutes

Student Task Statement

You are told that you will have to wait for 5 hours in a line with a group of other people. Determine whether:

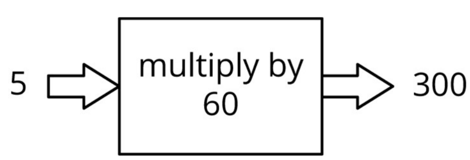
1. You know the number of minutes you have to wait.
2. You know how many people have to wait.

For each statement, if you answer yes draw an input-output diagram and write a statement that describes the way one quantity depends on another.

If you answer no give an example of 2 outputs that are possible for the same input.

Student Response

1. Yes, if you know how many hours you have to wait in line, then you can determine the number of minutes you have to wait in line. Answers vary. Sample response: You will have to wait 300 minutes, since each hour is 60 minutes, and $5 \times 60 = 300$.

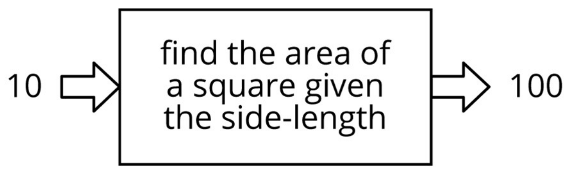


2. No, if you know how many hours you have to wait in line, you do not necessarily know how many people are in line. Answers vary. Sample response: The number of people who have to wait cannot be determined by the amount of time you have to wait. For example, there could be 50 people waiting, or there could be 100 people waiting.

Student Lesson Summary

Let's say we have an input-output rule that for each allowable input gives exactly one output. Then we say the output *depends* on the input, or the output is a **function** of the input.

For example, the area of a square is a function of the side length, because you can find the area from the side length by squaring it. So when the input is 10 cm, the output is 100 cm^2.



Sometimes we might have two different rules that describe the same function. As long as we always get the same, single output from the same input, the rules describe the same function.

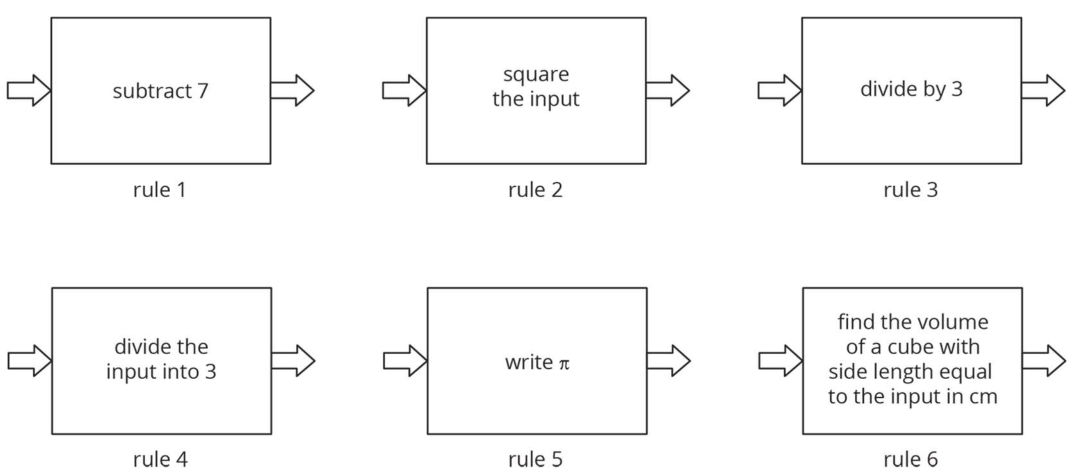
Glossary

- function

Lesson 2 Practice Problems

1. **Problem 1 Statement**

   Here are several function rules. Calculate the output for each rule when you use -6 as the input.



Solution

- Rule 1: -13
- Rule 2: 36
- Rule 3: -2
- Rule 4: -$\frac{1}{2}$
- Rule 5: π
- Rule 6: -6 is not a valid input for this rule since it doesn't make sense to express a side length with a negative number.

2. **Problem 2 Statement**

A group of students is timed while sprinting 100 metres. Each student’s speed can be found by dividing 100 m by their time. Is each statement true or false? Explain your reasoning.

a. Speed is a function of time.

b. Time is a function of distance.

c. Speed is a function of number of students racing.

d. Time is a function of speed.

Solution

a. True. For each time, one speed is generated.

b. False. For each distance (100 m), many times are generated.

c. False. The number of students racing does not affect any student’s speed, and the same speed may be reached for more than one student in a group of the same size.

d. True. For each speed calculated, there is only one possible time.

3. **Problem 3 Statement**

Diego’s history teacher writes a test for the class with 26 questions. The test is worth 123 points and has two types of questions: multiple choice worth 3 points each, and essays worth 8 points each. How many essay questions are on the test? Explain or show your reasoning.

Solution

9 essay questions. Explanations vary. Sample response: Use m to represent multiple choice questions and e for essay questions. Write the system as $m + e = 26$ and $3m + 8e = 123$, and solve it by substituting $m = 26 - e$ into the second equation.

4. **Problem 4 Statement**

These tables correspond to inputs and outputs. Which of these input and output tables could represent a function rule, and which ones could not? Explain or show your reasoning.

Table A:

input	output
-2	4
-1	1
0	0
1	1
2	4

Table B:

input	output
4	-2
1	-1
0	0
1	1
4	2

Table C:

input	output
1	0
2	0
3	0

Table D:

input	output
0	1
0	2
0	3

Solution

Table A and table C represent functions, but table B and table D do not. Explanations vary. Sample response: Tables B and D have multiple outputs for the same input, but functions take each input to only one output. On the other hand, it is okay for a function rule to take different inputs to the same output.