MEI Mathematics Education Innovation

Lesson 10: Calculating gradient

Goals

- Create a graph of a line using a verbal description of its features.
- Describe (orally) the graph of a line using formal or informal language precisely enough to identify a unique line.
- Generate a method to find gradient values given two points on the line.

Learning Targets

- I can calculate positive and negative gradients given two points on the line.
- I can describe a line precisely enough that another student can draw it.

Lesson Narrative

Students extend their work with gradient triangles to develop a method for finding the gradient of any line given the coordinates of two points on the line. They practise finding gradients this way and use a graph in order to check their answer (especially the sign).

Then students consider what information is sufficient to define (and accurately communicate) the position of a line in the coordinate plane. Lines with positive and negative gradient are examined as students move flexibly between coordinates of points on a line, the gradient of the line, and the graph showing the “uphill” or “downhill” orientation of the line. Many methods for describing the location of the lines are available, but students need to calculate carefully and use the coordinate grid in order to communicate the positions of the line clearly.

Building On

- Apply and extend previous understandings of operations with fractions to add, subtract, multiply, and divide rational numbers.

Addressing

- Understand the connections between proportional relationships, lines, and linear equations.
- Use similar triangles to explain why the gradient m is the same between any two distinct points on a non-vertical line in the coordinate plane; derive the equation $y = mx$ for a line through the origin and the equation $y = mx + c$ for a line intercepting the vertical axis at c.

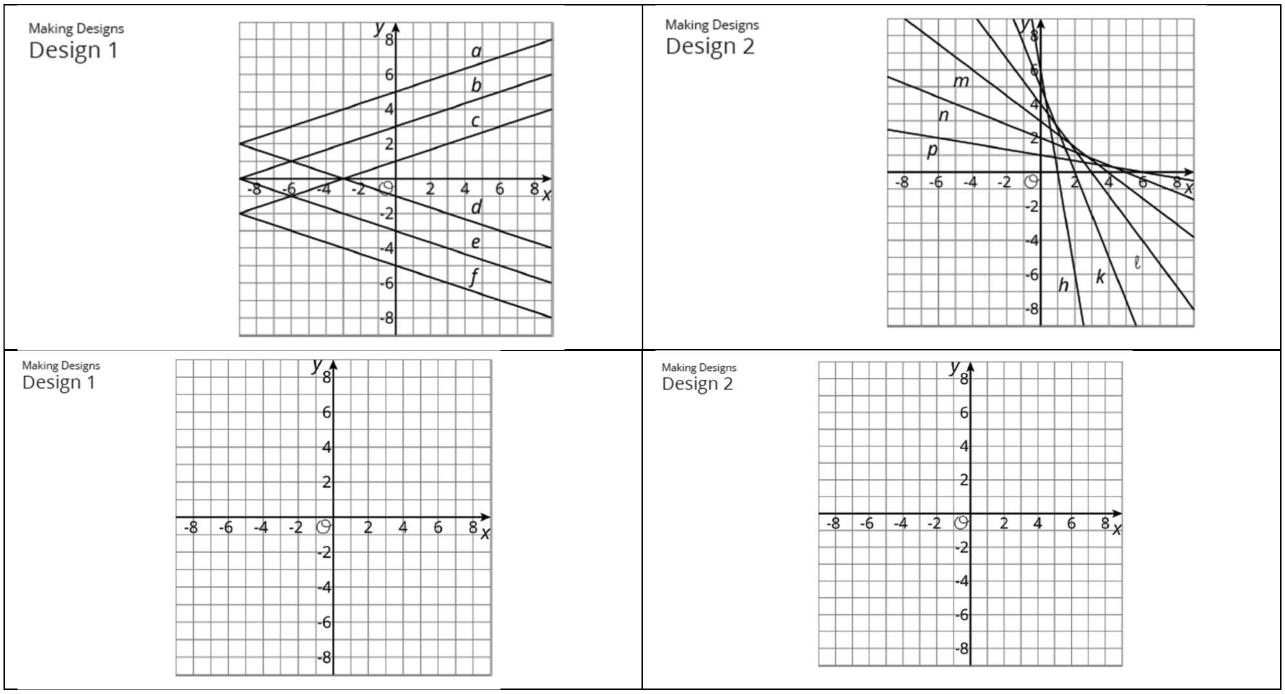
Instructional Routines

- Information Gap Cards
- Compare and Connect

- Number Talk

Required Materials

Graph paper
Pre-printed slips, cut from copies of the blackline master



Straightedges
A rigid edge that can be used for drawing line segments. Sometimes a ruler is okay to use as a straightedge, but sometimes it is preferable to use an unruled straightedge, like a blank index card.

Required Preparation

Print and cut up slips from the Making Designs blackline master. Prepare 1 copy for every 2 students.

Student Learning Goals

Let’s calculate gradient from two points.

10.1 Number Talk: Integer Operations

Warm Up: 5 minutes

Only three problems are given to allow time to discuss different values of a and b for each problem. It may not be possible to share every student’s responses, given limited time. Consider gathering only two or three different sets of values per problem. Each problem was chosen to elicit a different understanding of integer operations, so as students share theirs, ask how the information in the equation impacted their choices for a and b.

This warm-up is intended as an opportunity to review operations on positive and negative numbers.

Instructional Routines

- Number Talk

Launch

Display the problems all at once. Give students 1 minute of quiet think time per problem and ask them to give a signal when they have at least one set of values for each question. Follow with a whole-class discussion.

Representation: Internalise Comprehension. To support working memory, provide students with sticky notes or mini whiteboards.
Supports accessibility for: Memory; Organisation

Anticipated Misconceptions

Students may have forgotten that the quotient of two negative numbers is positive.

Student Task Statement

Find values for a and b that make each side have the same value.

$\frac{a}{b} = \text{-}2$

$\frac{a}{b} = 2$

$a - b = \text{-}2$

Student Response

- Answers vary. Sample responses: Any combination of two numbers with different signs for which, in absolute value, a is twice b. Examples: 6 ÷ -3, -6 ÷ 3
- Answers vary. Sample responses: Any combination of two numbers with the same sign in which, in absolute value, a is twice b. Examples: -6 ÷ -3, 6 ÷ 3
- Answers vary. Sample response: Any combination of two numbers in which b is two more than a. Examples: 6 − 8, -6 − (-4), 0 − 2, -2 − 0, -1 − 1

Activity Synthesis

Ask students to share their values for a and b for each problem. Include at least one set of values for each problem where a or b (or both) are negative. Record and display their values for a and b for all to see. Ask students how they decided on their values based on the information given in the equation. To involve more students in the conversation, consider asking:

- "Did anyone choose the same values?"
- "Who can restate ___'s reasoning in a different way?"
- "Did anyone choose different values?"
- "Does anyone want to add on to _____'s reasoning?"
- "Do you agree or disagree? Why?"

Speaking: Discussion Supports.: Display sentence frames to support students when they explain their strategy. For example, "First, I _____ because . . ." or "I noticed _____ so I . . . ." Some students may benefit from the opportunity to rehearse what they will say with a partner before they share with the whole class.
Design Principle(s): Optimise output (for explanation)

10.2 Toward a More General Gradient Formula

15 minutes

The purpose of this activity is to work out the gradients of different lines to get familiar with the formula "subtract y-coordinates, subtract x-coordinates, then divide." Students first calculate gradients for some lines with positive gradients, and then special attention is drawn to the fact that a line has a negative gradient. Students attend to what makes the same calculations have a negative result instead of a positive result.

Instructional Routines

- Compare and Connect

Launch

Have partners figure out the gradient of the line that passes through each pair of points.

1. (12,4) and (7,1) (answer: $\frac{3}{5}$)
2. (4,-11) and (7,-8) (answer: 1)
3. (1,2) and (600,3) (answer: $\frac{1}{599}$)
4. (37,40) and (30,33) (answer: 1)

Ask students to share their results and how they did it. Students may say they just found the difference between the numbers; making this more precise is part of the goal of the discussion for this activity. Ask students to complete the questions in the task and share their responses with a partner before class discussion. Provide access to graph paper and rulers.

Representation: Develop Language and Symbols. Create a display of important terms and vocabulary. Include the following terms and maintain the display for reference throughout the unit: gradient. Invite students to suggest language or diagrams to include on the display that will support their understanding of this term.
Supports accessibility for: Memory; Language

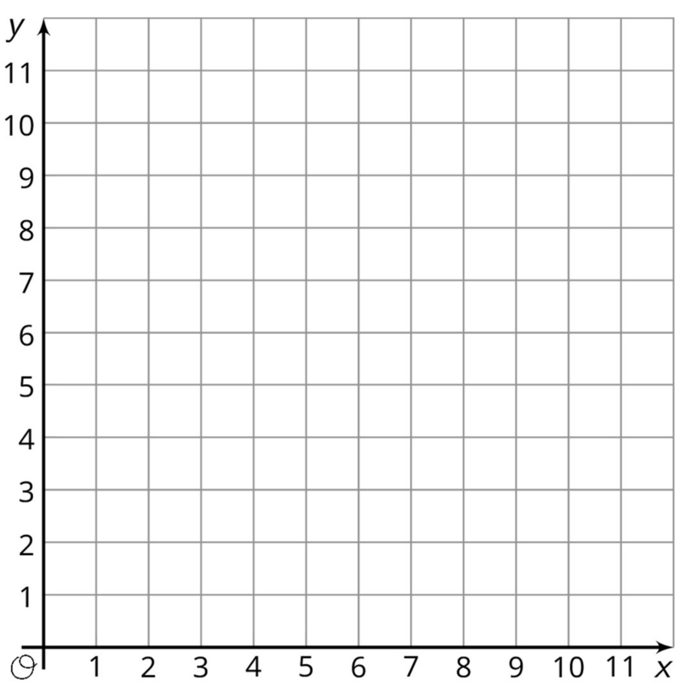
Anticipated Misconceptions

Students may struggle with operations on negative numbers and keeping everything straight if they try to take a purely algorithmic approach. Encourage them to plot both points and reason about the length of the vertical and horizontal portions of the gradient triangle and the sign of the gradient of the line, one step at a time. Sketching a graph of the line is also useful for verifying the sign of the line's gradient.

It is common for students to calculate a gradient and leave it in a form such as $\frac{-3}{-5}$. Remind students that this fraction is representing a division operation, and the quotient would be a positive value.

Student Task Statement

1. Plot the points (1,11) and (8,2), and use a ruler to draw the line that passes through them.
2. Without calculating, do you expect the gradient of the line through (1,11) and (8,2) to be positive or negative? How can you tell?
3. Calculate the gradient of this line.



Student Response

1. A line is graphed that passes through (1,11) and (8, 2).

2. Negative. Explanations vary. Sample responses: Because it's a downhill line; because y is getting smaller as x gets bigger.

3. $\frac{-9}{7}$, because $11 - 2 = 9$ and $1 - 8 = -7$, and $9 \div -7 = \frac{-9}{7}$.

Are You Ready for More?

Find the value of k so that the line passing through each pair of points has the given gradient.

1. $(k, 2)$ and $(11,14)$, gradient = 2

2. $(1, k)$ and $(4,1)$, gradient = -2

3. $(3,5)$ and $(k, 9)$, gradient = $\frac{1}{2}$

4. $(-1,4)$ and $(-3, k)$, gradient = $\frac{-1}{2}$

5. $(\frac{-15}{2}, \frac{3}{16})$ and $(\frac{-13}{22}, k)$, gradient = 0

Student Response

1. 5

2. 7

3. 11

4. 5

5. $\frac{3}{16}$

Activity Synthesis

When using two points to calculate the gradient of a line, care needs to be taken to subtract the x and y values in the same order. Using the first pair as an example, the gradient could be calculated either of these ways: $\frac{4-1}{12-7} = \frac{3}{5}$ or $\frac{1-4}{7-12} = \frac{-3}{-5} = \frac{3}{5}$ But if one of the orders were reversed, this would yield a negative value for the gradient when we know the gradient should have a positive value.

It is worth demonstrating, or having a student demonstrate an algorithmic approach to finding the negative gradient in the last part of the task. Draw attention to the fact that keeping the coordinates "in the same order" results in the numerator and denominator having opposite signs (one positive and one negative), so that their quotient must be negative. It might look like: $\frac{11-2}{1-8} = \frac{9}{-7} = -\frac{9}{7}$

Ask students how sketching a graph of the line will tell them whether the gradient is positive or negative. They should recognise that when it goes up, from left to right, the gradient is positive, and when it goes down, then the gradient is negative. Using a sketch of the graph can also be helpful to judge whether the magnitude of the (positive or negative) gradient is reasonable.

Representing, Speaking: Compare and Connect. Use this support as students calculate the gradient of the line between the points (1,11) and (8,2). Invite students to demonstrate their strategy using a visual or numerical representation. In pairs or groups, ask students to compare their strategies. Ask students to discuss how the strategies are the same and/or different, and then share with the whole class. This will help students connect how different approaches led to the same result of a negative gradient by keeping coordinates "in the same order."
Design Principle(s): Optimise output (for comparison); Maximise meta-awareness

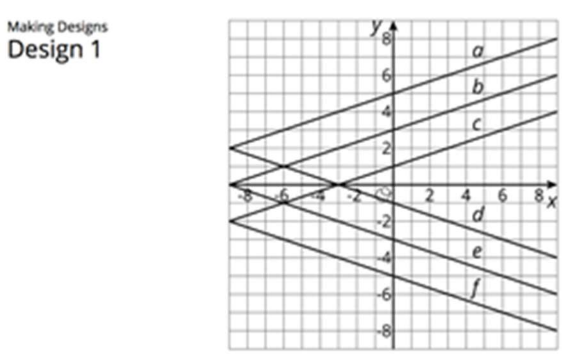
10.3 Making Designs

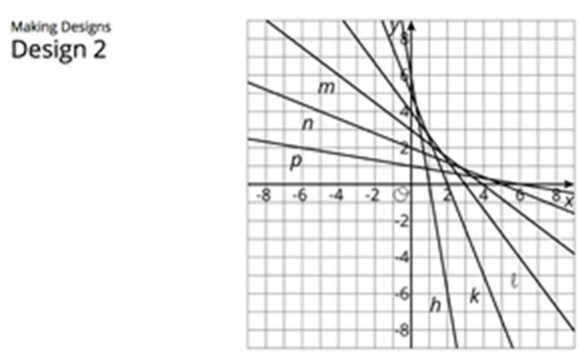
20 minutes

The goal of this activity is for students to recognise information that determines the location of a line in the coordinate plane, and to practise distinguishing between positive and negative gradients. In this activity, one partner has a design that they verbally describe to their partner, who then tries to draw it. The purpose of this activity is to provide an environment where students have to describe or interpret the gradient and locations of several lines. (Students are not expected to communicate by saying the equations of the lines, though there is nothing stopping them from doing so.) Students take turns describing and interpreting by doing this two times with two different designs.

Monitor for students who use language of gradient and vertical or horizontal intercepts to communicate the location of each line. Invite these students to share during the discussion. There are many other ways students might communicate the location of each line, but the recent emphasis on studying gradient and intercepts should make these choices natural.

The two designs in the blackline masters look like this:





You will need the Info Gap: Making Designs blackline master for this activity.

Thanks to Henri Picciotto for permission to use these designs.

Instructional Routines

- Information Gap Cards

Launch

Tell students they will describe some lines to a partner to try and get them to recreate a design. The protocol is described in the student task statement. Consider asking a student to serve as your partner to demonstrate the protocol to the class before distributing the designs and blank graphs.

Arrange students in groups of 2. Provide access to straightedges.

From the blackline master that you have copied and cut up ahead of time, give one partner the design, and the other partner a blank graph. Arrange the room to ensure that the partner drawing the design cannot peek at the design anywhere in the room. Once the first design has been successfully created, provide the second design and a blank graph to the other student in each partnership.

Representation: Provide Access for Perception. Display or provide students with a physical copy of the written directions and read them aloud. Check for understanding by inviting students to rephrase directions in their own words. Keep directions visible throughout the activity.
Supports accessibility for: Language; Memory Conversing: Use this modified version of *Information Gap* to give students an opportunity to describe (orally) the graph of a line using formal or informal language. Circulate and listen for common words or phrases students use to describe the designs. Record this language and display for all to see. Encourage students to borrow words or phrases from the display as needed.
Design Principle(s): Cultivate conversation

Student Task Statement

Your teacher will give you either a design or a blank graph. Do not show your card to your partner.

If your teacher gives you the design:

1. Look at the design silently and think about how you could communicate what your partner should draw. Think about ways that you can describe what a line looks like, such as its gradient or points that it goes through.
2. Describe each line, one at a time, and give your partner time to draw them.
3. Once your partner thinks they have drawn all the lines you described, only then should you show them the design.

If your teacher gives you the blank graph:

1. Listen carefully as your partner describes each line, and draw each line based on their description.

2. You are not allowed to ask for more information about a line than what your partner tells you.

3. Do not show your drawing to your partner until you have finished drawing all the lines they describe.

When finished, place the drawing next to the card with the design so that you and your partner can both see them. How is the drawing the same as the design? How is it different? Discuss any miscommunication that might have caused the drawing to look different from the design.

Pause here so your teacher can review your work. When your teacher gives you a new set of cards, switch roles for the second problem.

Student Response

Student designs should match those in the blackline master.

Activity Synthesis

After students have completed their work, ask students to discuss the process of communicating how to draw a line. Some guiding questions:

- "What details were important to pay attention to?"
- "How did you use coordinates to help communicate where the line is?"
- "How did you use gradient to communicate how to draw the line?"
- "Were there any cases where your partner did not give enough information to know where to draw the line? What more information did you need?"

Students might notice that the lines with the same gradient can be described in terms of translations (for example, line b is a vertical translation of line a down two units). This is an appropriate use of rigid motion language which recalls work done earlier in this unit. Students might also describe b as parallel to a and containing the point $(0,3)$. Finally, some students might use equations to communicate the location of the lines. All of these methods are appropriate. Keep the discussion focused on describing each line using gradient and coordinates for each individual line on its own.

Lesson Synthesis

In this lesson, students explored the interplay between the coordinates of points on a line and the gradient of that line, where the gradient could be positive or negative.

Ask students, “What information do you need to know exactly where a line is?” Valid responses might be the coordinates of two points on the line or the coordinates of one point and the line’s gradient. Demonstrate why knowing one point is not enough information (the line goes through it, but could have any gradient), and why only knowing the gradient is not enough information (you know at what gradient to draw the line, but it could be located anywhere—it could be any of a set of parallel lines). It can be helpful to use a metre rule to represent "the line" in this situation as you move it around a coordinate plane on the board.

Ask students, "If you know the coordinates of two points on a line, how can you tell if it has a positive or negative gradient?" Responses might include sketching a graph of the line to see if it's "uphill" or "downhill," or an algorithm involving subtraction and division, attending to keeping coordinates in "the same order" and performing operations correctly.

10.4 Different Gradients

Cool Down: 5 minutes

Students calculate the gradient of the line through two points. They are only given the coordinates of the points and are specifically directed not to graph the line. By now, students should have internalised an efficient method for finding gradient using the coordinates of two points on the line.

Student Task Statement

Without graphing, find the gradient of the line that goes through

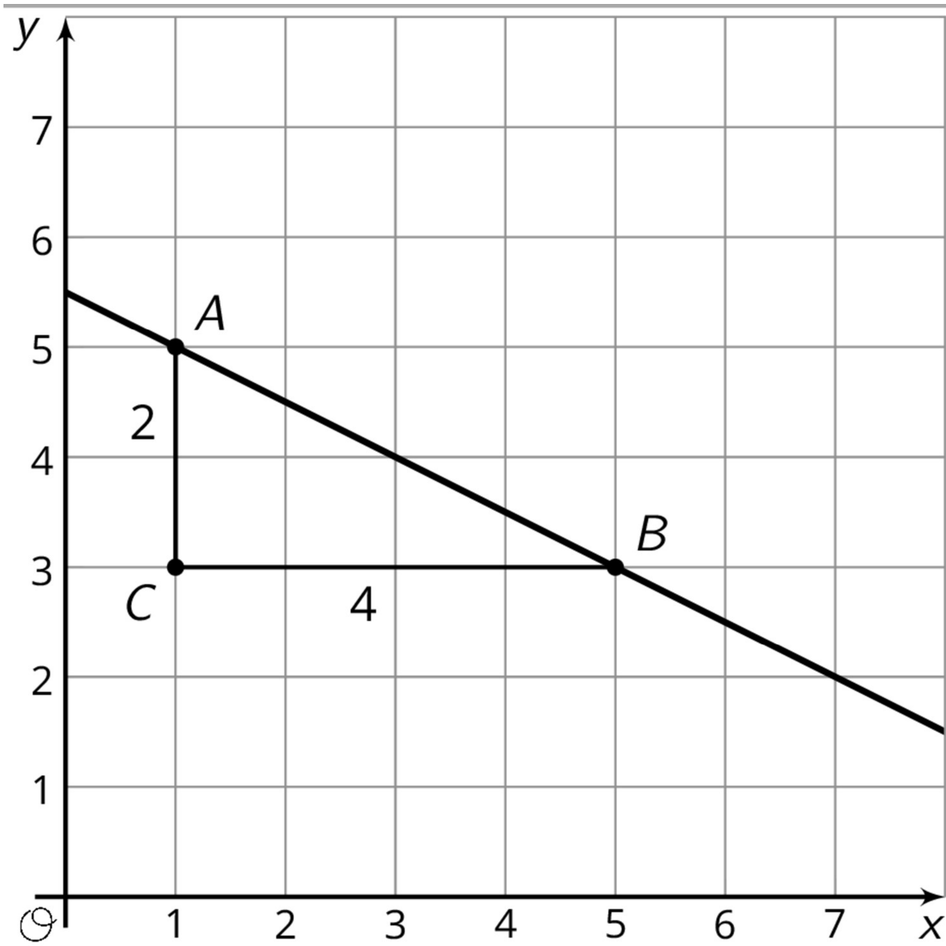
1. (0,5) and (8,2).
2. (2,-1) and (6,1).
3. (-3,-2) and (-1,-5).

Student Response

1. $\frac{-3}{8}$
2. $\frac{1}{2}$
3. $\frac{-3}{2}$

Student Lesson Summary

We learned earlier that one way to find the gradient of a line is by drawing a gradient triangle. For example, using the gradient triangle shown here, the gradient of the line is $-\frac{2}{4}$, or $-\frac{1}{2}$ (we know the gradient is negative because the line is decreasing from left to right).



But gradient triangles are only one way to calculate the gradient of a line. Let's calculate the gradient of this line a different way using just the points $A = (1,5)$ and $B = (5,3)$. Since we know the gradient is the vertical change divided by the horizontal change, we can calculate the change in the y-values and then the change in the x-values. Between points A and B, the y-value change is $3 - 5 = -2$ and the x-value change is $5 - 1 = 4$. This means the gradient is $-\frac{2}{4}$, or $-\frac{1}{2}$, which is the same as what we found using the gradient triangle.

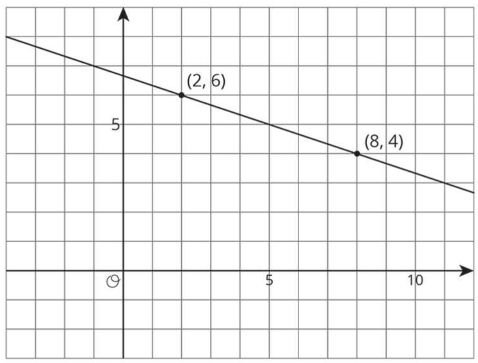
Notice that in each of the calculations, we subtracted the value from point A from the value from point B. If we had done it the other way around, then the y-value change would have been $5 - 3 = 2$ and the x-value change would have been $1 - 5 = -4$, which still gives us a gradient of $-\frac{1}{2}$. But what if we were to mix up the orders? If that had happened, we would think the gradient of the line is *positive* $\frac{1}{2}$ since we would either have calculated $\frac{-2}{-4}$ or $\frac{2}{4}$. Since we already have a graph of the line and can see it has a negative gradient, this is clearly incorrect. It we don't have a graph to check our calculation, we could think about how the point on the left, (1,5), is higher than the point on the right, (5,3), meaning the gradient of the line must be negative.

Lesson 10 Practice Problems

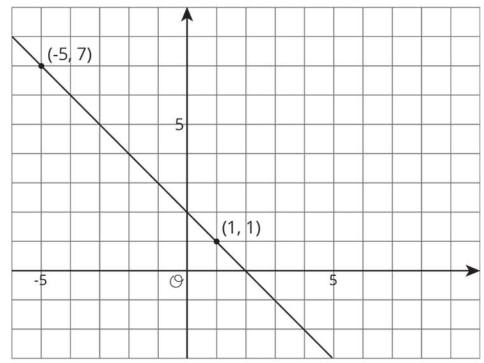
1. **Problem 1 Statement**

For each graph, calculate the gradient of the line.

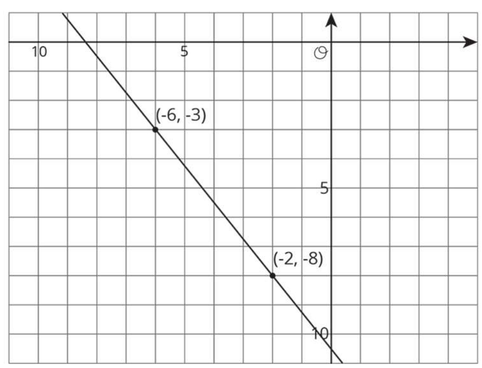
A



B



C



Solution

A: $\frac{-2}{6}$, B: -1, C: $\frac{-5}{4}$

2. **Problem 2 Statement**

Match each pair of points to the gradient of the line that joins them.

a. (9,10) and (7,2)

b. (-8,-11) and (-1,-5)

c. (5,-6) and (2,3)

d. (6,3) and (5,-1)

e. (4,7) and (6,2)

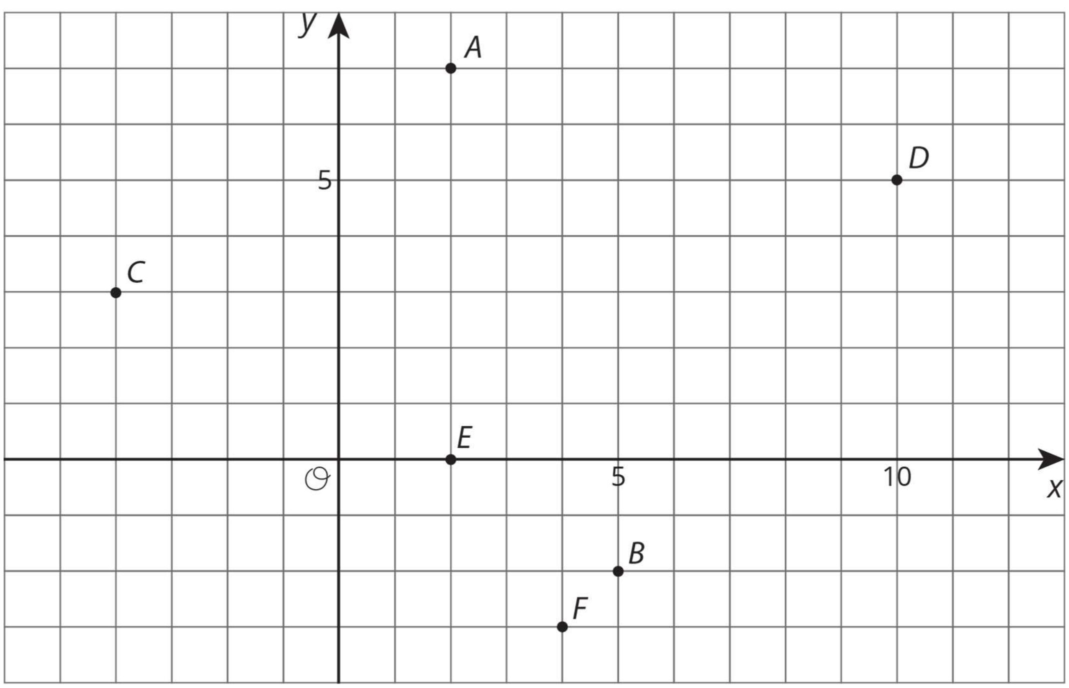
1. 4
2. -3
3. $-\frac{5}{2}$
4. $\frac{6}{7}$

Solution

- A: 1
- B: 4
- C: 2
- D: 1
- E: 3

3. **Problem 3 Statement**

Draw a line with the given gradient through the given point. What other point lies on that line?



a. Point A, gradient = -3

b. Point A, gradient = $\frac{-1}{4}$

c. Point C, gradient = $\frac{-1}{2}$

d. Point E, gradient = $\frac{-2}{3}$

Solution

a. Point B

b. Point D

c. Point E

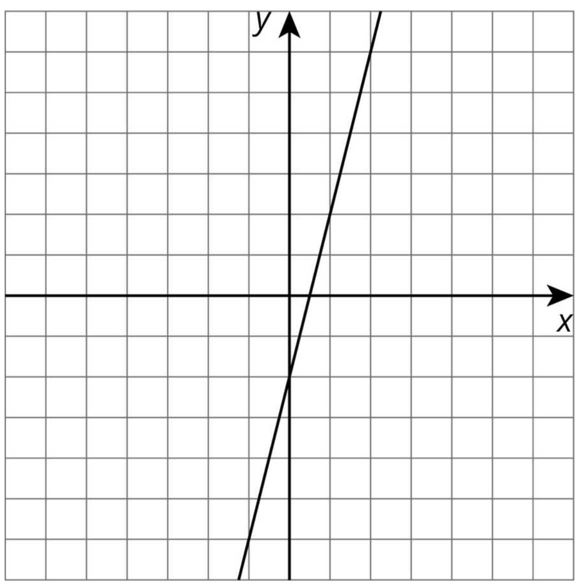
d. Point B

4. **Problem 4 Statement**

Make a sketch of a linear relationship with a gradient of 4 and a negative y-intercept. Show how you know the gradient is 4 and write an equation for the line.

Solution

Answers vary. Sample response:



The equation is $y = 4x - 2$. I can tell the gradient is 4 by looking at the points (0,-2) and (1,2) since $\frac{2--2}{1-0} = \frac{4}{1} = 4$.