Lesson 18: Scaling two dimensions

Goals

- Compare and contrast (orally) graphs of linear and nonlinear functions.
- Create an equation and a graph representing the volume of a cone as a function of its radius, and describe (orally and in writing) how a change in radius affects the volume.
- Describe (orally and in writing) how changing the input of a certain nonlinear function affects the output.

Learning Targets

- I can create a graph representing the relationship between volume and radius for all cylinders (or cones) with a fixed height.
- I can explain in my own words why changing the radius by a scale factor changes the volume by the scale factor squared.

Lesson Narrative

This is lesson optional. The previous lesson explored some proportional relationships that arise when we consider the volume of a cuboid or cone as a function of one of its dimensions, such as side length or height. Students studied what happens to the volume of the shape when you scale that dimension. In this lesson they see what happens to the volume when you scale two of the dimensions. They consider a cuboid on a square base where you keep the height constant and vary the side length of the base, and a cone where you keep the height constant and vary the radius of the base. In both cases you are really varying two dimensions, because both the length and the width of the base change at the same time. As in the previous lesson, they consider what happens when you scale the side length or the radius by a particular factor, and this time they discover that the volume scales by the square of the factor. For example, if you triple the side length of the square base of the cuboid, you multiply the volume by 9, which is 3^2. In general, if you scale the side length by a, you multiply the volume by a^2.

The main purpose of this lesson is to understand that if you scale two of the dimensions of a three-dimensional shape by the same factor, the volume scales by the square of that factor. A secondary purpose is to see some interesting examples of non-linear functions arising from geometry.

Building On

- Apply and extend previous understandings of arithmetic to algebraic expressions.

Addressing

- Interpret the equation $y = mx + c$ as defining a linear function, whose graph is a straight line; give examples of functions that are not linear. For example, the function

$A = s^2$ giving the area of a square as a function of its side length is not linear because its graph contains the points (1,1), (2,4) and (3,9), which are not on a straight line.

- Use functions to model relationships between quantities.
- Know the formulae for the volumes of cones, cylinders, and spheres and use them to solve real-world and mathematical problems.

Instructional Routines

- Clarify, Critique, Correct
- Discussion Supports

Student Learning Goals

Let’s change more dimensions of shapes.

18.1 Tripling Statements

Warm Up: 5 minutes

The purpose of this warm-up is for students to explore how scaling the addends or factors in an expression affects their sum or product. Students determine which statements are true and then create one statement of their own that is true. This will prepare students to see structure in the equations they will encounter in the lesson. Identify students who:

- choose the correct statements (b, c)
- pick numbers to test the validity of statements
- use algebraic structure to show that the statements are true

Ask these students to share during the discussion.

Launch

Arrange students in groups of 2. Give students 1–2 minutes of quiet work time followed by time to discuss their chosen statements with their partner. Follow with a whole-class discussion.

Student Task Statement

m, n, a, b, and c all represent positive integers. Consider these two equations: $m = a + b + c$ and $n = abc$

1. Which of these statements are true? Select **all** that apply.
    a. If a is tripled, m is tripled.
    b. If a, b, and c are all tripled, then m is tripled.

c. If a is tripled, n is tripled.

d. If a, b, and c are all tripled, then n is tripled.

2. Create a true statement of your own about one of the equations.

Student Response

1. b, c

2. Answers vary. Sample response: If a, b, and c are all tripled, then n is 27 times as large. When a, b, and c are tripled, the result is $3a \times 3b \times 3c$, which can be written as $(3 \times 3 \times 3)abc$ or $27abc$.

Activity Synthesis

Ask previously identified students to share their reasoning about which statements are true (or not true). Display any examples (or counterexamples) for all to see and have students refer to them while sharing. If using the algebraic structure is not brought up in students' explanations, display for all to see:

- If a, b, and c are all tripled, the expression becomes $3a + 3b + 3c$, which can be written as $3(a + b + c)$ by using the distributive property to factorise out the 3. So if all the addends are tripled, their sum, m, is also tripled.
- Looking at the third statement, if a is tripled, the expression becomes $(3a)bc$, which, by using the associative property, can be written as $3(abc)$. So if just a is tripled, then n, the product of a, b, and c is also tripled.

18.2 A Square Base

Optional: 15 minutes

This activity is optional. The purpose of this activity is for students to examine how changing the input of a non-linear function changes the output. In this activity, students consider how the volume of a cuboid with a square base and a known height of 11 units changes if the edge lengths of the base triple. By studying the structure of the equation representing the volume function, students see that tripling the input leads to an output that is 9 times greater. In the following activity, students will continue this thinking with the volume function for cylinders.

Identify students who make sketches of the two cuboids or write expressions of the form $99s^2$ or $11(3s)^2$ to describe the volume of the tripled cuboid.

Instructional Routines

- Discussion Supports

Launch

Give students quiet work time. Leave 5–10 minutes for a whole-class discussion and follow-up questions.

Action and Expression: Develop Expression and Communication. To help get students started, display sentence frames such as "Han is ____ because ____" or "I agree with ____ because ____"
Supports accessibility for: Language; Organisation

Anticipated Misconceptions

Some students might think that because you triple s then you are only tripling one dimension. Encourage these students to make a sketch of the cuboid before and after the doubling and to label all three edge lengths to help them see that two dimensions are tripling.

Student Task Statement

Clare sketches a cuboid with a height of 11 and a square base and labels the edges of the base s. She asks Han what he thinks will happen to the volume of the cuboid if she triples s.

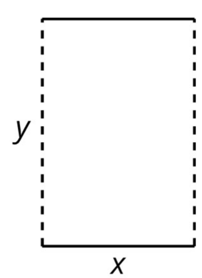
Han says the volume will be 9 times bigger. Is he right? Explain or show your reasoning.

Student Response

Yes, Han is right. Sample reasoning: Han is right because when the edge length of the square base is tripled, the area of the base is multiplied by 9. This makes the volume equation, $99s^2$, nine times as large as the original since $V = 11(3s)^2 = 11 \times 3s \times 3s = 11 \times 9s^2 = 99s^2$.

Are You Ready for More?

A cylinder can be constructed from a piece of paper by curling it so that you can glue together two opposite edges (the dashed edges in the diagram).



1. If you wanted to increase the volume inside the resulting cylinder, would it make more sense to double x, y, or does it not matter?

2. If you wanted to increase the surface area of the resulting cylinder, would it make more sense to double x, y, or does it not matter?

3. How would your answers to these questions change if we made a cylinder by gluing together the solid lines instead of the dashed lines?

Student Response

1. Double length x. Since x represents the circumference of the circular base, this would result in doubling the radius of the cylinder as well. Doubling the radius will result in 4 times the volume. Doubling y, the height of the cylinder, results in doubling the volume.

2. Double length x. Doubling x will quadruple the area of the circular bases as well as double the area of the curved surface. Doubling y will only double the curved surface.

3. If the cylinder was created by connecting the solid edges, doubling length y would result in both a larger volume and surface area because y would be the length related to the radius of the cylinder.

Activity Synthesis

Select previously identified students to share whether they think Han is correct. If possible, begin with students who made sketches of the two cuboids to make sense of the problem.

Ask students: "If this equation was graphed with edge length s on the horizontal axis and the volume of the cuboid on the vertical axis, what would the graph look like?" Suggest that they complete a table showing the volume of the cuboid when s equals 1, 2, 3, 4, and 5 units (the corresponding values of volume are 11, 44, 99, 176, and 275 cubic units) and then sketch a graph using these points. Give students quiet work time and then select a student to display their table and graph for all to see. Ask students what they notice about the graph when compared to the graphs from the previous lesson (the graph is non-linear—the volume increases by the square of whatever the base edge-length increases by).

Speaking: Discussion Supports. As students explain whether they agree with Han, invite students to restate their reasoning using mathematical language. Press for details inviting other students to challenge or elaborate on an idea. This will help students produce and make sense of the language needed to communicate their own ideas when reasoning about volume and the effects of changing dimensions.
Design Principle(s): Support sense-making; Optimise output (for generalisation)

18.3 Playing with Cones

Optional: 15 minutes (there is a digital version of this activity)

This activity is optional. In this activity, students continue working with function representations to investigate how changing the radius affects the volume of a cone with a fixed height. Students represent the relationship between the volume of the cone and the length of its radius with an equation and graph. They use these representations to justify what they think will happen when the radius of the cylinder is tripled. The work students did in the previous activity prepared them to use the equation to see the effect that tripling the radius has on the volume of the cone.

Identify students who use the equation versus the graph to answer the last question.

Instructional Routines

- Clarify, Critique, Correct

Launch

Give students 4–7 minutes of quiet work time followed by a whole-class discussion.

For students using the digital activity, they can generate their graph using an applet.

Anticipated Misconceptions

While students calculate the volume (or write an equation) they might mistake $(3r)^2$ as $2 \times 3r$, remind students what squaring a term involves and encourage them to expand the term if they need to see that $(3r)^2$ is $3r \times 3r$.

If students struggle to use the equation to see how the volume changes, encourage students to make a sketch of the cone.

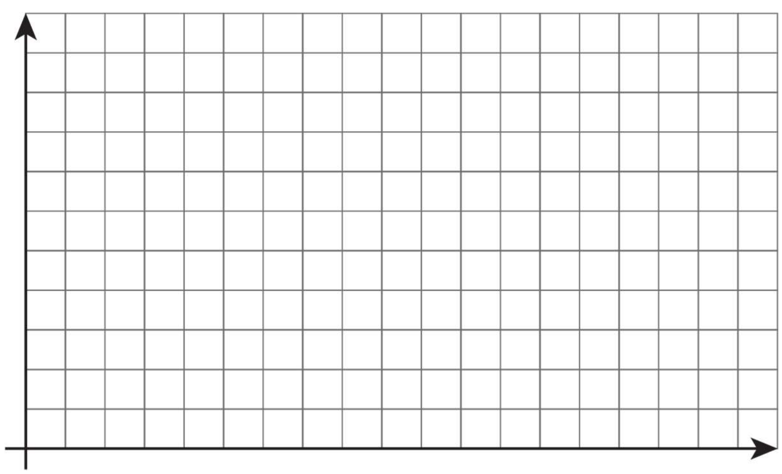
Student Task Statement

There are many cones with a height of 7 units. Let r represent the radius and V represent the volume of these cones.

1. Write an equation that expresses the relationship between V and r. Use 3.14 as an approximation for π.

2. Predict what happens to the volume if you triple the value of r.

3. Graph this equation.



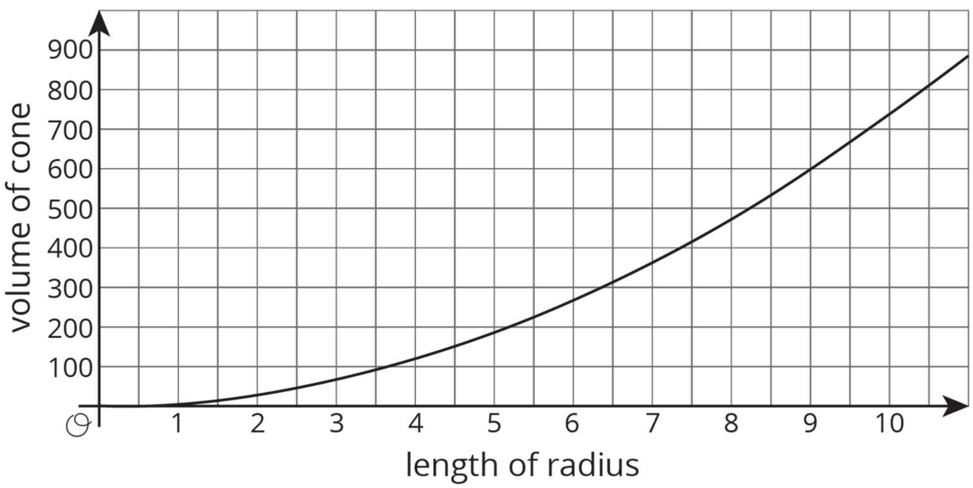
4. What happens to the volume if you triple r? Where do you see this in the graph? How can you see it algebraically?

Student Response

1. $V = 7.33r^2$

2. Answers vary. Sample response: The volume is 9 times bigger if the value of r is tripled.

3.



4. Answers vary. Sample response: When the radius, r, is tripled, the volume is 9 times as large. Since this is a not a proportional relationship, in the graph it can be seen that when the radius triples from 1 cm to 3 cm, the volume changes from 7.33 cm^3 to 65.94 cm^3, a value 9 times as large. Algebraically, if the radius triples from r to $3r$, then the volume changes from $7.33r^2$ to $7.33(3r)^2 = 9 \times 7.33r^2 = 65.94r^2$.

Activity Synthesis

The purpose of this discussion is for students to use the graph and equation to see that when you triple the radius you get a volume that is 9 times as large.

Ask previously identified students to share their graphs and equations. Display both representations for all to see, and ask students to point out where in each representation we see that the volume is 9 times as large. Ask students:

- "If the radius was quadrupled (made 4 times as large), how many times as large would the volume be?" (The volume would be 16 times as large since $\frac{1}{3}\pi(4r)^2 = \frac{1}{3}\pi r^2 4^2 = 16 \times \frac{1}{3}\pi r^2$.)

- "If the radius was halved, how many times as large would the volume be?" (The volume would be $\frac{1}{4}$ times as large since $\frac{1}{3}\pi\left(\frac{1}{2}r\right)^2 = \frac{1}{3}\pi r^2\left(\frac{1}{2}\right)^2 = \frac{1}{4}\frac{1}{3}\pi r^2$).

- "If the radius was scaled by an unknown factor a, how many times as large would the volume be?" (The volume would be a^2 times as large since $\frac{1}{3}\pi(ar)^2 = \frac{1}{3}\pi r^2 a^2 = a^2 \frac{1}{3}\pi r^2$.

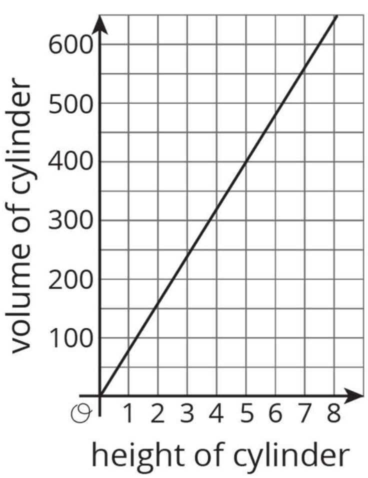
If students do not see the connection between scaling the radius length with a known value like 4 and an unknown value a, use several known values to help students generalise that scaling the radius by a scales the volume by a^2.

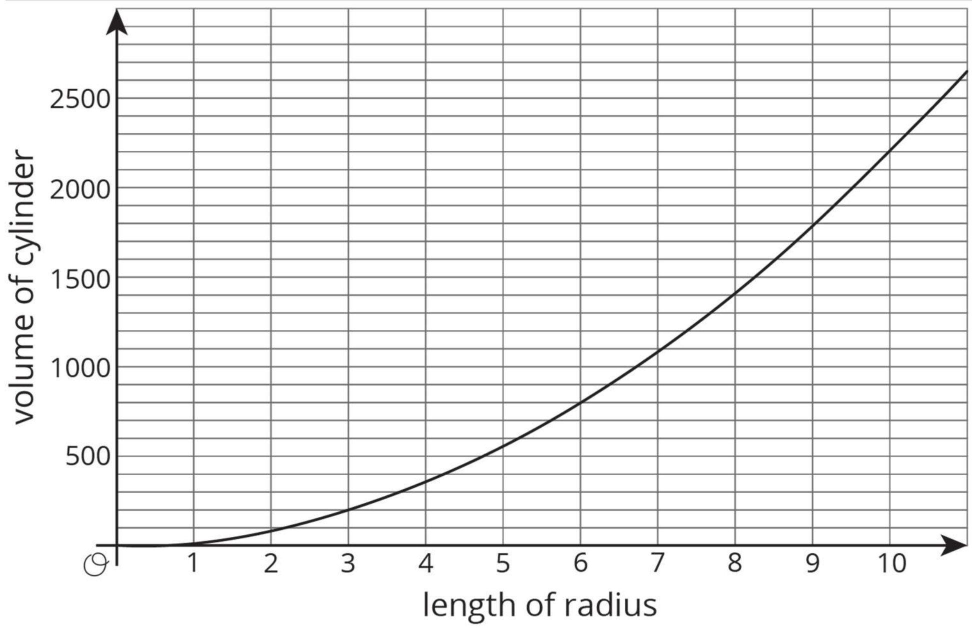
If time allows, ask students to compare this activity to the previous. How do the equations compare? How do the graphs compare? (In the last activity, the graph was sketched during the discussion.)

Writing: Clarify, Critique, Correct. Present an incorrect prediction describing what happens to the volume of the cone when the radius is tripled that reflects a possible misunderstanding from the class. For example, "The volume will be 3 times greater because that's what 'triples' means." Prompt students to critique the reasoning by asking, "Do you agree with the reasoning? Why or why not?" Invite students to write feedback to the author that identifies the misconception and how to improve on his/her work. Listen for students who tie their feedback to the formula and the describe the distinction between $(3r)^2$ and $3 \times r^2$. This will help students evaluate, and improve on, the written mathematical arguments of others and reason about the effect that tripling the radius has on the volume of the cone.
Design Principle(s): Maximise meta-awareness; Optimise output (for generalisation)

Lesson Synthesis

Display these graphs for all to see and give students a minute to consider what they represent.





Ask students:

- "What do these graphs represent? How are these graphs similar? Different?" (The first graph shows the relationship between the height and volume of all the cylinders with a fixed radius. The second graph shows the relationship between the radius and volume of all the cylinders with a fixed height. The first is linear, the second is non-linear.)
- "Think about what happens when a cube's edge lengths are doubled or tripled. What happens to the volume?" (The volume is increased by $2^3 = 8$, or by $3^3 = 27$.)
- "Why do you think changing the radius of a cylinder results in a graph that is not proportional?" (Two dimensions are changing when you change the radius of a cylinder.)

18.4 Halving Dimensions

Cool Down: 5 minutes

Student Task Statement

There are many cylinders for which the height and radius are the same value. Let c represent the height and radius of a cylinder and V represent the volume of the cylinder.

1. Write an equation that expresses the relationship between the volume, height, and radius of this cylinder using c and V.
2. If the value of c is halved, what must happen to the value of the volume V?

Student Response

1. $V = \pi c^3$

2. If the value of c is halved, then the value of the volume would be $\frac{1}{8}$ of the original volume since $\pi\left(\frac{1}{2}c\right)^3 = \pi c^3\left(\frac{1}{2}\right)^3 = \frac{1}{8}\pi c^3$

Student Lesson Summary

There are many cuboids that have a length of 4 units and width of 5 units but differing heights. If h represents the height, then the volume V of such a cuboid is

$V = 20h$

The equation shows us that the volume of a cuboid with a base area of 20 square units is a linear function of the height. Because this is a proportional relationship, if the height gets multiplied by a factor of a, then the volume is also multiplied by a factor of a:

$V = 20(ah)$

What happens if we scale *two* dimensions of a cuboid by a factor of a? In this case, the volume gets multiplied by a factor of a twice, or a^2.

For example, think about a cuboid with a length of 4 units, width of 5 units, and height of 6 units. Its volume is 120 cubic units since $4 \times 5 \times 6 = 120$. Now imagine the length and width each get scaled by a factor of a, meaning the new cuboid has a length of $4a$, width of $5a$, and a height of 6. The new volume is $120a^2$ cubic units since $4a \times 5a \times 6 = 120a^2$.

A similar relationship holds for cylinders. Think of a cylinder with a height of 6 and a radius of 5. The volume would be 150π cubic units since $\pi \times 5^2 \times 6 = 150\pi$. Now, imagine the radius is scaled by a factor of a. Then the new volume is $\pi \times (5a)^2 \times 6 = \pi \times 25a^2 \times 6$ or $150a^2\pi$ cubic units. So scaling the radius by a factor of a has the effect of multiplying the volume by a^2!

Why does the volume multiply by a^2 when only the radius changes? This makes sense if we imagine how scaling the radius changes the base area of the cylinder. As the radius increases, the base area gets larger in two dimensions (the circle gets wider and also taller), while the third dimension of the cylinder, height, stays the same.

Mathematics Education Innovation

Lesson 18 Practice Problems

Problem 1 Statement

There are many cylinders with a height of 18 metres. Let r represent the radius in metres and V represent the volume in cubic metres.

a. Write an equation that represents the volume V as a function of the radius r.

b. Complete this table, giving three possible examples.

r	V
1	

c. If the radius of a cylinder is doubled, does the volume double? Explain how you know.

d. Is the graph of this function a line? Explain how you know.

Solution

a. $V = 18\pi r^2$
b. Answers vary. Sample response:

r	V
1	18π
2	72π
4	288π

c. No, the volume does not double, it is multiplied by four. Explanations vary.
d. It is *not* a line. The three points in the table do not lie on a straight line.

Problem 2 Statement

As part of a competition, Diego must spin around in a circle 6 times and then run to a tree. The time he spends on each spin is represented by s and the time he spends running is r. He gets to the tree 21 seconds after he starts spinning.

a. Write an equation showing the relationship between s and r.

b. Rearrange the equation so that it shows r as a function of s.

c. If it takes Diego 1.2 seconds to spin around each time, how many seconds did he spend running?

Solution

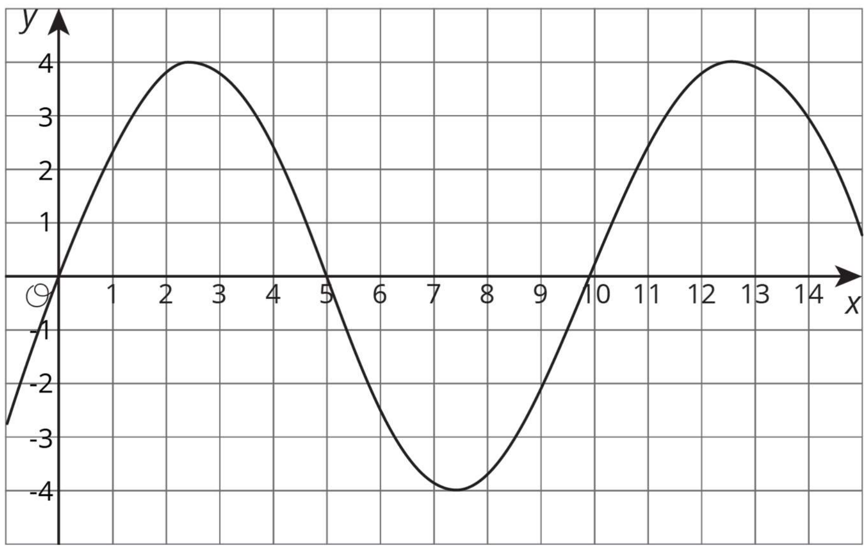
a. $6s + r = 21$

b. $r = 21 - 6s$

c. 13.8 seconds

Problem 3 Statement

The table and graph represent two functions. Use the table and graph to answer the questions.



x	1	2	3	4	5	6
y	3	-1	0	4	5	-1

a. For which values of x is the output from the table less than the output from the graph?

b. In the graphed function, which values of x give an output of 0?

Solution

a. 2 and 3

b. 0, 5, and 10

Problem 4 Statement

A cone has a radius of 3 units and a height of 4 units.

a. What is this volume of this cone?

b. Another cone has quadruple the radius, and the same height. How many times larger is the new cone's volume?

Solution

a. 12π cubic units

b. 16 times larger (The new cone's volume is $V = 192\pi$, 4 times larger. Quadrupling the radius makes the volume 4^2 times larger.)