Lesson 20: The volume of a sphere

Goals

- Calculate the volume of a sphere, cylinder, and cone which have a radius of r and height of $2r$, and explain (orally) the relationship between their volumes.
- Create an equation to represent the volume of a sphere as a function of its radius, and explain (orally and in writing) the reasoning.

Learning Targets

- I can find the volume of a sphere when I know the radius.

Lesson Narrative

The purpose of this lesson is for students to recognise that the volume of a sphere with radius r is $\frac{4}{3}\pi r^3$ and begin to use the formula. Students inspect an image of a sphere that snugly fits inside a cylinder (they each have the same radius, and the height of the cylinder is equal to the diameter of the sphere), and use their intuition to guess about how the volume of the sphere relates to the volume of the cylinder, building on the work in the previous lesson. Then, they watch a video that shows a sphere inside a cylinder, and the contents of a cone (with the same base and height as the cylinder) are poured into the remaining space. This demonstration shows that for these shapes, the cylinder contains the volumes of the sphere and cone together. From this observation, the volume of a specific sphere is computed. Then, the formula $\frac{4}{3}\pi r^3$ for the volume of a sphere is derived. (At this point, this is taken to be true for any sphere even though we only saw a demonstration involving a particular sphere, cone, and cylinder. A general proof of the formula for the volume of a sphere would require mathematics beyond this level.)

Addressing

- Solve real-world and mathematical problems involving volume of cylinders, cones, and spheres.
- Know the formulae for the volumes of cones, cylinders, and spheres and use them to solve real-world and mathematical problems.

Instructional Routines

- Co-Craft Questions
- Compare and Connect
- Notice and Wonder

Required Preparation

For the A Sphere in a Cylinder activity, students will need to view a video.

Student Learning Goals

Let's explore spheres and their volumes.

20.1 Sketch a Sphere

Warm Up: 5 minutes

The purpose of this activity is for students to practise sketching spheres and labelling the radius and diameter of the sphere.

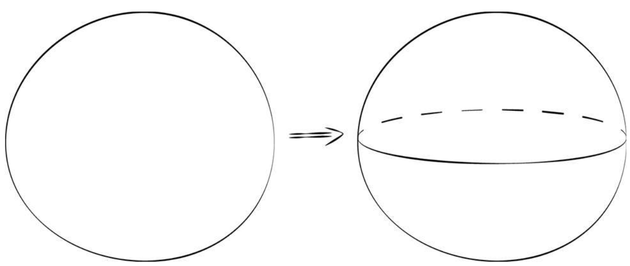
Launch

Give students 1–2 minutes of quiet work time, followed by a whole-class discussion.

Student Task Statement

Here is a method for quickly sketching a sphere:

- Draw a circle.
- Draw an oval in the middle whose edges touch the sphere.



1. Practise sketching some spheres. Sketch a few different sizes.
2. For each sketch, draw a radius and label it r.

Student Response

1. Answers vary.
2. Answers vary.

Activity Synthesis

Invite students to share their sketches. Ask students to share what the diameter would look like if they did not already draw one in. Remind students that sketches can be used to help visualise a problem where an image might not be provided. In today's lesson, they will be working with activities that might or might not have images provided and they should sketch or label any images provided to use as a tool to help understand the problem thoroughly.

20.2 A Sphere in a Cylinder

20 minutes

In this activity, students begin by looking at an image of a sphere in a cylinder. The sphere and cylinder have the same radius and the height of the cylinder is equal to the diameter of the sphere. Students consider the image and reason about how the volumes of the two shapes compare to get a closer estimate of the volume of the sphere.

Then students watch a video that shows a sphere inside a cylinder set up like the image. A cone with the same base and height as the cylinder is introduced and its contents poured into the sphere, completely filling the empty space between the sphere and the cylinder. Students are asked to record anything they notice and wonder as they watch the video and a list is created as a class.

Once students are told that the cylinder, cone, and sphere all have the same radius length and the cylinder and cone have equal heights, they are asked to figure out the volumes of the cylinder and cone and are given time to discuss with a partner how to find the volume of the sphere. They are asked to write an explanation in words first before doing actual calculations so that they can reason about the relationships shown in the video and how they can use those relationships to figure out the unknown volume.

Identify students who discuss either method for calculating the volume of the sphere:

- subtract the volume of the cone from the volume of the cylinder.
- make the connection that a cone is $\frac{1}{3}$ of the cylinder so the sphere must be the $\frac{2}{3}$ that fills up the rest of the cylinder.

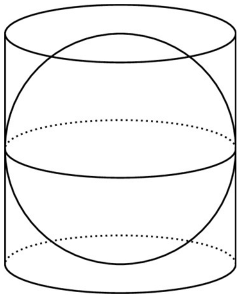
Instructional Routines

- Compare and Connect
- Notice and Wonder

Launch

Arrange students in groups of 2. Display for all to see:

A sphere fits snugly into a cylinder so that its circumference touches the curved surface of the cylinder and the top and bottom touch the bases of the cylinder.



Ask, “In the previous lesson we thought about hemispheres in cylinders. Here is a sphere in a cylinder. Which is bigger, the volume of the cylinder or the volume of the sphere? Do you think the bigger one is twice as big, more than twice as big, or less than twice as big?” then give students 1 minute of quiet think time. Invite students to share their responses and keep their answers displayed for all to see throughout the lesson so that they can be referred to during the Lesson Synthesis.

Show the video and tell students to write down anything they notice or wonder while watching. Ask students to share the things they noticed and wondered. Record and display their responses for all to see. If not mentioned by students, be sure these things are brought up:

- Notice:
  - The sphere fits inside the cylinder.
  - The sphere is filled up.
  - There is space around the sphere inside the cylinder.
  - It takes the volume of one cone to fill up the remaining spaces in the cylinder.
- Wonder:
  - Do the sphere and the cylinder have the same radius?
  - Do the cone and cylinder have the same radius?
  - Do the cone and cylinder have the same height?

Tell students that the sphere inside the cylinder seen in the video is the same as the one in the picture shown previously. Ask students: “does this give us any answers to the list of wonders?” (Yes, this tells us that the sphere and cylinder have the same radius.)

Tell students that the cone and cylinder have the same height and base area. Ask students:

- “Does this give us any more answers to the list of wonders?” (Yes, the cone and cylinder have the same height and radius.)
- “What does that mean about the volume of the cone and the volume of the cylinder in the video?” (The volume of the cone is $\frac{1}{3}$ the volume of the cylinder.)

Show the video one more time and ask students to think about how we might calculate the volume of the sphere if we know the radius of the cone or cylinder. Give students 1 minute of quiet think time followed by time for a partner discussion. Give students time to work on the task followed by a whole-class discussion.

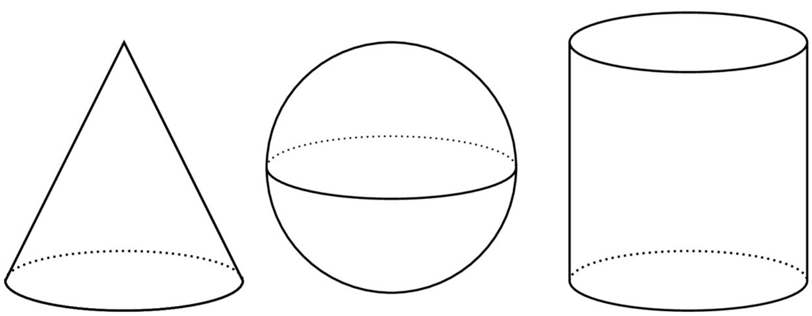
Video 'Volume of a Cylinder, Sphere, and Cone' available here: https://player.vimeo.com/video/304138133.

Representation: Internalise Comprehension. Demonstrate and encourage students to use colour coding and annotations to highlight connections between representations in a problem. For example, ask students to colour code the radii and heights in each of the shapes and when calculating volume.
Supports accessibility for: Visual-spatial processing Representing: Compare and Connect. Use this routine to compare and contrast the different ways students calculated the volume of the sphere. Ask students to consider what is the same and what is different about each method used. Draw students' attention to the different calculations (e.g., $2\left(\frac{250}{3}\right)\pi$ or $250\pi - \left(\frac{250}{3}\right)\pi = \left(\frac{500}{3}\right)\pi$) that equate to the volume of the sphere and how these calculations are related to the volume of the cone and cylinder. In this discussion, emphasise language used to help students make sense of strategies used to calculate the volume of the sphere. These exchanges strengthen students' mathematical language use and reasoning of volume.
Design Principle(s): Maximise meta-awareness; Support sense-making

Anticipated Misconceptions

If students struggle to keep track of all the dimensions of the different shapes, encourage them to label the images with the appropriate dimensions.

Student Task Statement



Here are a cone, a sphere, and a cylinder that all have the same radii and heights. The radius of the cylinder is 5 units. When necessary, express all answers in terms of π.

1. What is the height of the cylinder?
2. What is the volume of the cylinder?
3. What is the volume of the cone?
4. What is the volume of the sphere? Explain your reasoning.

Student Response

1. 10 units. The top of the sphere touches the top of the cylinder, so the diameter of the sphere is the height of the cylinder.
2. 250π cubic units because $V = \pi 5^2 \times 10$.

3. $\frac{250}{3}\pi$ cubic units because $V = \frac{1}{3}\pi 5^2 \times 10$.

4. $\frac{500}{3}\pi$ cubic units. Answers vary. Sample response: Subtracting the volume of the cone from the volume of the cylinder gives the volume of the sphere. So the volume of the sphere is $\frac{500}{3}\pi$ cubic units because $250\pi - \frac{250}{3}\pi = \frac{500}{3}\pi$.

Activity Synthesis

Select previously identified students to share their methods for calculating the volume of the sphere. Ask students to compare the two methods mentioned in the narrative:

- "What is different about these two methods?" (One is using the fact that the volume of the cone is $\frac{1}{3}$ of the volume of the cylinder so the sphere's volume must make up the other $\frac{2}{3}$. The other subtracts the volumes we know in order to get the unknown volume of sphere.)
- "What do the two methods have in common?" (Both methods are calculating the same amount but in different ways.)

Display for all to see. $\pi 5^2 \times 10 - \frac{1}{3}\pi 5^2 \times 10$

Ask students: "What does this expression represent?" (The volume of the cylinder minus the volume of the cone.)

Draw students' attention back to the guesses they made at the start of the activity about how much bigger the cylinder's volume is than the sphere. Ask students if we can answer that question now. (Note: if students do not make the connection that the sphere's volume is $\frac{2}{3}$ the volume of the cylinder, they will have another chance to look at the relationship in the next activity.)

20.3 Spheres in Cylinders

10 minutes

The purpose of this activity is to build from the concrete version in the previous activity to a generalised formula of a sphere with an unknown radius. The previous activity prepared students with strategies to work through this task where they must manipulate the variables in the volume equations. Students first calculate the volume of the cylinder and cone in the activity and use what they learned in the previous activity to calculate the volume of the sphere. Finally, they are asked about the relationship between the volume of the cylinder and sphere, which connects back to the discussion of the previous activity.

Identify students who:

- recognise that the volume of the sphere is $\frac{2}{3}$ the volume of the cylinder and use that to easily come up with the general formula for volume of a sphere $\frac{4}{3}\pi r^3$.
- use the subtraction method discussed in the previous activity.

Instructional Routines

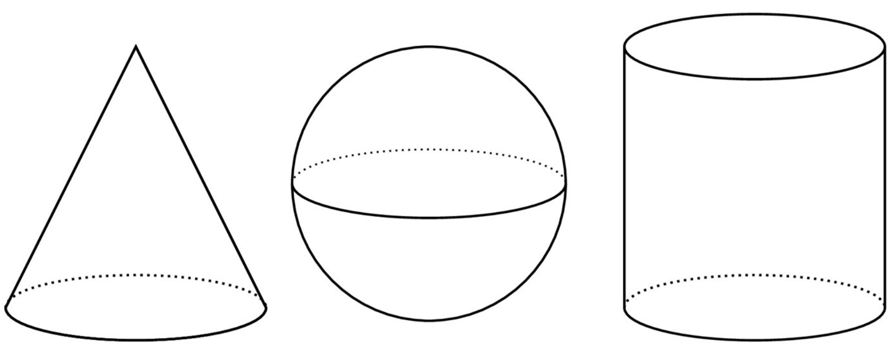
- Co-Craft Questions

Launch

Tell students that they are going to consider a different sphere inside of a cylinder along with a cone of the same height and radius as the sphere. This is similar to the previous activity; however, in this activity, the length of the radius r is unknown. Give students 4–6 minutes of quiet work time followed by a whole-class discussion.

Writing, Conversing: Co-Craft Questions. Display the three images of the shapes along with the task statement without revealing the questions that follow. Ask students to write possible questions that could be answered about this situation. Invite students to share their questions with a partner, and select 2–3 groups to share with the class. Listen for questions that require students to reason about the relationships between the volume of the three different shapes. Next, reveal the questions of the activity. This helps students produce the language of mathematical questions and talk about the relationships between the volumes of the different shapes in this task.
Design Principle(s): Maximise meta-awareness; Support sense-making

Student Task Statement



Here are a cone, a sphere, and a cylinder that all have the same radii and heights. Let the radius of the cylinder be r units. When necessary, express answers in terms of π.

1. What is the height of the cylinder in terms of r?
2. What is the volume of the cylinder in terms of r?
3. What is the volume of the cone in terms of r?
4. What is the volume of the sphere in terms of r?

5. A volume of the cone is $\frac{1}{3}$ the volume of a cylinder. The volume of the sphere is what fraction of the volume of the cylinder?

Student Response

1. $2r$ because the diameter of the sphere is the height of the cylinder

2. $2\pi r^3$ because $V = \pi r^2 2r$

3. $\frac{2}{3}\pi r^3$ because $V = \frac{1}{3}\pi r^2 2r$

4. $\frac{4}{3}\pi r^3$ because $2\pi r^3 - \frac{2}{3}\pi r^3$

5. $\frac{2}{3}$

Activity Synthesis

Select previously identified students to share their methods for calculating the volume of a sphere. Display for all to see the two different strategies side by side and ask students:

- "Which method did you use to calculate the volume of the sphere?"
- "Look at the method that you did not use. Explain to a partner why that method works."
- "Which method do you prefer? Why?"

If students did not use both of the methods described in the narrative and outlined below, add them to the list of methods for students to compare before asking the questions. Display both methods side by side for all to see and ask the same questions.

The $\frac{2}{3}$ method:

volume of the sphere $= \frac{2}{3}$(volume of the cylinder) $= \frac{2}{3}(2\pi r^3) = \frac{4}{3}\pi r^3$

Subtraction method:

volume of the sphere $=$ volume of the cylinder $-$ volume of the cone $= 2\pi r^3 - \frac{2}{3}\pi r^3$

$$= (2 - \frac{2}{3})\pi r^3$$

$$= \frac{4}{3}\pi r^3$$

Although either method works, there are reasons students might choose one over the other. The subtraction method is a bit more involved as it requires the distributive property to combine like terms and subtracting $\frac{2}{3}$ from 2. It might make more sense to students, however, since it describes the video they saw in that the volume of the sphere is the difference between the volumes of the cylinder and cone. The $\frac{2}{3}$ method is a bit simpler in terms of manipulating expressions, but students might not fully understand why the volume of the sphere is $\frac{2}{3}$ the volume of the cylinder.

Add the formula $V = \frac{4}{3}\pi r^3$ and a diagram of a sphere to your classroom displays of the formulas being developed in this unit.

Lesson Synthesis

Display for all to see the equation $V \approx 4r^3$. Tell students "A quick estimate for the volume of a sphere of radius r that you can use if you don't have a calculator is $V \approx 4r^3$. (No fraction or π!) How good of an approximation do you think this is? Can you come up with a better one?" Ask students to calculate the volume of a sphere with a radius of 10 inches using:

- the actual volume formula $V = \frac{4}{3}\pi r^3$ (4 188.79 cubic inches)
- the approximation formula $V \approx 4r^3$ (4 000 cubic inches)
- their own approximation formula. (Possible formula: $V \approx 4 \times r^3 \times 1.05$. 4 200 cubic inches)

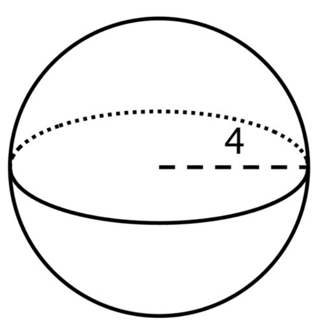
Give students quiet think time, then time to compare their improved approximations with a partner and decide which of their formulas is the 'best approximation.' Invite partners to share their choices with the class. Record and display student-created volume of a sphere approximation formulas for all to see.

20.4 Volumes of Spheres

Cool Down: 5 minutes

Student Task Statement

Recall that the volume of a sphere is given by the formula $V = \frac{4}{3}\pi r^3$.



1. Here is a sphere with radius 4 feet. What is the volume of the sphere? Express your answer in terms of π.

2. A spherical balloon has a diameter of 4 feet. Approximate how many cubic feet of air this balloon holds. Use 3.14 as an approximation for π, and give a numerical answer.

Student Response

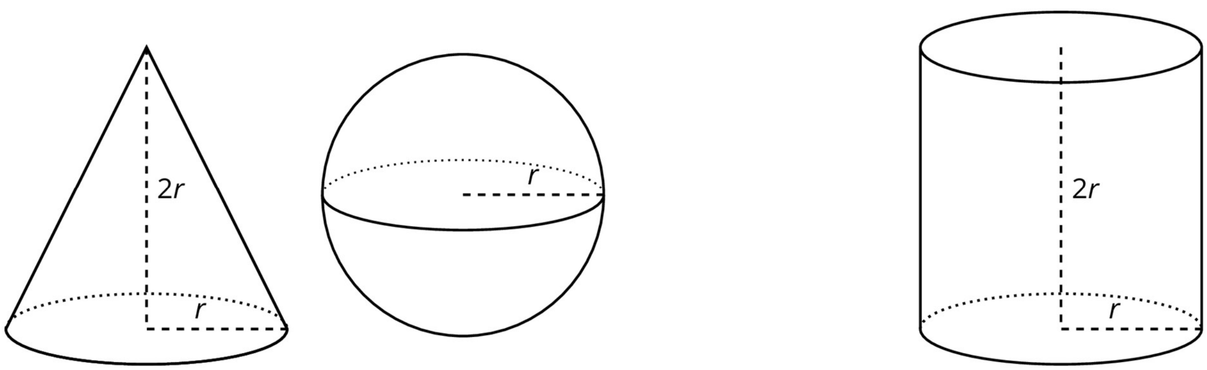
1. $\frac{256}{3}\pi$ or 85.33π cubic feet because $V = \frac{4}{3}\pi(4)^3$

2. 33.49 cubic feet because $V = \frac{4}{3}\pi(2)^3$

Student Lesson Summary

Think about a sphere with radius r units that fits snugly inside a cylinder. The cylinder must then also have a radius of r units and a height of $2r$ units. Using what we have learned about volume, the cylinder has a volume of $\pi r^2 h = \pi r^2 \times (2r)$, which is equal to $2\pi r^3$ cubic units.

We know from an earlier lesson that the volume of a cone with the same base and height as a cylinder has $\frac{1}{3}$ of the volume. In this example, such a cone has a volume of $\frac{1}{3} \times \pi r^2 \times 2r$ or just $\frac{2}{3}\pi r^3$ cubic units.



If we filled the cone and sphere with water, and then poured that water into the cylinder, the cylinder would be completely filled. That means the volume of the sphere and the volume of the cone add up to the volume of the cylinder. In other words, if V is the volume of the sphere, then

$$V + \frac{2}{3}\pi r^3 = 2\pi r^3$$

This leads to the formula for the volume of the sphere,

$$V = \frac{4}{3}\pi r^3$$

Lesson 20 Practice Problems

Problem 1 Statement

a. A cube's volume is 512 cubic units. What is the length of its edge?

b. If a sphere fits snugly inside this cube, what is its volume?

c. What fraction of the cube is taken up by the sphere? What percentage is this? Explain or show your reasoning.

Solution

a. 8 units

b. $\frac{256}{3}\pi$ cubic units

c. $\frac{\pi}{6}$, which is slightly more than 50%. Sample explanation: The volume of the sphere as a fraction of the volume of the cube is $\frac{\frac{4}{3}\pi\times4^3}{512}$. To make this fraction easier to work with, note that $\frac{\frac{4}{3}\pi\times4^3}{512}=\frac{\frac{4}{3}\pi\times4^3}{8^3}=\frac{4}{3}\pi\times\left(\frac{1}{2}\right)^3=\frac{4\pi}{24}=\frac{\pi}{6}$. Since π is slightly more than 3, then $\frac{\pi}{6}$ is slightly more than 50%.

Problem 2 Statement

Sphere A has radius 2 cm. Sphere B has radius 4 cm.

a. Calculate the volume of each sphere.

b. The radius of Sphere B is double that of Sphere A. How many times greater is the volume of B?

Solution

a. Sphere A: $\frac{32}{3}\pi$ cm^3, Sphere B: $\frac{256}{3}\pi$ cm^3

b. The volume of Sphere B is 8 times greater than the volume of Sphere A, which is 2^3 times.

Problem 3 Statement

Three cones have a volume of 192π cm^3. Cone A has a radius of 2 cm. Cone B has a radius of 3 cm. Cone C has a radius of 4 cm. Find the height of each cone.

Solution

– Cone A has a height of 144 cm.

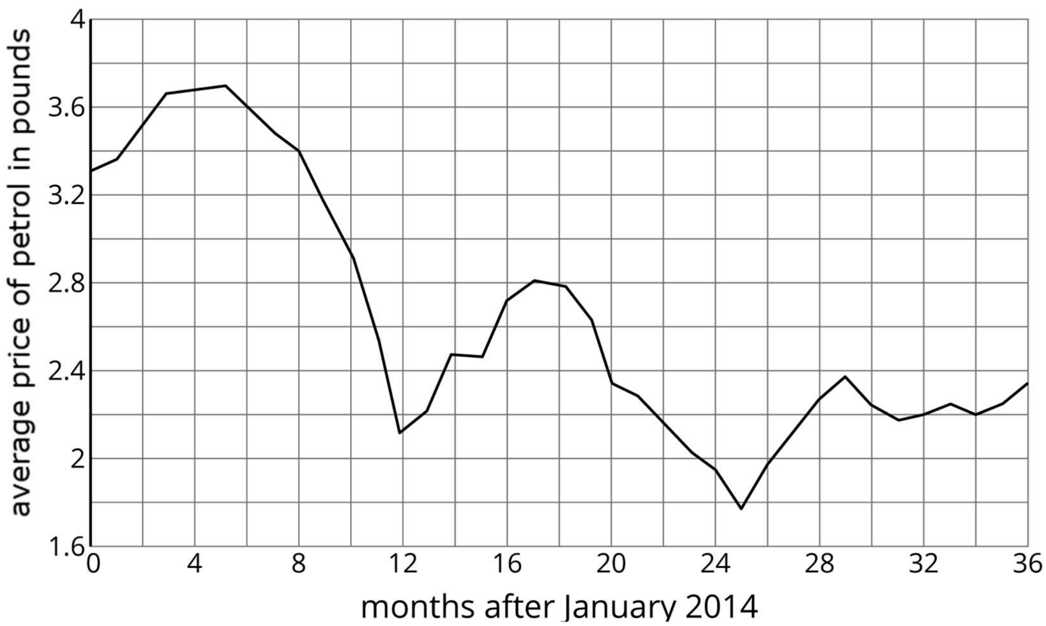
– Cone B has a height of 64 cm.

– Cone C has a height of 36 cm.

In each case, the height can be found by solving the formula $192\pi = \frac{1}{3}\pi \times r^2 h$ for h.

Problem 4 Statement

The graph represents the average price of petrol in pounds as a function of the number of months after January 2014.



a. How many months after January 2014 was the price of petrol the greatest?

b. Did the average price of petrol ever get below £2?

c. Describe what happened to the average price of petrol in 2014.

Solution

a. 5

b. Yes, in the 24th, 25th, and 26th months

c. Answers vary. Sample response: The average price of petrol rose from January until 5 months later (June) and then decreased for the rest of the year.

Problem 5 Statement

Match the description of each sphere to its correct volume.

A. Sphere A: radius of 4 cm

B. Sphere B: diameter of 6 cm

C. Sphere C: radius of 8 cm

D. Sphere D: radius of 6 cm

1. 288π cm^3

2. $\frac{256}{3}\pi$ cm^3

3. 36π cm^3

4. $\frac{2048}{3}\pi$ cm3

Solution

- A: 2
- B: 3
- C: 4
- D: 1

Problem 6 Statement

While conducting an inventory in their bicycle shop, the owner noticed the number of bicycles is 2 fewer than 10 times the number of tricycles. They also know there are 410 wheels on all the bicycles and tricycles in the store. Write and solve a system of equations to find the number of bicycles in the store.

Solution

$b = 10t - 2, 3t + 2b = 410$. There are 178 bicycles in the store.