Lesson 12: Using equations for lines

Goals

- Create an equation of a line with positive gradient on a coordinate grid using knowledge of similar triangles.
- Generalise (orally) a process for enlarging a gradient triangle ABC on a coordinate plane with centre of enlargement A and scale factor s.
- Justify (orally) that a point(x, y) is on a line by verifying that the values of x and y satisfy the equation of the line.

Learning Targets

- I can find an equation for a line and use that to decide which points are on that line.

Lesson Narrative

In the previous two lessons, students saw that all gradient triangles for a line give the same gradient value, and this value is called the gradient of the line. They also began writing relationships satisfied by all points (x, y) on a line. In this lesson, they continue to write equations but with less scaffolding, that is no similar triangles are selected so students need to figure out what to do given a line and a few points on the line.

The properties of gradient triangles that make the gradient of a line meaningful have to do with enlargements. In particular, enlargements do not change the quotient of the vertical side length and horizontal side length of a gradient triangle. Students return to enlargements in this lesson, applied to a single gradient triangle with varying scale factor. This gives a different way of seeing how the coordinates of points on a line vary.

Both techniques, using equations and studying all of the enlargements of a single gradient triangle, give expressions representing points on a line.

Addressing

- Use similar triangles to explain why the gradient m is the same between any two distinct points on a non-vertical line in the coordinate plane; derive the equation $y = mx$ for a line through the origin and the equation $y = mx + c$ for a line intercepting the vertical axis at c.
- Understand congruence and similarity using physical models, transparencies, or geometry software.
- Describe the effect of enlargements, translations, rotations, and reflections on two-dimensional shapes using coordinates.

Building Towards

- Use similar triangles to explain why the gradient m is the same between any two distinct points on a non-vertical line in the coordinate plane; derive the equation $y =$

mx for a line through the origin and the equation $y = mx + c$ for a line intercepting the vertical axis at c.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Clarify, Critique, Correct
- Compare and Connect
- Think Pair Share

Required Materials

Geometry toolkits

tracing paper, graph paper, coloured pencils, scissors, and an index card to use as a straightedge or to mark right angles, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Student Learning Goals

Let’s write equations for lines.

12.1 Missing centre

Warm Up: 5 minutes

Given a point, the image of the point under an enlargement, and the scale factor of the enlargement, students must identify the centre of the enlargement. This requires thinking about the meaning of enlargements and the fact that the centre of enlargement, the point enlarged, and the image, are collinear.

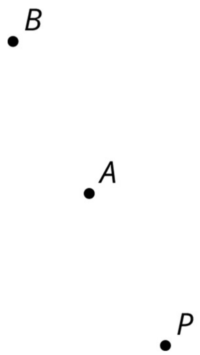
Launch

Provide access to geometry toolkits. (In particular, a ruler or index card is needed.)

Student Task Statement

An enlargement with scale factor 2 sends A to B. Where is the centre of the enlargement?

Student Response



The point, marked as P in the diagram, is on the same line as A and B, the same distance from B to A, but on the other side of A.

Activity Synthesis

Ask students what they know about the centre of enlargement that helps to solve this problem. The key fact is that the centre of enlargement lies on the same line as A and B. The scale factor is 2 so if P is the centre of enlargement, then the length of line segment PB is twice the length of line segment PA.

12.2 Writing Relationships from Two Points

10 minutes

In the previous lesson, students found an equation satisfied by the points on a line using properties of gradient triangles and a general point, labelled (x, y), on the line. In this activity, they pursue this work but the scaffold of the given gradient triangles has been removed. Once students draw appropriate gradient triangles, this is an opportunity to practise and consolidate learning. In addition, students use the equation (satisfied by points on the line) to check whether or not specific points lie on the line.

Note that the y-intercept was intentionally left off of this diagram, so that students who may have seen some of this material are discouraged from jumping straight to $y = mx + c$ and encouraged to engage with thinking about similar triangles.

There are many gradient triangles that students can draw, but the one joining (5,3) and (7,7) is the most natural for calculating the gradient and then (x, y) and either (5,3) or (7,7) can be used to find an equation. Monitor for students who make these choices (and use them to decide whether or not the given points lie on the line) and invite them to present during the discussion.

Instructional Routines

- Clarify, Critique, Correct
- Think Pair Share

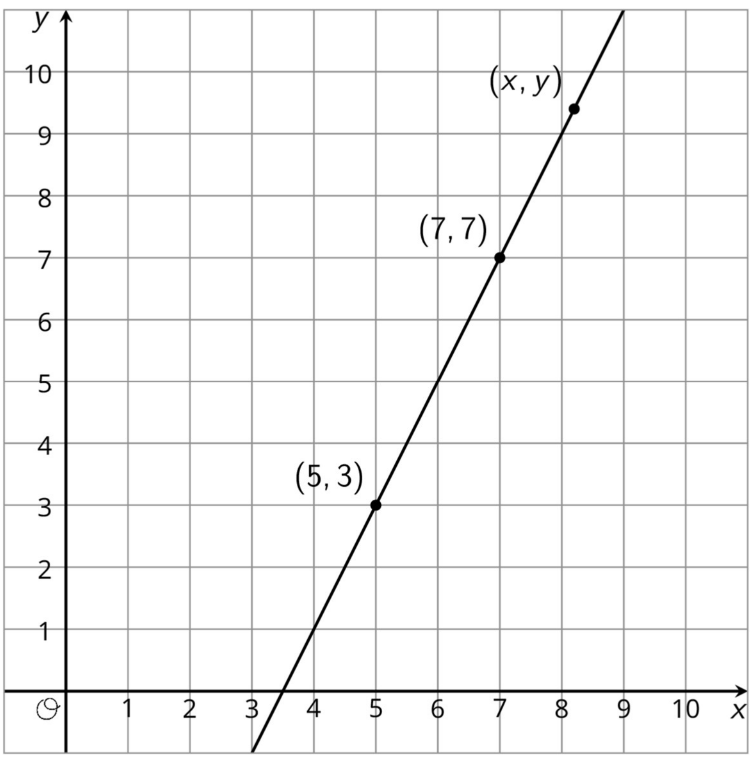
Launch

Provide access to geometry toolkits (in particular, a straightedge is helpful). Give 2–3 minutes of quiet work time. Then ask them to share their responses and reasoning with a partner, followed by a whole-class discussion.

Representation: Internalise Comprehension. Begin with a physical demonstration of using gradient triangles to find the equation of a line to support connections between new situations and prior understandings. Consider using these prompts: "What does this demonstration have in common with the graph in this activity?" or "How can you apply your reasoning with gradient triangles to this graph?"
Supports accessibility for: Conceptual processing; Visual-spatial processing

Student Task Statement

Here is a line.



1. Using what you know about similar triangles, find an equation for the line in the diagram.
2. What is the gradient of this line? Does it appear in your equation?
3. Is (9,11) also on the line? How do you know?
4. Is (100,193) also on the line?

Student Response

1. $\frac{y-7}{x-7} = 2$ or equivalent.

2. 2. Answers vary based on equation.

3. Yes. Possible strategies:

   – For every 1 to the right, you go 2 up. I started at (7,7) and counted until I got to (9,11).

   – When y is 11 and x is 9, it makes my equation true.

4. Yes. When y is 193 and x is 100, it satisfies my equation.

Are You Ready for More?

There are many different ways to write down an equation for a line like the one in the problem. Does $\frac{y-3}{x-6} = 2$ represent the line? What about $\frac{y-6}{x-4} = 5$? What about $\frac{y+5}{x-1} = 2$? Explain your reasoning.

Student Response

No. The equation $\frac{y-3}{x-6} = 2$ represents a line of gradient 2 containing the point (6,3). Since our line does not contain the point (6,3), this is not our line. The equation $\frac{y-6}{x-4} = 5$ represents a line of gradient 5 containing the point (4,6). Since our line has gradient 2, this is not our line. The equation $\frac{y+5}{x-1} = 2$ represents a line of gradient 2 containing the point (1,-5). Since our line contains the point (1,-5) and has gradient 2, this is another possible equation for our line.

Activity Synthesis

Invite selected students to show how they arrived at their equation. Also, ask them how the equation helps to determine whether or not the points in the last two questions lie on the line. Emphasise that the graph, as shown, is not helpful for checking whether or not (100,193) is on the line, but the equation is true for these numbers, so this point must be on the line.

Highlight that using (x, y) and (5,3) for a gradient triangle gives an equation such as $\frac{y-3}{x-5} = 2$ while using (x, y) and (7,7) gives the equation $\frac{y-7}{x-7} = 2$. These equations look different, but they both work to check whether or not a point (x, y) is on the line. Using algebra to show that these two equations are equivalent is not necessary, but students can see from the picture that either equation can be used to test whether or not a point (x, y) is on the line. If students have not done so already when they share their solutions, draw and label the two gradient triangles that correspond to these two equations.

Reading, Writing, Speaking: Clarify, Critique, Correct. Before students share whether the point with coordinates (100, 193) is on the line, present an incorrect answer and explanation. For example, "The equation of the line is $\frac{x-5}{y-3} = 2$. When x is 100 and y is 193, it does not satisfy my equation because $\frac{100-5}{193-} = \frac{95}{190} = \frac{1}{2}$. Since $\frac{1}{2}$ is not equal to 2, then the point (100, 193) is not on the line." Ask students to identify the error, critique the reasoning, and write a correct explanation. Prompt students to share their critiques and corrected explanations with the class. Listen for and amplify the language students use to clarify the meaning of gradient and explain why the gradient of the line is equivalent to $\frac{y-3}{x-5}$. This routine will engage students in meta-awareness as they critique and correct a common misconception about the gradient of a line.
Design Principle(s): Optimise output (for explanation); Maximise meta-awareness

12.3 Enlargements and Gradient Triangles

15 minutes

This activity investigates the coordinates of points on a line from the point of view of enlargements. At the beginning of this unit, students experimented with enlargements and made numerous important discoveries including

- Enlargements change distances between points by a scale factor s.
- Enlargements preserve angles.
- Enlargements take lines to lines.

Having developed the key ideas of *similar triangles* and *gradient,* this activity returns to enlargements, applying them systematically to a single gradient triangle. All of these enlargements of the triangle are similar, their long sides all lie on the same line, and the coordinates of the points on that line have a structure intimately linked with the enlargements used to produce them.

For the third question, monitor for students who

- look for and express a pattern for the coordinates of the points from earlier questions; a scale factor of 1 gives $C = (2,2)$, a scale factor of 2 gives $C = (4,3)$, a scale factor of 2.5 gives $C = (5,3.5)$, so the x-coordinate appears to be twice the scale factor while the y-coordinate appears to be one more than the scale factor.
- use the structure of $\triangle ABC$ and the definition of enlargements.

For the final question, students can either

- reason through from scratch, looking at the x-coordinate or y-coordinate for example or
- use the coordinates they find for a scale factor of s in the third question.

Monitor for both approaches and invite students to share during the discussion.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Compare and Connect

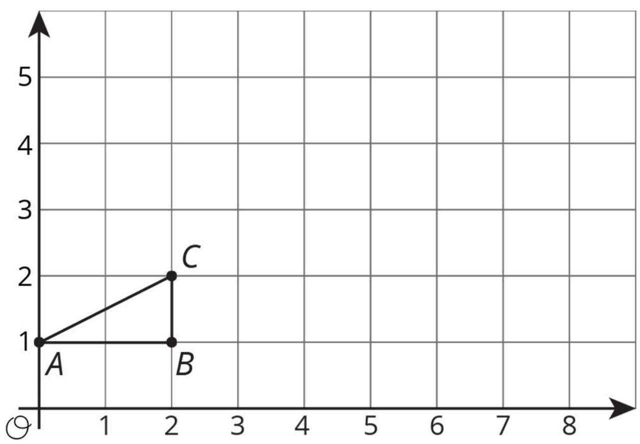
Launch

Access to geometry toolkits. Give 2–3 minutes of quiet work time, then ask students to share their reasoning with a partner. Follow with a whole-class discussion.

Representation: Develop Language and Symbols. Maintain a display of important terms and vocabulary. During the launch, take time to review terms students will need to access for this activity. Invite students to suggest language or diagrams to include that will support their understanding of: enlargement and scale factor.
Supports accessibility for: Conceptual processing; Language

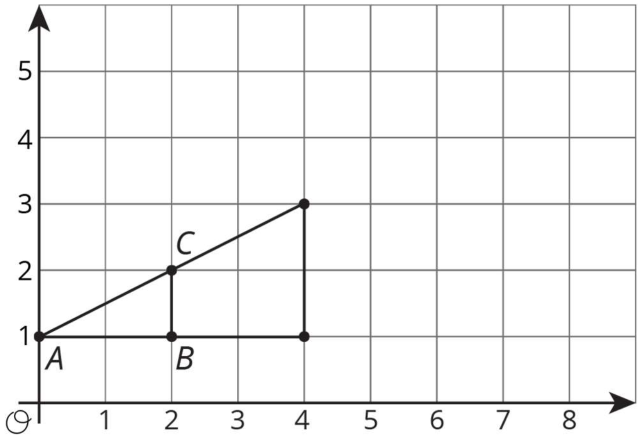
Student Task Statement

Here is triangle ABC.

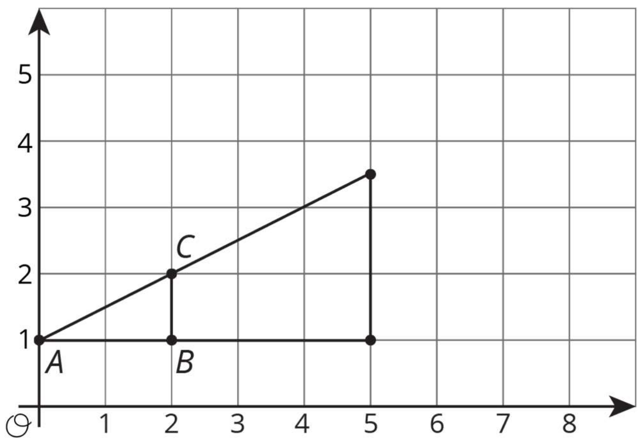


1. Draw the enlargement of triangle ABC with centre $(0,1)$ and scale factor 2.
2. Draw the enlargement of triangle ABC with centre $(0,1)$ and scale factor 2.5.
3. Where is C mapped by the enlargement with centre $(0,1)$ and scale factor s?
4. For which scale factor does the enlargement with centre $(0,1)$ send C to $(9,5.5)$? Explain how you know.

Student Response



1.



2.

3. $(2s, s+1)$. To get from $(0,1)$ to $(2,2)$, we move to the right by 2 units and up 1 unit. Applying a scale factor of s will multiply each of these side lengths by s. So C will go to the point $2s$ units to the right and s units up from $(0,1)$. That is the point $(2s, s+1)$.

4. $s = 4.5$. We want $(2s, s+1)$ to be $(9,5.5)$. Solve either the equation $2s = 9$ or $s + 1 = 5.5$ to get $s = 4.5$.

Activity Synthesis

First, focus on the third question, inviting selected students to present, in this sequence: first those who identified a pattern from the first two questions and then those who studied the impact of an enlargement with scale factor s on $\triangle ABC$. Point out that the argument looking at where C is taken by the enlargement with scale factor s and centre A explains *why* the x coordinate doubles and the y coordinate is 1 more than the scale factor.

Next, invite selected students to share their answers to the last question.

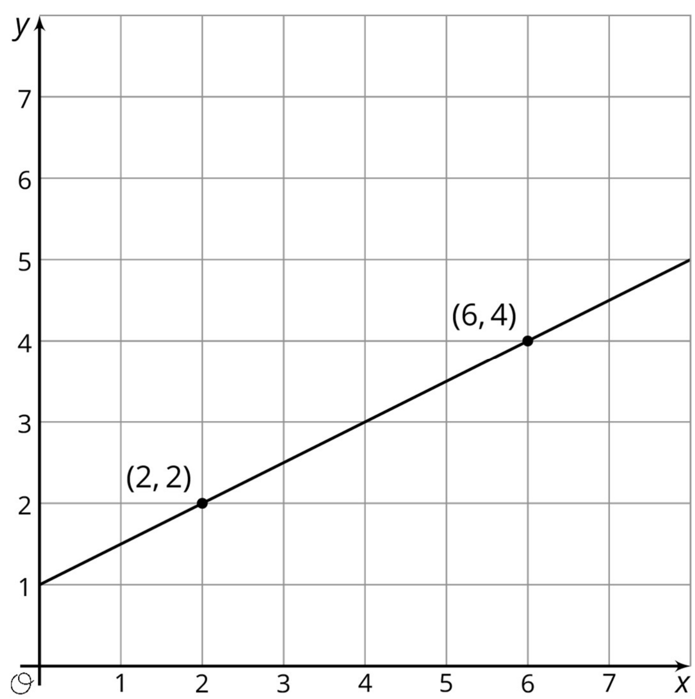
In previous activities, students have used similar triangles to show that the points (x, y) on the line containing the long side of triangle ABC satisfy the relationship $\frac{y-1}{x} = \frac{1}{2}$. In this activity, they find that points (x, y) on this line are of the form $(2s, s + 1)$ where s is a (positive) real number. Make sure that students understand that the equation $\frac{y-1}{x} = \frac{1}{2}$ is true if we take $y - 1 = s + 1$ and $x = 2s$. Also, emphasise the key role that enlargements play in these arguments.

The big take away from this lesson is that the structure of the coordinates of points on a line can be derived from properties of enlargements.

Speaking, Listening: Compare and Connect. When discussing where C is mapped, some students may need support to generalise that C will be mapped to $(2s, s + 1)$ after an enlargement with centre (0,1) and scale factor s and how these distances relate to the visual image. Ask students to write a short explanation or create a visual display of how they determined where C is mapped. Provide students time to examine the different approaches or representations of their classmates. Ask students what worked well in different approaches. Next, ask students to explain how the lengths in their explanations relate to the generalised form. Ask students where they see $2s$ and $s + 1$ represented in the gradient triangles. Be sure to demonstrate asking questions that students can ask each other, rather than asking questions to test understanding.
Design Principle(s): Maximise meta-awareness, Cultivate conversation

Lesson Synthesis

The coordinates of points on a line have a nice structure that is useful for checking whether or not a given point is on a line. Here is a line with a couple of labelled points.



- What is the gradient of this line? It's $\frac{1}{2}$ because a gradient triangle (draw in the gradient triangle) for the two labelled points has horizontal side length 4 and vertical side length 2.
- What is an equation for the line? One example is $\frac{y-2}{x-2} = \frac{1}{2}$. Label a general point on the line (x, y), and draw in a gradient triangle to show this relationship.

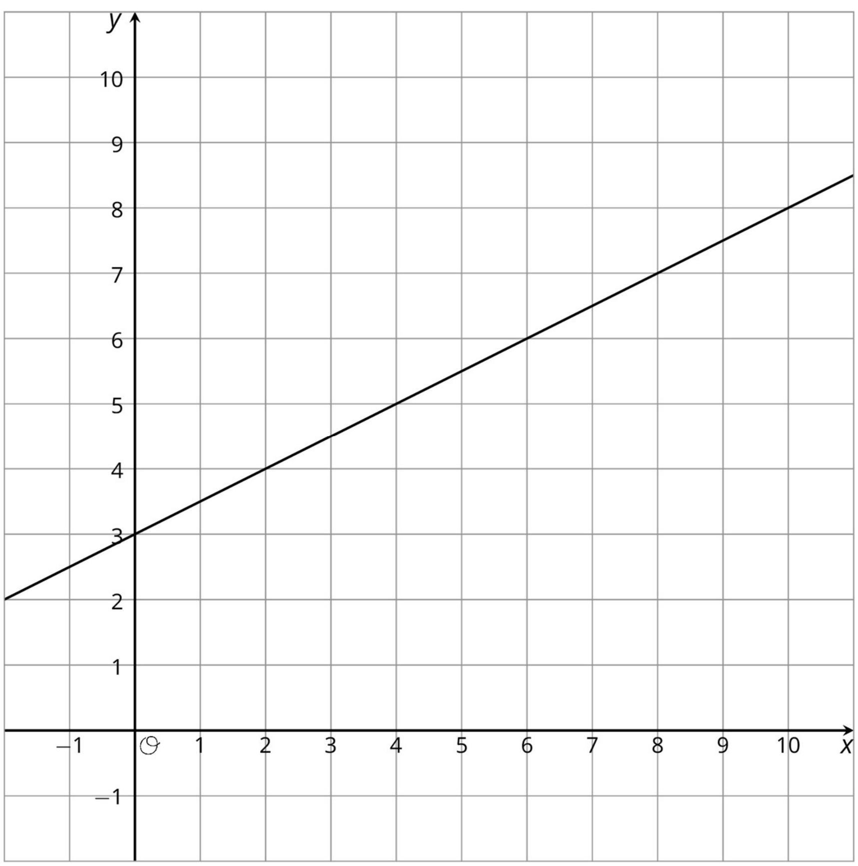
How can we find out whether or not the point (72,37) is on this line? The points on the line satisfy the equation $\frac{y-2}{x-2} = \frac{1}{2}$. Since $\frac{37-2}{72-2} = \frac{1}{2}$, the point (72,37) is on the line!

12.4 Is the Point on the Line?

Cool Down: 5 minutes

Students determine whether or not a point with given coordinates lies on a line. The line is presented graphically with no scaffolding and the coordinates of the point are too large to check whether or not it lies on the line by examining the graph. Students are expected to write an equation satisfied by points on the line and apply this.

Student Task Statement



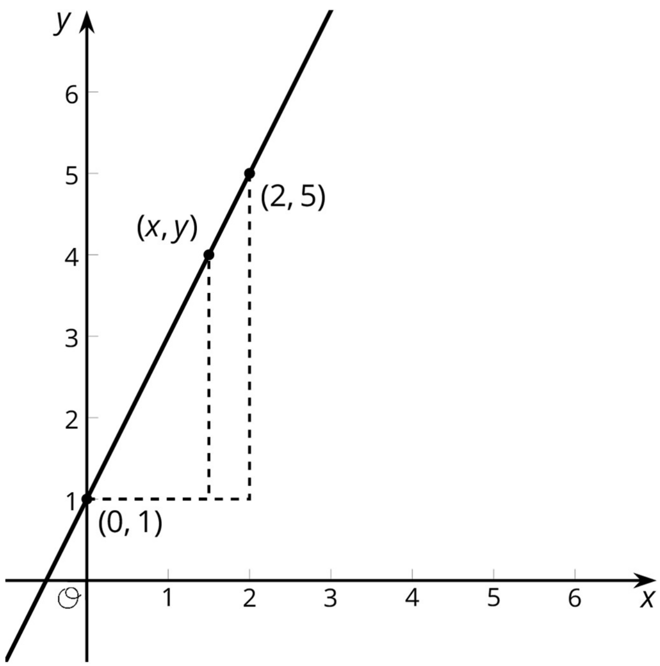
Is the point (20,13) on this line? Explain your reasoning.

Student Response

Yes. The gradient of the line is $\frac{1}{2}$: moving to the right 2 units and up 1 unit gives a point on the line. Since (0,3) is on the line, moving right 20 units and up 10 units gives another point on the line. This is the point (20,13).

Student Lesson Summary

We can use what we know about gradient to decide if a point lies on a line. Here is a line with a few points labelled.



The gradient triangle with vertices (0,1) and (2,5) gives a gradient of $\frac{5-1}{2-0} = 2$. The gradient triangle with vertices (0,1) and (x, y) gives a gradient of $\frac{y-1}{x}$. Since these gradients are the same, $\frac{y-1}{x} = 2$ is an equation for the line. So, if we want to check whether or not the point (11,23) lies on this line, we can check that $\frac{23-1}{11} = 2$. Since (11,23) is a solution to the equation, it is on the line!

Lesson 12 Practice Problems

1. **Problem 1 Statement**

   Select **all** the points that are on the line through (0,5) and (2,8).

   a. (4,11)

   b. (5,10)

   c. (6,14)

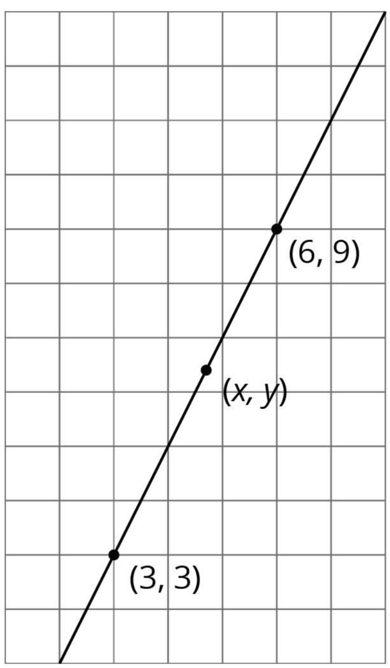
   d. (30,50)

   e. (40,60)

   Solution ["A", "C", "D"]

2. **Problem 2 Statement**

   All three points displayed are on the line. Find an equation relating x and y.

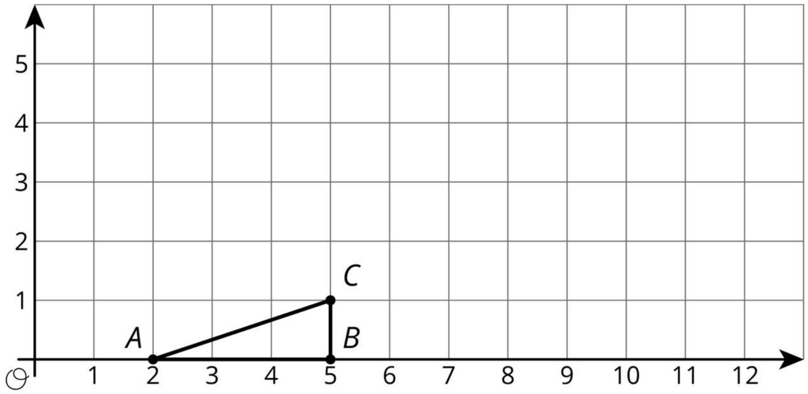


   Solution

   Answers vary. Sample response: $\frac{y-3}{x-3} = 2$ (or $y = 2x - 3$)

3. **Problem 3 Statement**

   Here is triangle ABC.



a. Draw the enlargement of triangle ABC with centre (2,0) and scale factor 2.

b. Draw the enlargement of triangle ABC with centre (2,0) and scale factor 3.

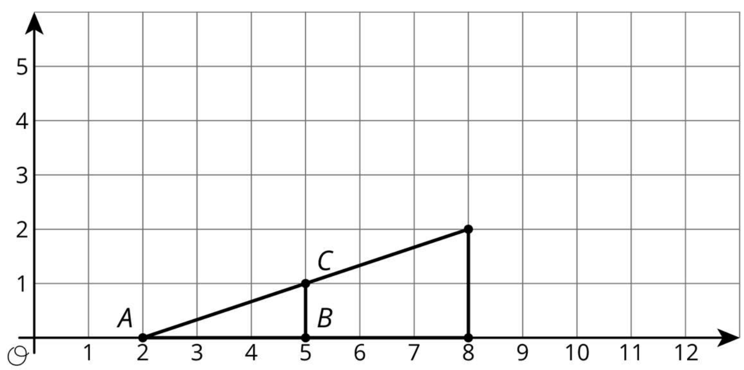
c. Draw the enlargement of triangle ABC with centre (2,0) and scale factor $\frac{1}{2}$.

d. What are the coordinates of the image of point C when triangle ABC is enlarged with centre (2,0) and scale factor s?

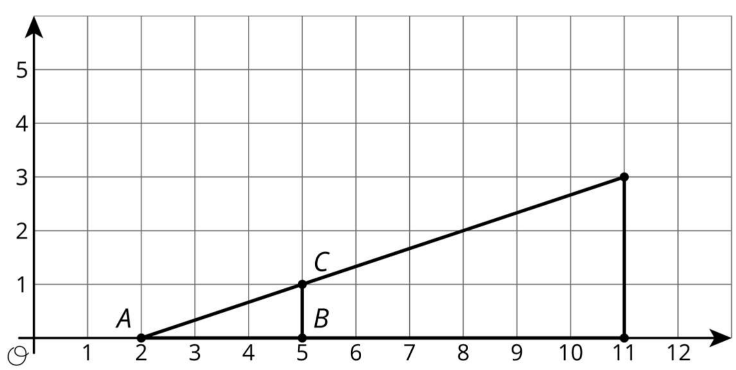
e. Write an equation for the line containing all possible images of point C.

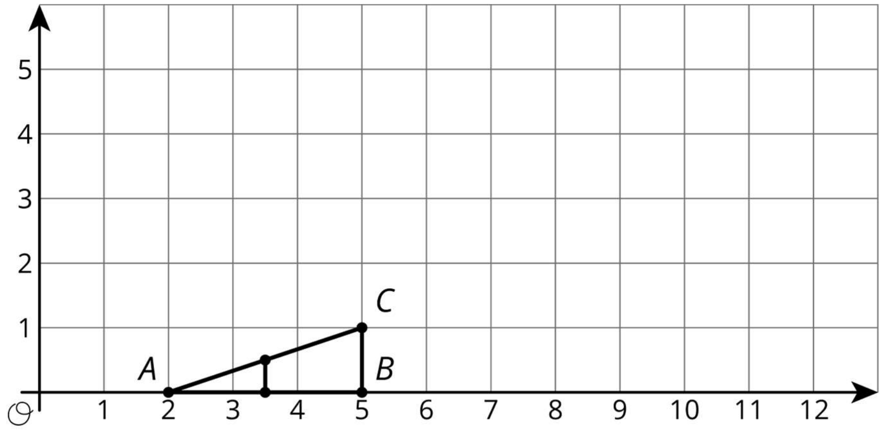
Solution

a.



b.



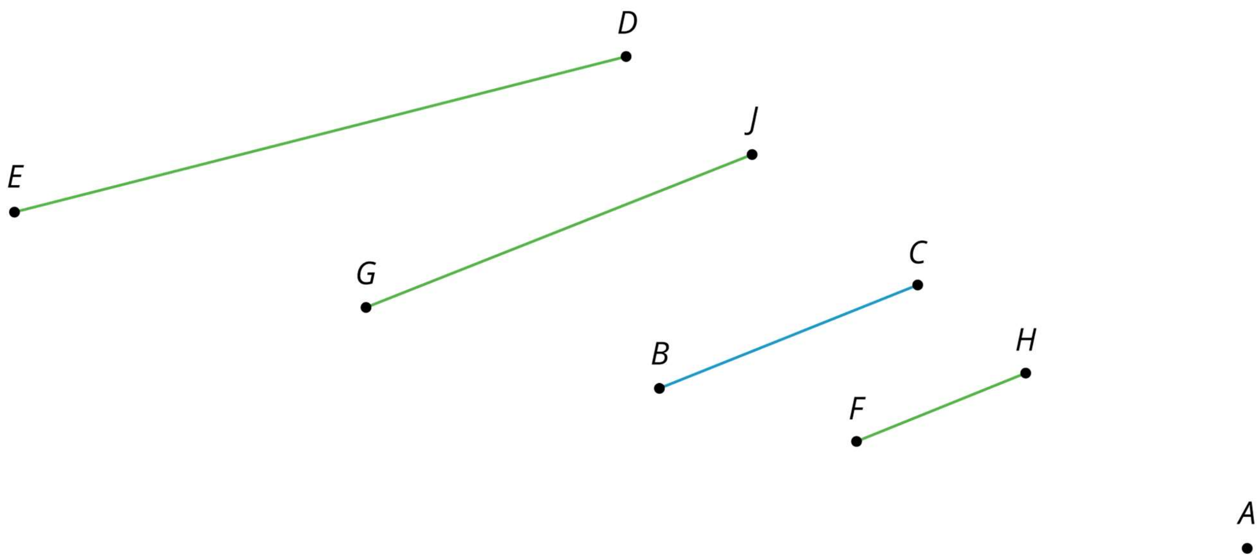
c. 

d. $(2 + 3s, s)$

e. $\frac{y}{x-2} = \frac{1}{3}$ (or equivalent)

4. **Problem 4 Statement**

Here are some line segments.



a. Which line segment is an enlargement of $\overline{BC}$ using A as the centre of enlargement and a scale factor of $\frac{2}{3}$?

b. Which line segment is an enlargement of $\overline{BC}$ using A as the centre of enlargement and a scale factor of $\frac{3}{2}$?

c. Which line segment is not an enlargement of $\overline{BC}$, and how do you know?

Solution

a. Line segment $\overline{FH}$ (A scale factor of $\frac{2}{3}$ produces a parallel line segment with shorter length.)

b. Line segment $\overline{GJ}$ (A scale factor of $\frac{3}{2}$ will produce a parallel line segment with longer length.)

c. Line segment $\overline{DE}$ (Enlargements take lines to parallel lines, and $\overline{DE}$ is not parallel to $\overline{FH}$.)