Lesson 8: Translating to $y = mx + c$

Goals

- Coordinate (orally) features of the equation $y = c + mx$ to the graph, including lines with a negative y-intercept.
- Create and compare (orally and in writing) graphs that represent linear relationships with the same rate of change but different initial values.

Learning Targets

- I can explain where to find the gradient and vertical intercept in both an equation and its graph.
- I can write equations of lines using $y = mx + c$.

Lesson Narrative

This lesson develops a third way to understand an equation for a line in the coordinate plane. In previous lessons, students wrote an equation of a line by generalising from repeated calculations using their understanding of similar triangles and gradient. They have also written an equation of a linear relationship by reasoning about initial values and rates of change and have graphed the equation as a line in the plane. This lesson introduces the idea that any line in the plane can be considered a vertical translation of a line through the origin.

In the previous lesson, the terms in the expression are more likely to be arranged $c + mx$ because the situation involves a starting amount and then adding on a multiple. In this lesson, $mx + c$ is more likely because the situation involves starting with a relationship that includes $(0,0)$ and shifting up or down. Students continue to only consider lines with positive gradients, but in this lesson, the notion of a negative y-intercept (not in a context) is introduced.

In addition, students match lines presented in many different forms: equation, graph, description, table. This combines much of what they have learned about lines in this unit, including gradient and vertical intercept.

Addressing

- Understand the connections between proportional relationships, lines, and linear equations.
- Verify experimentally the properties of rotations, reflections, and translations:

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Compare and Connect

• Discussion Supports

Required Materials

Geometry toolkits

tracing paper, graph paper, coloured pencils, scissors, and an index card to use as a straightedge or to mark right angles, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Pre-printed cards, cut from copies of the blackline master Translating a line

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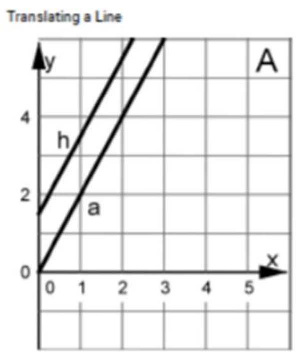
<td>
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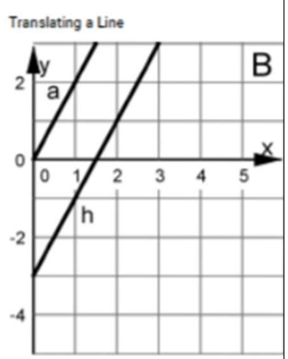
<td>
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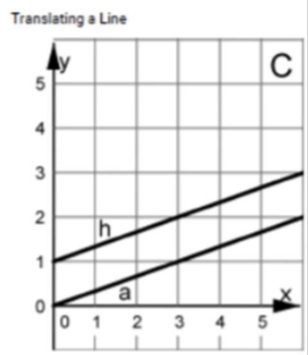
<td>
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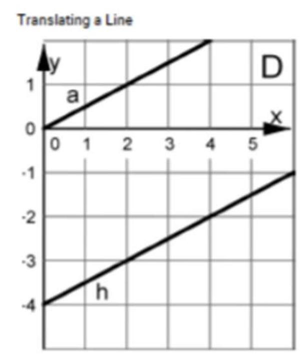
<td>
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<tr>
<td><table><tr><th>x</th><th>y</th></tr><tr><td>0</td><td>1.5</td></tr><tr><td>1</td><td>3.5</td></tr><tr><td>1.5</td><td>4.5</td></tr><tr><td>2.75</td><td>7</td></tr></table></td>
<td>The y-coordinate is three less than double the x-coordinate.</td>
<td>The y-coordinate is one more than a third of the x-coordinate.</td>
<td><table><tr><th>x</th><th>y</th></tr><tr><td>0</td><td>-4</td></tr><tr><td>2</td><td>-3</td></tr><tr><td>4</td><td>-2</td></tr><tr><td>8</td><td>0</td></tr></table></td>
</tr>
<tr>
<td>$y = \frac{3}{2} + 2x$</td>
<td>$2x - 3 = y$</td>
<td>$y = \frac{1}{3}x + 1$</td>
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Required Preparation

Print and cut up slips from the Translating a Line blackline master. Prepare 1 set of cards for every 2 students (this is not needed if doing the digital version).

Student Learning Goals

Let's see what happens to the equations of translated lines.

8.1 Lines that Are Translations

Warm Up: 5 minutes

The purpose of this warm-up is to remind students that the translation of a line is parallel to the original line, and to plant the seed that a line can be taken to a parallel line by translating it. They inspect several lines to decide which could be translations of a given line. Then they describe the translations by specifying the number of units and the direction the original line should be translated.

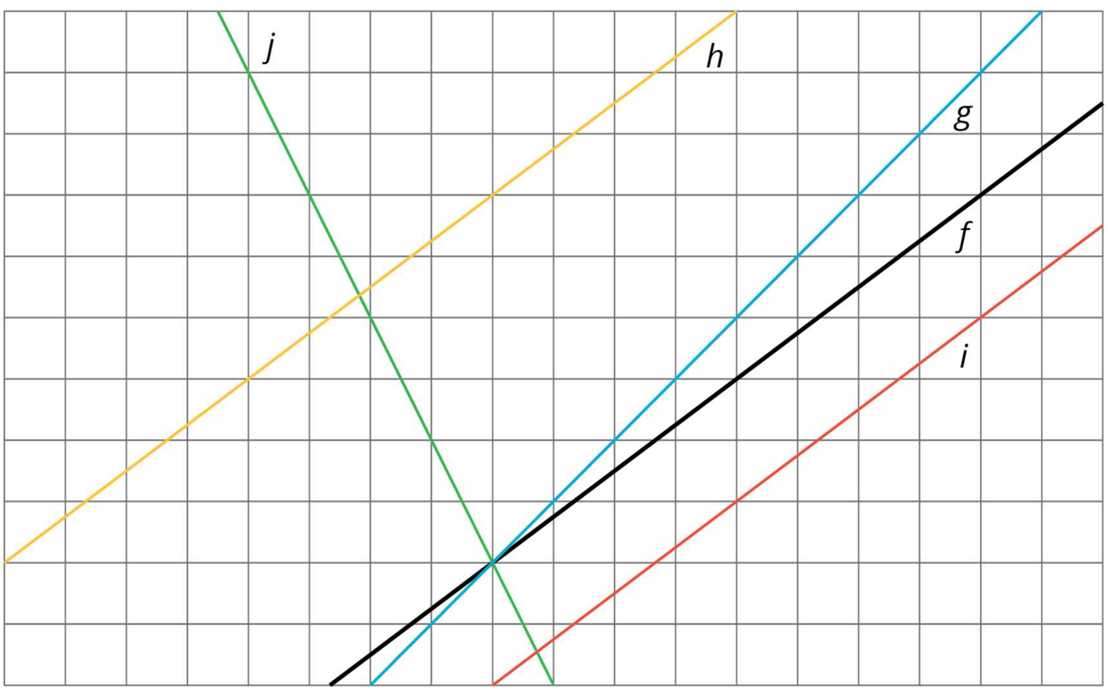
Launch

Arrange students in groups of 2. Give students 2 minutes of quiet think time and access to geometry toolkits. Ask them to share their responses with a partner afterwards.

Anticipated Misconceptions

Students may think that lines i and h can't be images of line f because the part of i and h we can see is shorter than the part of f we can see. Tell these students that all of the lines go on indefinitely in both directions.

Student Task Statement



The diagram shows several lines. You can only see part of the lines, but they actually continue forever in both directions.

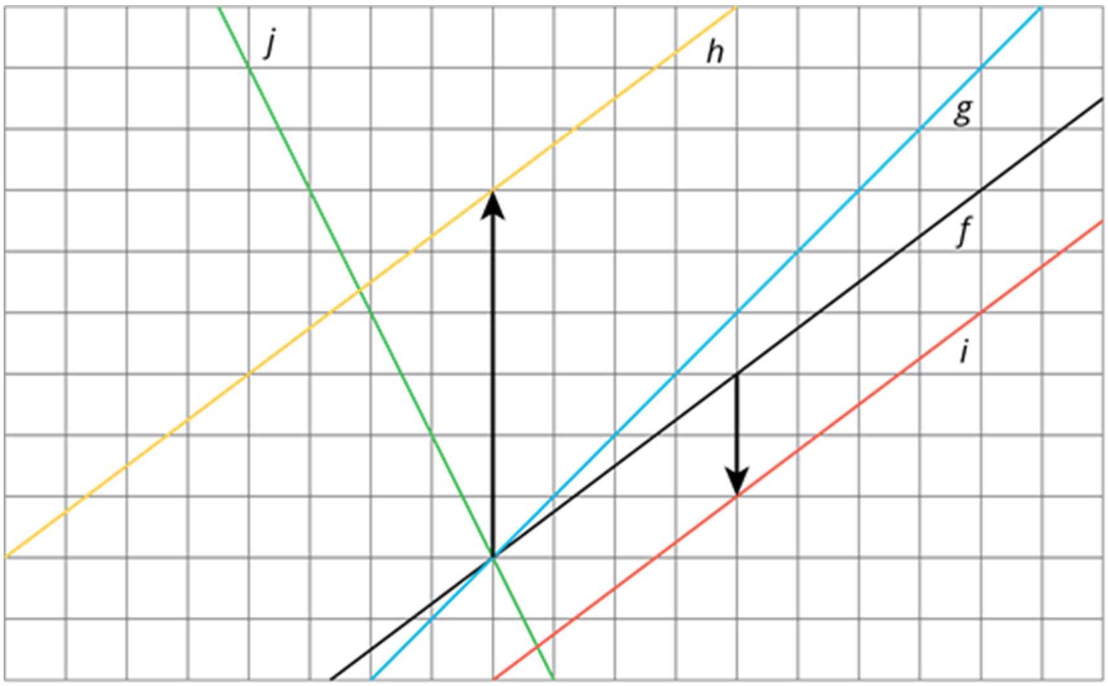
1. Which lines are images of line f under a translation?

2. For each line that is a translation of f, draw an arrow on the grid that shows the vertical translation distance.

Student Response

1. h and i. They appear to be parallel to f, and translated lines are parallel to the original.

2. h is f translated up 6 units. i is f translated down 2 units.



Activity Synthesis

Invite students to share how they know that lines h and i are translations of f. The main reason to bring out is that they are parallel to f. It might be worth demonstrating with a transparency that we can translate f to match up with h and i, but to match up with the other lines, we would need to rotate (or reflect) the transparency. If using a transparency to demonstrate the number of units to translate f up or down, it is helpful to draw a dot on a specific point on both the underlying graph and on the transparency.

8.2 Increased Savings

15 minutes (there is a digital version of this activity)

The goal of this activity is to get students to think about translations of lines in a context. Students examine two scenarios. In the first, there is a proportional relationship between the number of hours Diego works and his total earnings. In the second, Diego starts with £30 saved and then saves all of his earnings. Graphically, the two lines showing these relationships are parallel (the second is a vertical translation of the first for the extra £30 Diego has already saved). The lines have the same gradient (Diego's hourly rate of pay is the same) but different y-intercepts (one has a y-intercept of 0 while the other has a y-

intercept of 30). Students will observe this structure in the equations that they write for the two lines.

Even though babysitters are most often paid for increments of 1 hour or $\frac{1}{2}$ hour and in increments of £1, this uses a continuous line to represent this relationship. If a student brings up the idea that it would be better to represent the relationship using discrete points than a line, acknowledge the observation but suggest that a continuous line is an acceptable representation. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation.

Monitor for students who use these approaches as they graph the two earnings scenarios:

- Making a table (showing earnings for different numbers of hours worked) and then graphing
- Plotting points directly
- Writing an equation that represents the situation and then graphing the solutions to the equation

Select students using these approaches and invite them to present during the discussion.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Compare and Connect

Launch

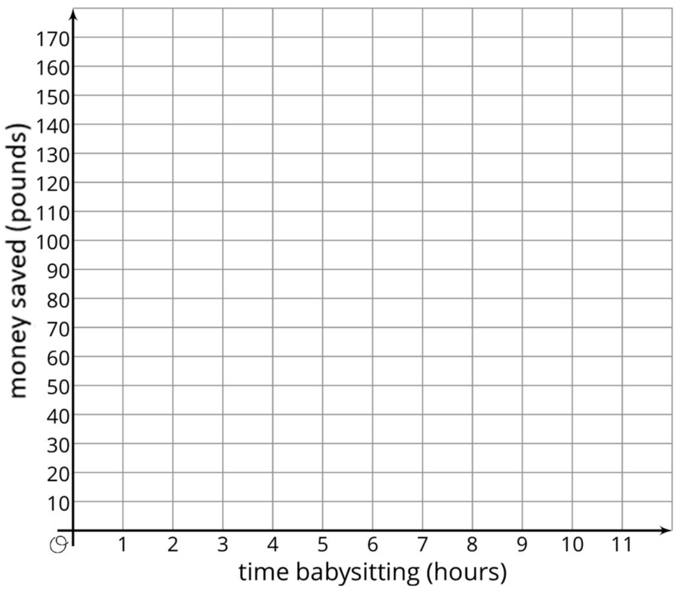
Give students 5 minutes of quiet work time and tell them to pause after the first 3 questions. After they have discussed their responses with a partner, ensure that students have graphed both situations correctly and can articulate that the second graph can be viewed as a vertical translation of the first graph. Then, instruct students to complete the remaining question either independently or with a partner.

If using the digital activity, the mathematical thinking and reasoning remains the same. The applet, however, will assist students in creating graphs to model the babysitting scenarios. Furthermore, some students may start exploring translations in general by creating multiple rays. The structure for the lesson will remain the same for print and digital.

Engagement: Develop Effort and Persistence. Encourage and support opportunities for peer interactions. Invite students to talk about their ideas with a partner before writing them down. Display sentence frames to support students when they explain their strategy. For example, “First I ____ because….” or “One thing that is different about the two lines is ….”
Supports accessibility for: Language; Social-emotional skills

Student Task Statement

1. Diego earns £10 per hour babysitting. Assume that he has no money saved before he starts babysitting and plans to save all of his earnings. Graph how much money, y, he has after x hours of babysitting.



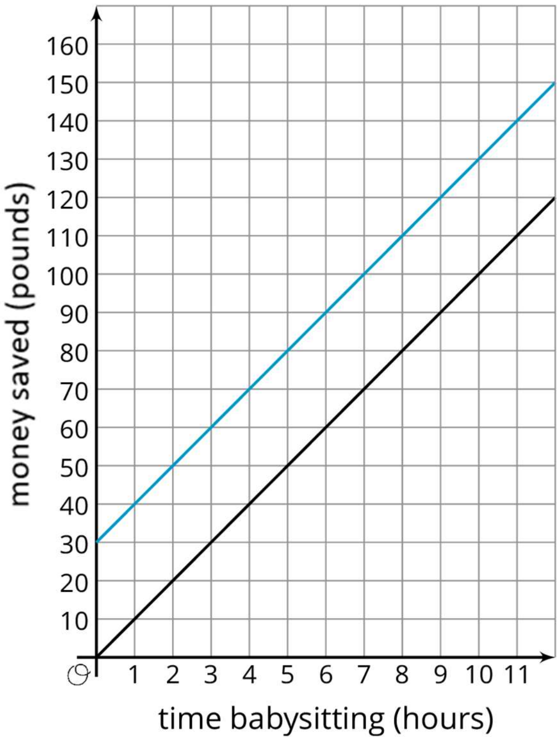
2. Now imagine that Diego started with £30 saved before he starts babysitting. On the same set of axes, graph how much money, y, he would have after x hours of babysitting.

3. Compare the second line with the first line. How much *more* money does Diego have after 1 hour of babysitting? 2 hours? 5 hours? x hours?

4. Write an equation for each line.

Student Response

1 and 2. The line through $(0,0)$ is the solution to first question, and the line through $(0,30)$ is the solution to the second question:



3. He has £30 more in the second case, no matter how many hours he babysits. Can be seen as a vertical translation up by 30 pounds.

4. $y = 10x$ for the first question, and $y = 30 + 10x$ or equivalent for the second question.

Activity Synthesis

Invite selected students to share their methods for graphing Diego's earnings, sequenced in this order:

- Making a table (showing earnings for different numbers of hours worked) and then graphing
- Plotting points directly
- Writing an equation that represents the situation and then graphing the solutions to the equation

Notice that making a table has the advantage of revealing the *arithmetic* relationship between Diego's earnings in the two situations. No matter how many hours, Diego works, he has £30 more in the second situation. Plotting the points directly shows geometrically that the vertical distance is always the same between pairs of points for the same number of hours worked (this vertical distance represents the £30). The equations are the most abstract and contain all of the arithmetic and geometric information once we can interpret them. If h is the number of hours Diego works and m is how much money he has saved, then the two equations can be written as $m = 10h$ and $m = 30 + 10h$. The graphs of these lines have gradient 10 (and they are parallel!), but the second equation has a y-

intercept of 30 while the first has a y-intercept of 0. The graph of $m = 30 + 10h$ is the same as the graph of $m = 10h$ except that every point is moved up by the same amount (this explains why they are parallel).

Have students connect the equations and situations they have graphed in the activity by asking students what the graph of $m = 20 + 10h$ would look like (it would lie between the graphs of $m = 30 + 10h$ and $m = 10h$). What situation does this equation represent? (Diego started with £20 saved and then earned £10 for each hour of babysitting.)

Representing, Conversing: Compare and Connect. To begin the whole-class discussion, arrange students in groups of 2, and invite students to compare the methods they used to graph Diego's earnings. Ask students to discuss what is the same and what is different about their approaches. Once the class has discussed the various methods, spend some time focusing on the connections between the equations and other representations. Ask students, "Where does the '$10x$' in each equation come from?" "How is it represented in each representation?" Listen for connections that relate to the differences in methods students discussed prior to starting the synthesis. This will help students better understand the connection between the multiple representations.
Design Principle(s): Optimise output (for comparison); Cultivate conversation

8.3 Translating a Line

15 minutes (there is a digital version of this activity)

The previous activity examined parallel lines and equations that define them, focusing on their common attributes (gradient) and their different attributes (y-intercept). This activity further examines parallel lines, including situations where the y-intercept is negative. In addition, students match lines represented in many different ways including

- graphically
- verbal description
- table of values
- equations

You will need the Translating a Line blackline master for this activity if not using the interactive version.

Instructional Routines

- Discussion Supports

Launch

Arrange students in groups of 2. Students first identify equations that describe a line with a negative y-intercept, then they complete a matching activity involving graphs, equations, and tables.

Display the image in the activity. Ask students how the pair of lines is the same and different from the lines in the previous activity. Instead of being translated up, line a was translated down. Ask students to explain how to change $y = \frac{1}{4}x$, an equation representing line a, to represent line h. They should readily come up with $y = \frac{1}{4}x - 5$. Ask students for another way to write the same equation, for example, $\frac{1}{4}x - 5 = y$. The first part of the activity is about finding different ways to write this equation.

Tell students to complete the first part of the activity and then pause for discussion. Ensure that all understand why the incorrect equations are incorrect, then distribute the cards to sort. Tell students to take turns identifying cards that go together, and for each choice they make, explain to their partner why they think they go together. Once they have sorted them into groups, they write an equation on the blank card to represent the line that does not have an equation.

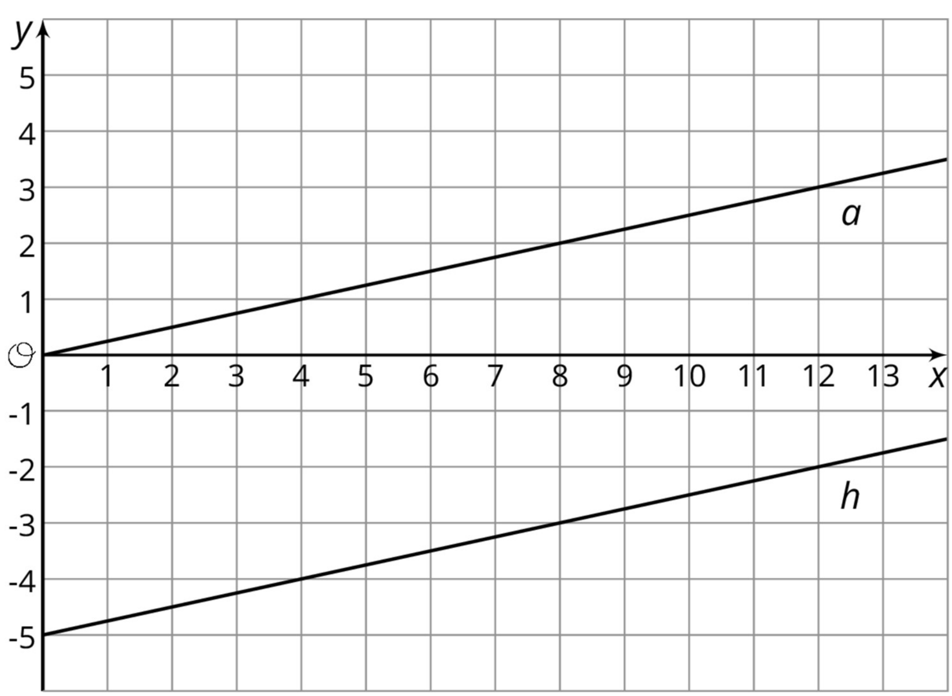
Representation: Internalise Comprehension. Demonstrate and encourage students to use colour coding and annotations to highlight connections between graphs, equations and tables to justify their matches. For example, use the same colour to highlight connections between line h and the matching equation.
Supports accessibility for: Visual-spatial processing

Student Task Statement

This graph shows two lines.

Line a goes through the origin (0,0).

Line h is the image of line a under a translation.



1. Select all of the equations whose graph is line h.

   a. $y = \frac{1}{4}x - 5$

   b. $y = \frac{1}{4}x + 5$

   c. $\frac{1}{4}x - 5 = y$

   d. $y = \text{-}5 + \frac{1}{4}x$

   e. $\text{-}5 + \frac{1}{4}x = y$

   f. $y = 5 - \frac{1}{4}x$

2. Your teacher will give you 12 cards. There are 4 pairs of lines, A–D, showing the graph, a, of a proportional relationship and the image, h, of a under a translation. Match each line h with an equation and either a table or description. For the line with no matching equation, write one on the blank card.

Student Response

1. Equations A, C, D, E describe line h.

2. The blackline master shows the correct matching. The equation students must write is $y = \frac{1}{2}x - 4$ or equivalent.

Are You Ready for More?

A student says that the graph of the equation $y = 3(x + 8)$ is the same as the graph of $y = 3x$, only translated upwards by 8 units. Do you agree? Why or why not?

Student Response

Disagree. Using the distributive property on $3(x + 8)$ gives $3x + 24$, so the equation is $y = 3x + 24$. This is the graph of $y = 3x$ translated up by 24 units.

Activity Synthesis

Once all groups have completed the matching, discuss the following:

- "Were any of the matches more difficult than others? What made them difficult?"
- "Did any groups have to adjust an initial guess that turned out to be wrong? What adjustments were made and why?"
- "What clues did you look for to see which equation went with a graph?"

Solicit all versions of the missing equation. Ensure that students understand that $y = \frac{1}{2}x - 4$ and $y =$ -4 $+ \frac{1}{2}x$ are equivalent.

Speaking: Discussion Supports. Use this routine to support students in producing statements to describe the reasons for their matches. Provide sentence frames such as: "The equation ______ matches line h on this card because…." or "This description matches line h on this card because…." Encourage students to press for details from their partners by challenging their explanations and/or encouraging mathematical vocabulary. These exchanges strengthen students' use of mathematical language related to representations of translations.
Design Principle(s): Support sense-making, Optimise output (for explanation)

Lesson Synthesis

Display a graph of two lines on the same set of axes: one of the form $y = mx$ and the other of the form $y = mx + c$. Discuss:

- "How can we think of one of these lines as a transformation of the other?"
- "What is the equation of the line that goes through the origin?" (Discuss how you need to figure out the gradient.)
- "How is the equation of the line that does not go through the origin different?" (Make sure to bring out that the c in $mx + c$ gives the vertical translation to get from the graph of $y = mx$ to the graph of $y = mx + c$; the translation is up when $c > 0$ and down when $c < 0$.)

8.4 Similarities and Differences in Two Lines

Cool Down: 5 minutes

Student Task Statement

Describe how the graph of $y = 2x$ is the same and different from the graph of $y = 2x - 7$. Explain or show your reasoning.

Student Response

Answers vary. Students may or may not sketch graphs as part of their solution.

Possible responses to how they are the same:

- They have the same gradient.
- They both have a gradient of 2.
- They are parallel to each other.

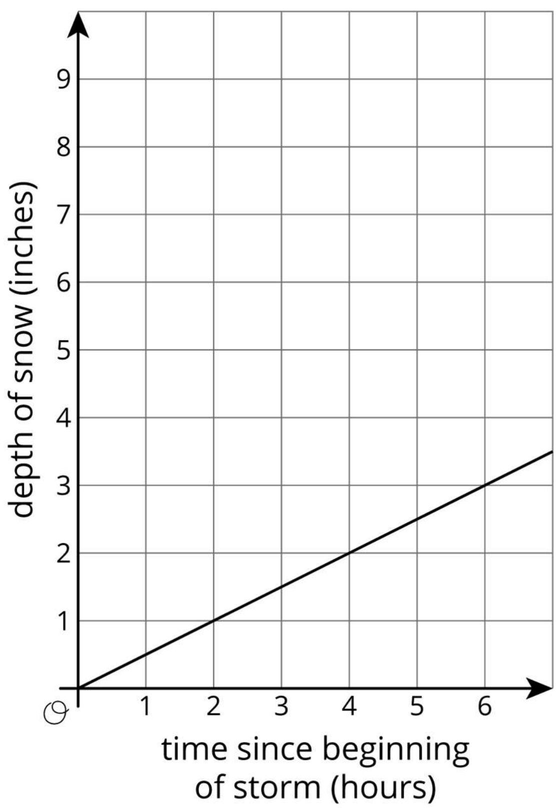
Possible responses to how they are different:

- They are in a different location.
- One is a translation of the other.
- They cross the y-axis (or x-axis) at a different point.
- They don't go through any of the same points.

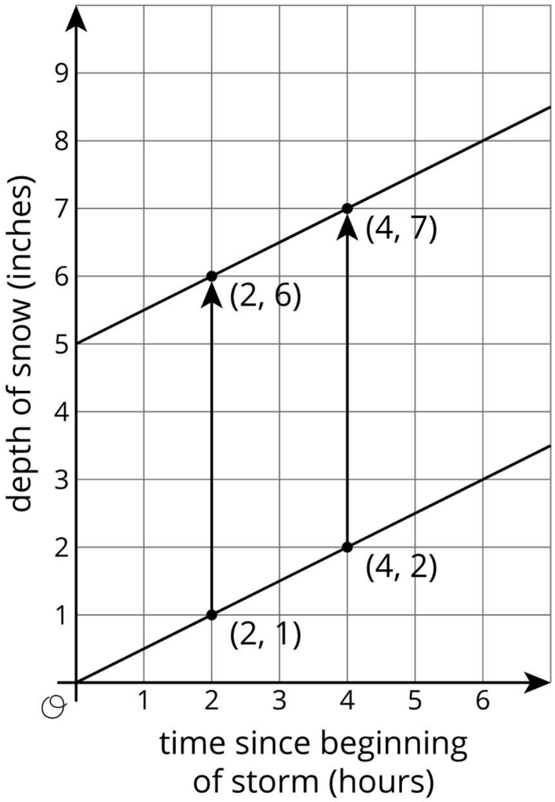
Student Lesson Summary

During an early winter storm, the snow fell at a rate of $\frac{1}{2}$ inch per hour. We can see the rate of change, $\frac{1}{2}$, in both the equation that represents this storm, $y = \frac{1}{2}x$, and in the gradient of the line representing this storm.

In addition to being a linear relationship between the time since the beginning of the storm and the depth of the snow, we can also call this as a proportional relationship since the depth of snow was 0 at the beginning of the storm.



During a mid-winter storm, the snow again fell at a rate of $\frac{1}{2}$ inch per hour, but this time there was already 5 inches of snow on the ground.



We can graph this storm on the same axes as the first storm by taking all the points on the graph of the first storm and translating them up 5 inches.

Two hours after each storm begins, 1 inch of new snow has fallen. For the first storm, this means there is now 1 inch of snow on the ground. For the second storm, this means there are now 6 inches of snow on the ground.

Unlike the first storm, the second is not a proportional relationship since the line representing the second storm has a vertical intercept of 5. The equation representing the storm, $y = \frac{1}{2}x + 5$, is of the form $y = mx + c$, where m is the rate of change, also the gradient of the graph, and c is the initial amount, also the vertical intercept of the graph.

Lesson 8 Practice Problems

1. **Problem 1 Statement**

   Select **all** the equations that have graphs with the same y-intercept.

   a. $y = 3x - 8$

   b. $y = 3x - 9$

   c. $y = 3x + 8$

   d. $y = 5x - 8$

   e. $y = 2x - 8$

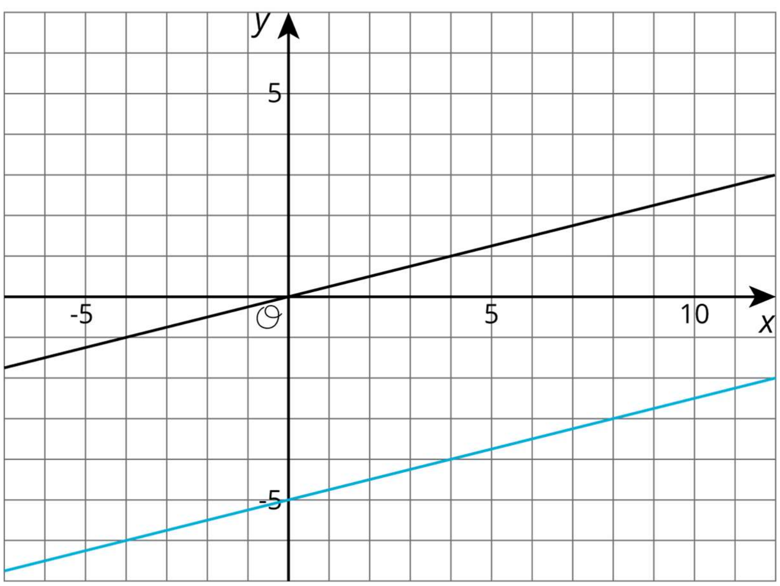
   f. $y = \frac{1}{3}x - 8$

   Solution ["A", "D", "E", "F"]

2. **Problem 2 Statement**

   Create a graph showing the equations $y = \frac{1}{4}x$ and $y = \frac{1}{4}x - 5$. Explain how the graphs are the same and how they are different.

   Solution



   Answers vary. Sample response: The graphs have the same gradient of $\frac{1}{4}$ but different y-intercepts. The first is 0, and the second is -5. Another sample response: Each (x, y) on the first graph is translated down by 5 to get a corresponding point on the second graph.

3. **Problem 3 Statement**

A cable company charges £70 per month for cable service to existing customers.

a. Find a linear equation representing the relationship between x, the number of months of service, and y, the total amount paid in pounds by an existing customer.

b. For new customers, there is an additional one-time £100 service fee. Repeat the previous problem for new customers.

c. When the two equations are graphed in the coordinate grid, how are they related to each other geometrically?

Solution

a. $y = 70x$

b. $y = 70x + 100$

c. The two lines are parallel, with the second line being the first line translated vertically 100 units upwards.

4. **Problem 4 Statement**

A mountain road is 5 miles long and gains height at a constant rate. After 2 miles, the height is 5 500 feet above sea level. After 4 miles, the height is 6 200 feet above sea level.

a. Find the height of the road at the point where the road begins.

b. Describe where you would see the point in part (a) on a graph where y represents the height in feet and x represents the distance along the road in miles.

Solution

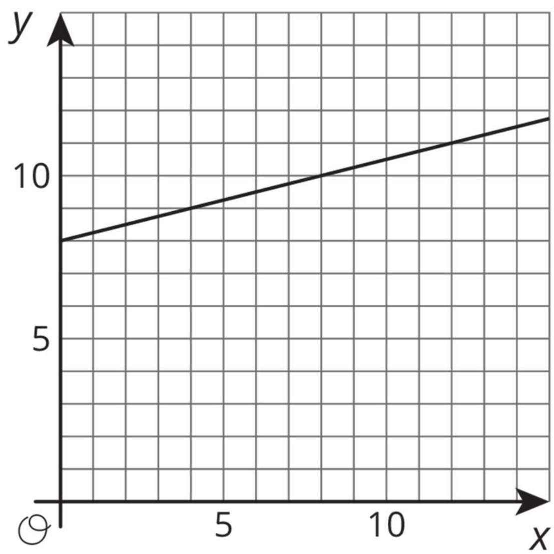
a. 4 800 feet above sea level

b. The point would be $(0, 4\,800)$ located on the y-axis.

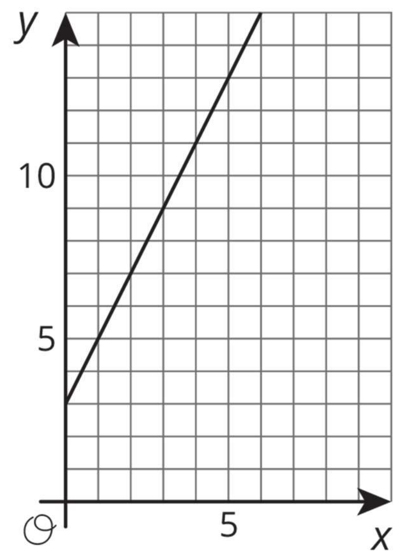
5. **Problem 5 Statement**

Match each graph to a situation.

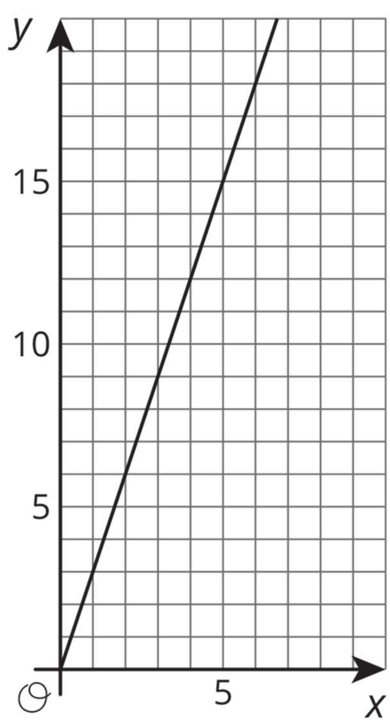
A



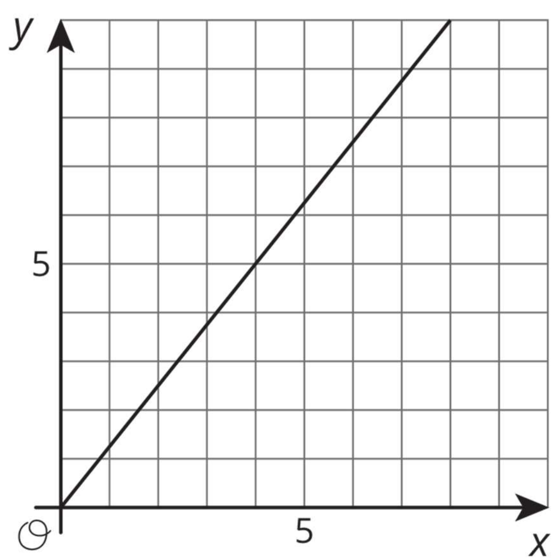
B



C



D



a. Graph A

b. Graph B

c. Graph C

d. Graph D

1. The graph represents the perimeter, y, in units, for an equilateral triangle with side length of x units. The gradient of the line is 3.

2. The amount of money, y, in a cash box after x tickets are purchased for carnival games. The gradient of the line is $\frac{1}{4}$.

3. The number of chapters read, y, after x days. The gradient of the line is $\frac{5}{4}$.

4. The graph shows the cost in pounds, y, of a muffin delivery and the number of muffins, x, ordered. The gradient of the line is 2.

Solution

- A: 2
- B: 4
- C: 1
- D: 3