Lesson 1: Understanding proportional relationships

Goals

- Comprehend that for the equation of a proportional relationship given by $y = kx$, k represents the constant of proportionality.
- Create graphs and equations of proportional relationships in context, including an appropriate scale.
- Interpret diagrams or graphs of proportional relationships in context.

Learning Targets

- I can graph a proportional relationship from a story.
- I can use the constant of proportionality to compare the pace of different animals.

Lesson Narrative

This lesson is the first of four where students work with proportional relationships from a more sophisticated perspective. Embedded alongside their work with proportional relationships, students learn about graphing from a blank set of axes. Attending to precision in labelling axes, choosing an appropriate scale, and drawing lines are skills students work with in this lesson and refine over the course of this unit and in units that follow.

The purpose of this lesson is to get students thinking about what makes a "good" graph by first considering what are the components of a graph (e.g., labels, scale) and then adding scale to graphs of the pace of two bugs. Students also graph a line based on a verbal description of a relationship and compare the newly graphed line to already graphed proportional relationships.

This lesson includes graphs with elapsed time in seconds on the vertical axis and distance travelled in centimetres on the horizontal axis. It is common for people to believe that time is always the independent variable, and should therefore always be on the horizontal axis. This is a really powerful heuristic. The problem is, it isn't true.

In general, a context that involves a relationship between two quantities does not dictate which quantity is the independent variable and which is the dependent variable: that is a choice made by the modeller. Consider this situation: A runner is travelling one mile every 10 minutes. There is more than one way to represent this situation.

- We can say the number of miles travelled, d, depends on the number of minutes that have passed, t, and write $d = 0.1t$. This way of expressing the relationship might be more useful for questions like, "How far does the runner travel in 35 minutes?"
- We can also say that the number of minutes that have passed, t, depends on the number of miles travelled, d, and write $t = 10d$. This way of expressing the

relationship might be more useful for questions like, "How long does it take the runner to travel 2 miles?"

These are both linear relationships. The rate of change in the first corresponds to speed (0.1 miles per minute), and the rate of change in the second corresponds to pace (10 minutes per mile). Both have meaning, and both could be of interest. It is up to the modeller to decide what kinds of questions she wants to answer about the context and which way of expressing the relationship will be most useful in answering those questions.

Building On

- Recognise and represent proportional relationships between quantities.

Addressing

- Understand the connections between proportional relationships, lines, and linear equations.

Building Towards

- Graph proportional relationships, interpreting the unit rate as the slope of the graph. Compare two different proportional relationships represented in different ways. For example, compare a distance-time graph to a distance-time equation to determine which of two moving objects has greater speed.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Clarify, Critique, Correct
- Co-Craft Questions
- Notice and Wonder

Student Learning Goals

Let’s study some graphs.

1.1 Notice and Wonder: Two Graphs

Warm Up: 5 minutes

The purpose of this warm-up is to get a conversation started about what features a graph needs. In the following activities, students will put these ideas to use by adding scale to some axes with two proportional relationships graphed on it.

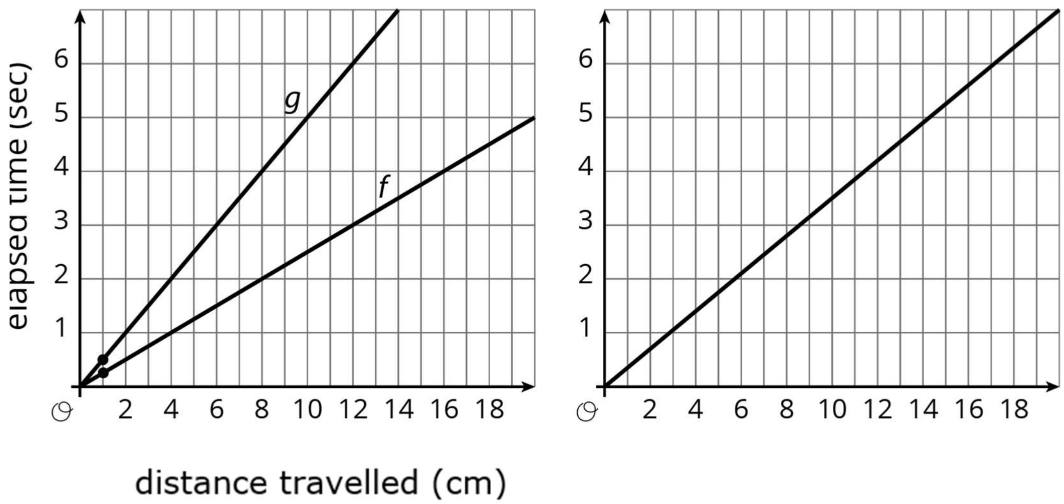
Instructional Routines

- Notice and Wonder

Launch

Tell students they will look at a picture, and their job is to think of at least one thing they notice and at least one thing they wonder about the picture. Display the problem for all to see and ask students to give a signal when they have noticed or wondered about something.

Student Task Statement



What do you notice? What do you wonder?

Student Response

Things students may notice:

- The second set of axes are not labelled
- If the first graph is about speed, then f is twice as fast as g.
- Graph g is something going a speed of 2 centimetres every second
- Graph f is something going a pace of about 0.25 seconds per 1 centimetre.

Things students may wonder:

- What do the two points mean?
- Why does one graph show two lines while the other only has one?
- What do g and f represent?

Activity Synthesis

Invite students to share the things they noticed and wondered. Record and display their responses for all to see. If possible, record the relevant reasoning on or near the image. After each response, ask the class if they agree or disagree and to explain alternative ways of thinking, referring back to the images each time. If the missing labels are not mentioned, make sure to bring them up.

1.2 Moving Through Representations

15 minutes

In this activity, students investigate the paces of two different bugs. Using the chart at the start of the activity, students answer questions about pace, decide on a scale for the axes, and mark and label the time needed to travel 1 cm for each bug (unit rate).

Identify students who use different scales on the axes to share during the Activity Synthesis. For example, some students may count by 0.5 on the distance axis while others may count by 1.

Instructional Routines

- Co-Craft Questions

Launch

Arrange students in groups of 2. Before students start working, ensure that they understand that each bug's position is measured at the front of their head. So for example, in the second diagram, the ladybug has moved 4 centimetres and the ant has moved 6 centimetres.

Ask students to review the images and the first problem in the activity and give a signal when they have finished. Invite students to share their ideas about which bug is represented by line u and which bug is represented by line v. (The ladybug is u, the ant is v.) If not brought up in students' explanations, draw attention to how the graph shows the *pace* of the two bugs—that is, the graph shows how much time it takes to go a certain distance, which is different than a graph of speed, which shows how much distance you go for a certain amount of time.

Give students work time to complete the remaining problems with their partner followed by a whole-class discussion.

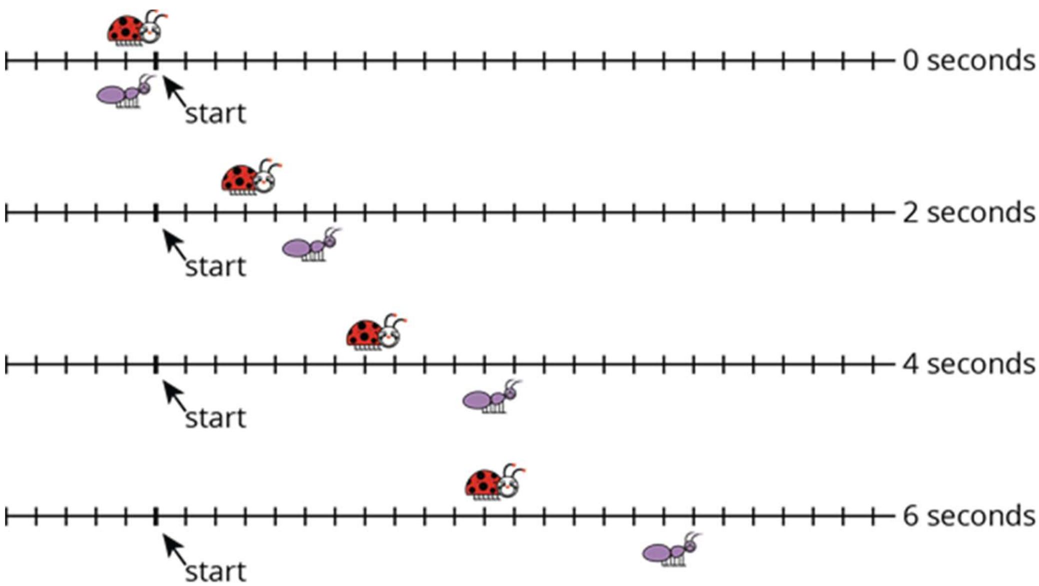
Action and Expression: Develop Expression and Communication. Invite students to talk about their ideas with a partner before writing them down. Display sentence frames to support students when they explain their ideas. For example, "Based on the diagrams, line u represents ___ because…."
Supports accessibility for: Language; Organisation Writing, Speaking: Co-Craft Questions. Use this routine to help students interpret the first image, and to increase awareness of language used to make comparisons about speed and pace. Display only the prompt and images (without the line graphs). Invite students to write possible mathematical questions about the situation. When students share their questions with the class, highlight those that wonder about distance, time and the meaning of tick marks in the diagrams. Reveal the graph and ask students to work on the questions that follow. This helps students produce the language of mathematical questions about different representations for speed.
Design Principle(s): Optimise output; Cultivate conversation

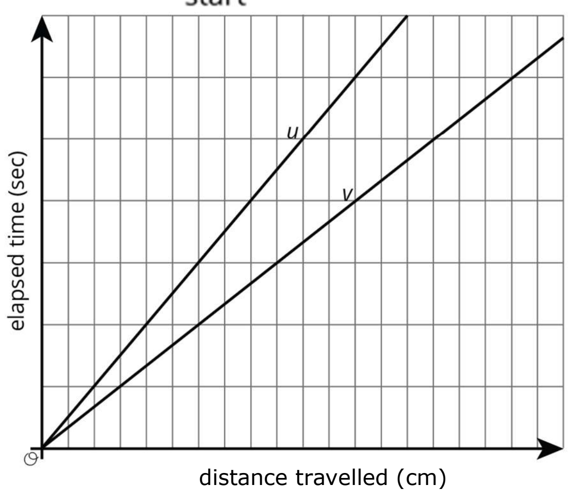
Anticipated Misconceptions

Students might confuse pace with speed and interpret a steeper line as meaning the ladybug is moving faster. Monitor students to ensure that they attend to the time and distance on the tick mark diagrams and plot points as $(distance, time)$ with time on the y-axis and distance on the x-axis. Reinforce language of how many seconds per a given interval of distance. Make explicit that twice as fast means half the pace.

Student Task Statement

A ladybug and ant move at constant speeds. The diagrams with tick marks show their positions at different times. Each tick mark represents 1 centimetre.





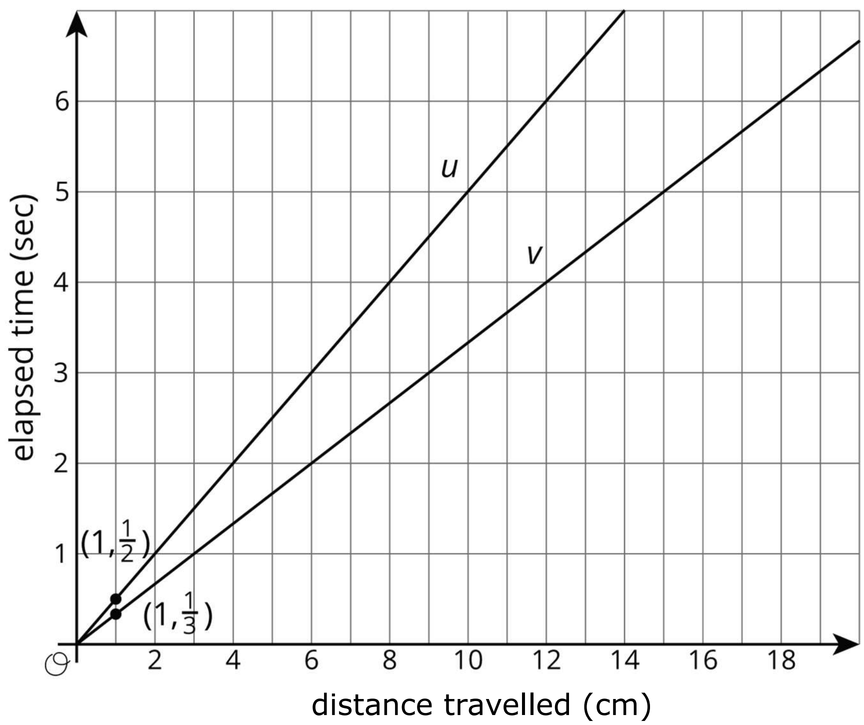
1. Lines u and v also show the positions of the two bugs. Which line shows the ladybug's movement? Which line shows the ant's movement? Explain your reasoning.

2. How long does it take the ladybug to travel 12 cm? The ant?

3. Scale the vertical and horizontal axes by labelling each grid line with a number. You will need to use the time and distance information shown in the tick-mark diagrams.

4. Mark and label the point on line u and the point on line v that represent the time and position of each bug after travelling 1 cm.

Student Response



1. Ladybug: line u, ant: line v

2. Ladybug: 6 seconds, ant: 4 seconds

3. See graph.

4. See graph.

Are You Ready for More?

1. How fast is each bug travelling?

2. Will there ever be a time when the ant is twice as far away from the start as the ladybug? Explain or show your reasoning.

Student Response

1. The red bug (ladybug) is travelling at 2 cm/sec and the purple bug (ant) is travelling at 3 cm/sec.

2. No, the purple bug (ant) is always half as much again as far from the start as the red bug (lady bug).

Activity Synthesis

Display the images from the problem for all to see. Begin the discussion by inviting students to share their solutions for how long it takes each bug to travel 12 cm. Encourage students to reference one or both of the images as they explain their thinking.

Ask previously selected students to share their graphs with added scale and how they decided on what scale to use. If possible, display these graphs for all to see. There are many correct ways to choose a scale for this situation, though some may have made it difficult for students to plot the answer to the final problem. If this happened, highlight these graphs and encourage students to read all problems when they are making decisions about how to construct a graph. Since this activity had a problem asking for information about 1 cm, it makes sense to count by 1s (or even something smaller!) on the distance axis.

1.3 Moving Twice as Fast

15 minutes

In this activity, students use the representations from the previous activity and add a third bug that is moving twice as fast as the ladybug. Students are also asked to write equations for all three bugs. An important aspect of this activity is students making connections between the different representations.

Monitor for students using different strategies to write their equations. For example, some students may reason from the unit rates they can see on their graphs and write equations in the form of $y = kx$, where k is the unit rate (constant of proportionality). Others may write equations of the form $\frac{y}{x} = \frac{b}{a}$, where (a, b) is a point on the line. Select several of these students to share during the discussion.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Clarify, Critique, Correct

Launch

Keep students in the same groups. Give 5–7 minutes work time followed by a whole-class discussion.

Action and Expression: Internalise Executive Functions. Chunk this task into more manageable parts to support students who benefit from support with organisational skills in problem solving. For example, show only one question at a time, pausing to check for understanding before moving on.
Supports accessibility for: Organisation; Attention Speaking: Clarify, Critique, Correct. For the first question “Imagine a bug that is moving twice as fast as the ladybug. On each tick-mark diagram, mark the position of this bug,” display an incomplete statement like, “I looked at how far the ladybug went and made my bug go farther” or a flawed statement like “I put my bug 2 tick marks ahead of the ant.” Invite students to discuss with a partner possible ways

to correct or clarify each statement. This will give students an opportunity to use language to clarify their understanding of proportionality.
Design Principle(s): Maximise meta-awareness

Student Task Statement

Refer to the tick-mark diagrams and graph in the earlier activity when needed.

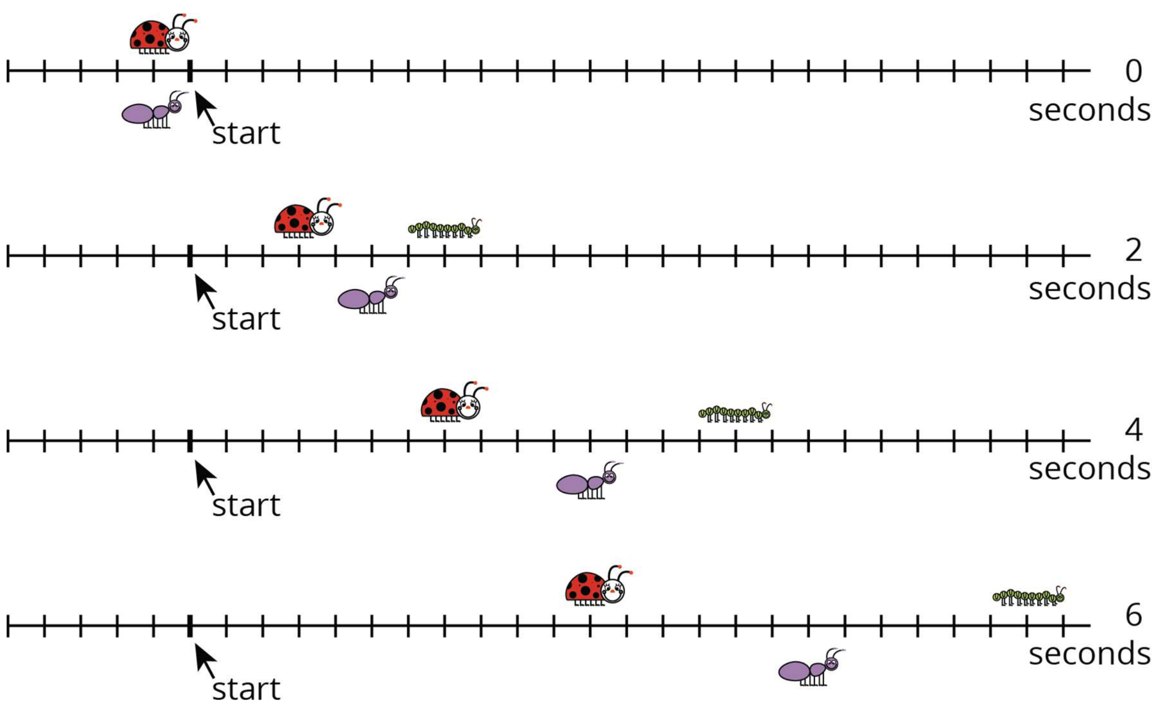
1. Imagine a bug that is moving twice as fast as the ladybug. On each tick-mark diagram, mark the position of this bug.

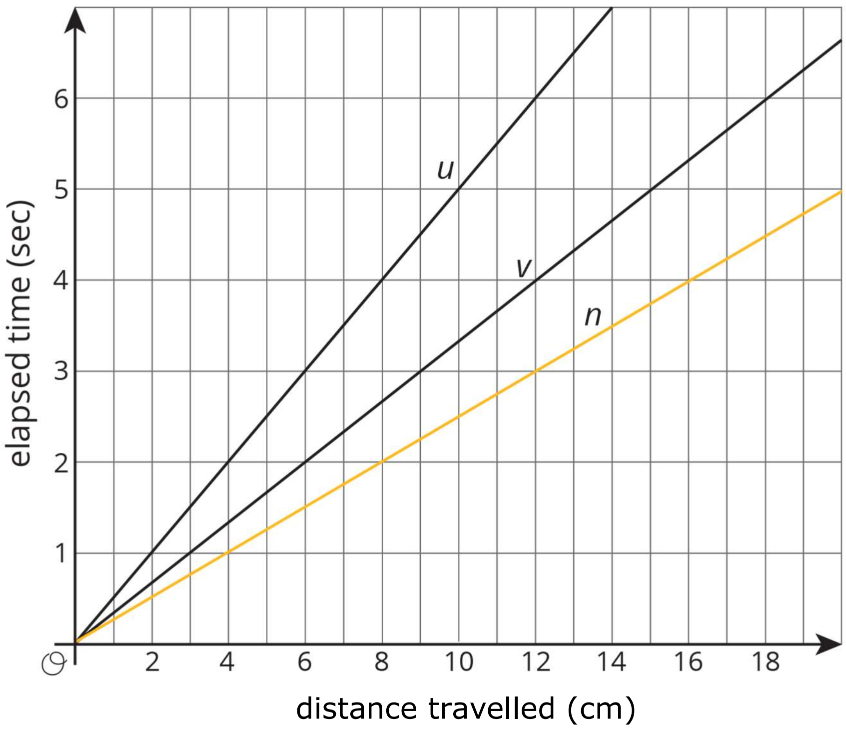
2. Plot this bug's positions on the coordinate axes with lines u and v, and connect them with a line.

3. Write an equation for each of the three lines.

Student Response

1.





2. See graph. Line n represents a bug moving double the ladybug's distance in the same amount of time.

3. Answers vary. Possible response: Equations are ladybug: $y = \frac{1}{2}x$, ant: $y = \frac{1}{3}x$, new bug: $y = \frac{1}{4}x$ (twice as fast as ladybug), where x represents the distance travelled and y represents elapsed time.

Activity Synthesis

Display both images from the previous task for all to see. Invite previously selected students to share their equations for each bug. Sequence students so that the most common strategies are first. Record the different equations created for each bug and display these for all to see.

As students share their reasoning about the equation for the third bug, highlight strategies that support using the equation (original is k and new one is $\frac{1}{2}k$) and graph (less steep, still constant proportionality, half point values). If no students write an equation of the form $y = kx$, do so and remind students of the usefulness of k, the constant of proportionality, when reasoning about proportional relationships.

Consider asking the following questions to help students make connections between the different representations:

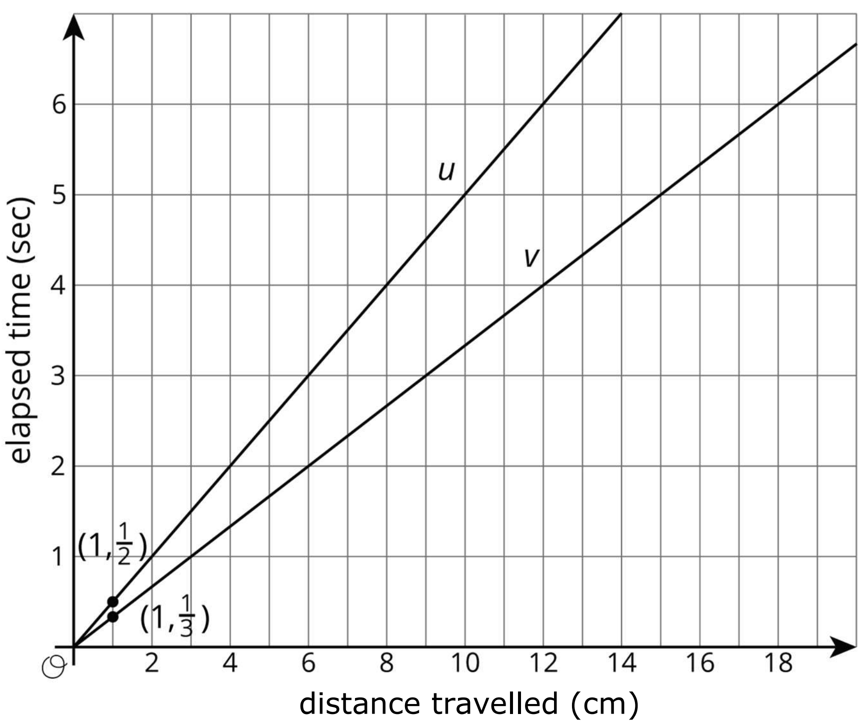
- "What features of the tick-mark diagrams, lines, and equations can you identify that would allow someone to figure out which bug is moving faster?" (The tick-mark diagrams give the coordinates of points that will go on the graph because they show

how far each bug has gone after each amount of time. We can see the positions of the bugs on the tick-mark diagrams so we know which is faster. The graph shows how far they went for any amount of time and the slope helps to show which is faster. Both help compare the movements of the two or three bugs.)

- “The tick-mark diagrams show some of the bugs’ movements, but not all of them. How can we use the graphs of the lines to get more complete information?” (The tick-mark diagrams only show time every 2 seconds. On the graph we can see the bugs’ positions at any point in time.)
- “Are you convinced that your graph (or equation) supports the fact that the new bug is going twice as fast as the ladybug?”

Lesson Synthesis

Display a scaled graph of the two bugs for all to see. Remind students that line u is the ladybug and that line v is the ant.



Ask students:

- “What would the graph of a bug going 3 times faster than the ant look like?” (It would go through the points (0,0), $(1,\frac{1}{9})$, and (9,1).)
- “What would an equation showing the relationship between the bugs' distance and time look like?” (Since it is going 4 times faster and goes through the point (9,1), it has a **constant of proportionality** of $\frac{1}{9}$, which means one equation is $y = \frac{1}{9}x$.)

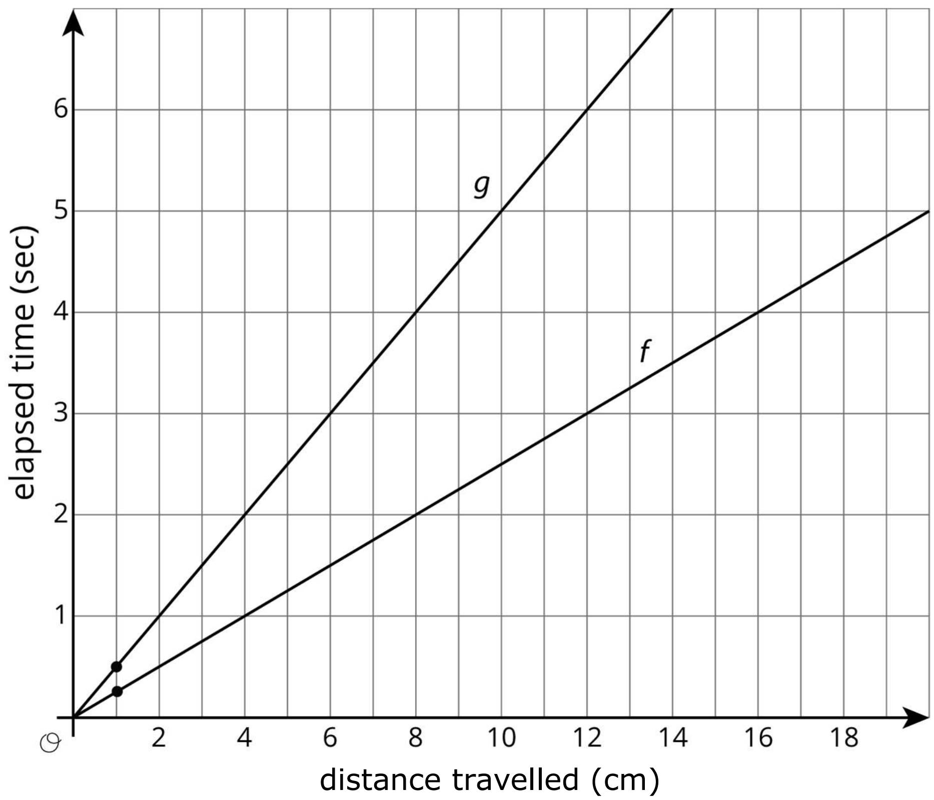
- "If we wanted to scale the graph so we could see how long it takes the ladybug to travel 50 cm, what numbers could we use on the vertical axis?" (The ladybug travels 50 cm in 25 seconds, so the vertical axis would need to extend to at least that value.)

1.4 Turtle Race

Cool Down: 5 minutes

Student Task Statement

This graph represents the positions of two turtles in a race.



1. On the same axes, draw a line for a third turtle that is going half as fast as the turtle described by line g.

2. Explain how your line shows that the turtle is going half as fast.

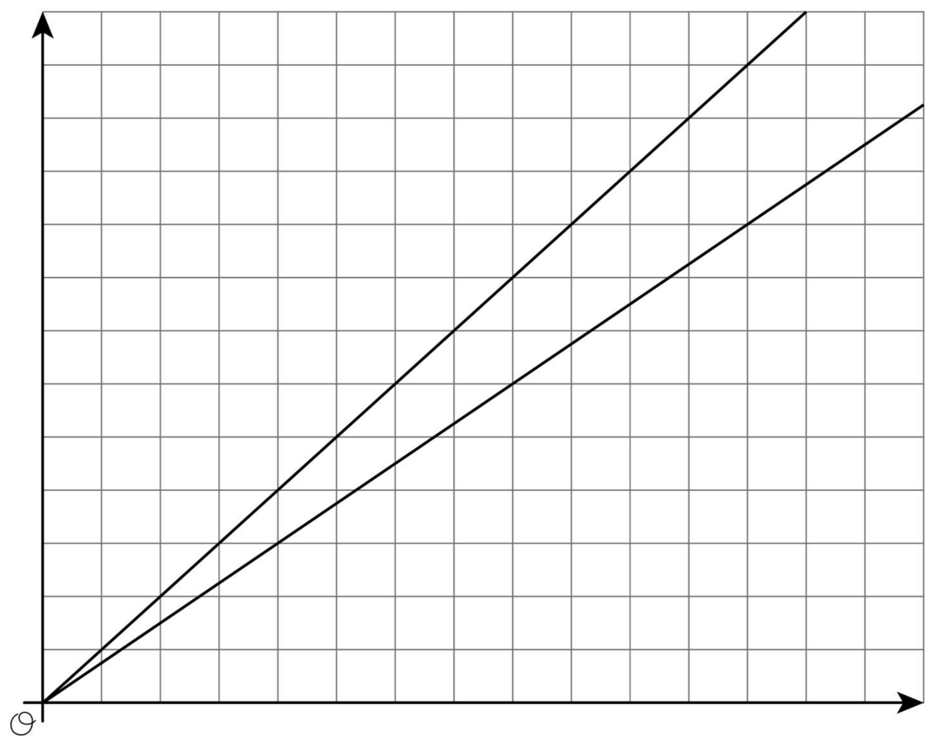
Student Response

1. A line through (0,0), (1,1), (2,2), etc.

2. Looking at the values for 2 seconds, turtle g moves 4 cm and the third turtle moves only 2 cm. This third turtle covers half the distance in the same amount of time.

Student Lesson Summary

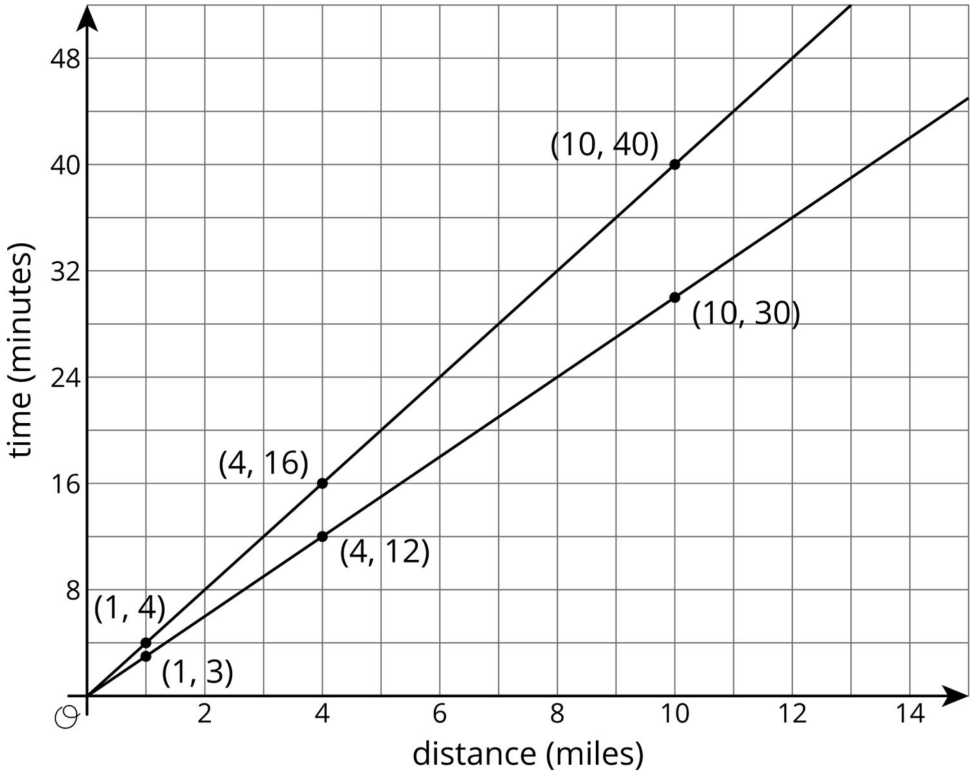
Graphing is a way to help us make sense of relationships. But the graph of a line on a coordinate axes without scale or labels isn't very helpful. For example, let's say we know that on longer bike rides Kiran can ride 4 miles every 16 minutes and Mai can ride 4 miles every 12 minutes. Here are the graphs of these relationships:



Without labels we can't even tell which line is Kiran and which is Mai! Without labels and a scale on the axes, we can't use these graphs to answer questions like:

1. Which graph goes with which rider?
2. Who rides faster?
3. If Kiran and Mai start a bike trip at the same time, how far are they after 24 minutes?
4. How long will it take each of them to reach the end of the 12 mile bike path?

Here are the same graphs, but now with labels and scale:



Revisiting the questions from earlier:

1. Which graph goes with each rider? If Kiran rides 4 miles in 16 minutes, then the point (4,16) is on his graph. If he rides for 1 mile, it will take 4 minutes. 10 miles will take 40 minutes. So the upper graph represents Kiran's ride. Mai's points for the same distances are (1,3), (4,12), and (10,30), so hers is the lower graph. (A letter next to each line would help us remember which is which!)

2. Who rides faster? Mai rides faster because she can ride the same distance as Kiran in a shorter time.

3. If Kiran and Mai start a bike trip at the same time, how far are they after 20 minutes? The points on the graphs at height 20 are 5 miles for Kiran and a little less than 7 miles for Mai.

4. How long will it take each of them to reach the end of the 12 mile bike path? The points on the graphs at a horizontal distance of 12 are 36 minutes for Mai and 48 minutes for Kiran. (Kiran's time after 12 miles is almost off the grid!)

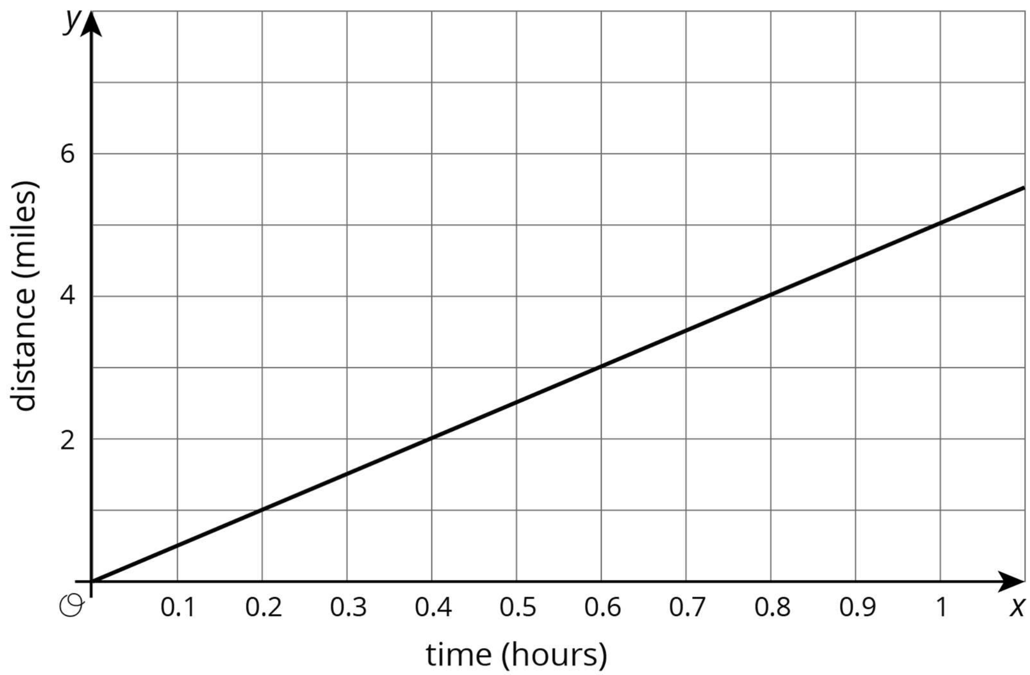
Glossary

- constant of proportionality

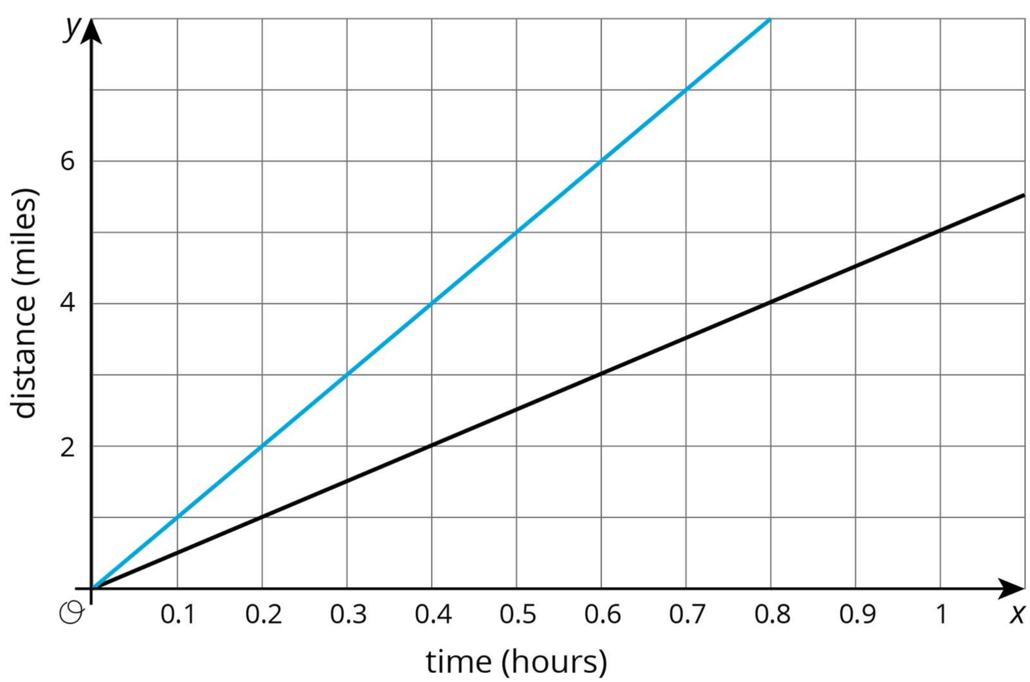
Lesson 1 Practice Problems

1. **Problem 1 Statement**

   Priya jogs at a constant speed. The relationship between her distance and time is shown on the graph. Diego bikes at a constant speed twice as fast as Priya. Sketch a graph showing the relationship between Diego's distance and time.



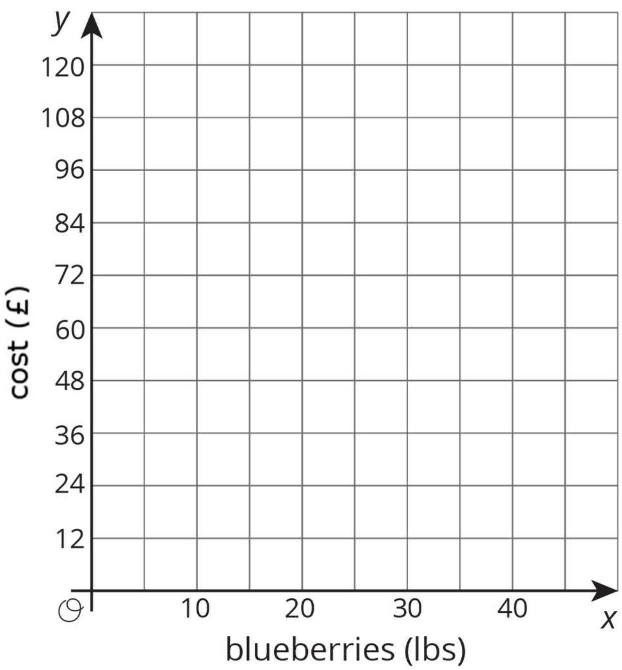
Solution



2. **Problem 2 Statement**

A pick-your-own blueberry farm offers 6 lbs of blueberries for £16.50.

Sketch a graph of the relationship between cost and pounds of blueberries.



Solution

A ray that passes through (0,0) and (6,16.5).

3. **Problem 3 Statement**

A line contains the points (-4,1) and (4,6). Decide whether or not each of these points is also on the line:

a. (0,3.5)

b. (12,11)

c. (80,50)

d. (-1,2.875)

Solution

a. On the line

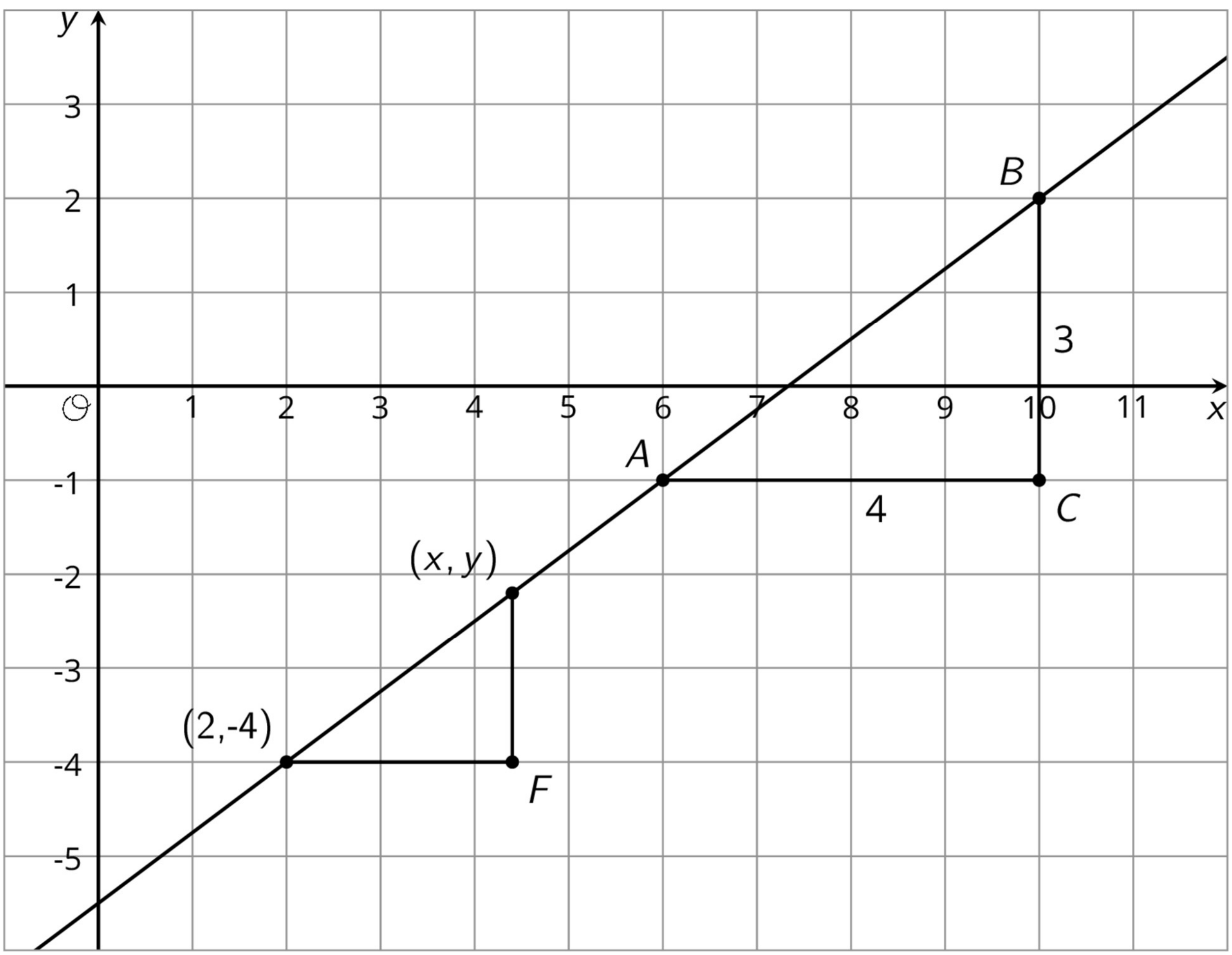
b. On the line

c. Not on the line

d. On the line

4. **Problem 4 Statement**

The points (2,-4), (x, y), A, and B all lie on the line. Find an equation relating x and y.



Solution

$\frac{y+4}{x-2} = \frac{3}{4}$ (or equivalent)