Lesson 10: Applications of Pythagoras’ theorem

Goals

- Describe (orally) situations that use right-angled triangles, and explain how Pythagoras’ theorem could help solve problems in those situations.
- Use Pythagoras’ theorem to solve problems within a context, and explain (orally) how to organise the reasoning.

Learning Targets

- I can use Pythagoras’ theorem to solve problems.

Lesson Narrative

In this lesson students use Pythagoras’ theorem and its converse as a tool to solve application problems. In the first activity, they solve a problem involving the distance and speed of two children walking and riding a bike along different sides of a triangular region. In the second activity they find internal diagonals of cuboids.

Addressing

- Use square root and cube root symbols to represent solutions to equations of the form $x^2 = p$ and $x^3 = p$, where p is a positive rational number. Evaluate square roots of small perfect squares and cube roots of small perfect cubes. Know that $\sqrt{2}$ is irrational.
- Apply Pythagoras’ theorem to determine unknown side lengths in right-angled triangles in real-world and mathematical problems in two and three dimensions.
- Know that there are numbers that are not rational, and approximate them by rational numbers.

Instructional Routines

- Co-Craft Questions
- Three Reads
- Discussion Supports
- Poll the Class
- Think Pair Share

Student Learning Goals

Let’s explore some applications of Pythagoras’ theorem.

10.1 Closest Estimate: Square Roots

Warm Up: 5 minutes

The purpose of this warm-up is for students to reason about square roots by estimating the value of each expression. The values given as choices are close in range to encourage students to use the square roots they know to help them estimate ones they do not. These understandings will be helpful for students in upcoming activities where they will be applying Pythagoras' theorem.

While four problems are given, it may not be possible to share every strategy. Consider gathering only two or three different strategies per problem.

Instructional Routines

- Discussion Supports

Launch

Display one problem at a time. Give students 30 seconds of quiet think time for each problem and ask them to give a signal when they have an answer and a strategy. Keep all previous problems displayed throughout the talk. Follow with a whole-class discussion.

Student Task Statement

Which estimate is closest to the actual value of the expression? Explain your reasoning.

1. $\sqrt{24}$
    - 4
    - 4.5
    - 5
2. $\sqrt{7}$
    - 2
    - 2.5
    - 3
3. $\sqrt{42}$
    - 6
    - 6.5
    - 7
4. $\sqrt{10} + \sqrt{97}$
    - 13
    - 13.25

- 13.5

Student Response

1. 5; Strategies may vary. Sample response: The square root of 25 is 5, and 24 is very close to 25.
2. 2.5; Strategies may vary. Sample response: 7 is just a little over halfway between 2^2 and 3^2
3. 6.5; Strategies may vary. Sample response: 42 is almost halfway between 6^2 and 7^2
4. 13; Strategies may vary. Sample response: The square root of 10 is a little more than 3 and the square root of 97 is just a little bit less than 10.

Activity Synthesis

Ask students to share their strategies for each problem. Record and display their responses for all to see. After each student shares, ask if their chosen estimate is more or less than the actual value of each expression.

Speaking: Discussion Supports.: Display sentence frames to support students when they explain their strategy. For example, "First, I _____ because . . ." or "I noticed _____ so I . . . ." Some students may benefit from the opportunity to rehearse what they will say with a partner before they share with the whole class.
Design Principle(s): Optimise output (for explanation)

10.2 Cutting Corners

15 minutes

The purpose of this activity is for students to use Pythagoras' theorem to reason about distances and speeds to figure out who will win a race. Students must translate between the context and the geometric representation of the context and back. Identify students whose work is clearly labelled and organised to share during the whole-class discussion.

Instructional Routines

- Three Reads
- Poll the Class

Launch

Arrange students in groups of 2. Provide students with access to calculators. Ask students to read and answer the first problem and then give a signal when they have done so. Next, poll the class to see who students think will win the race and post the results of the poll for all to see. Select 2–3 students to share why they chose who they chose. The conversation should illuminate a few key points: Mai travels farther than Tyler, but she is also going faster, so it is not immediately clear who will win.

Give 2 minutes of quiet work time for the second problem, followed by partner and then whole-class discussions.

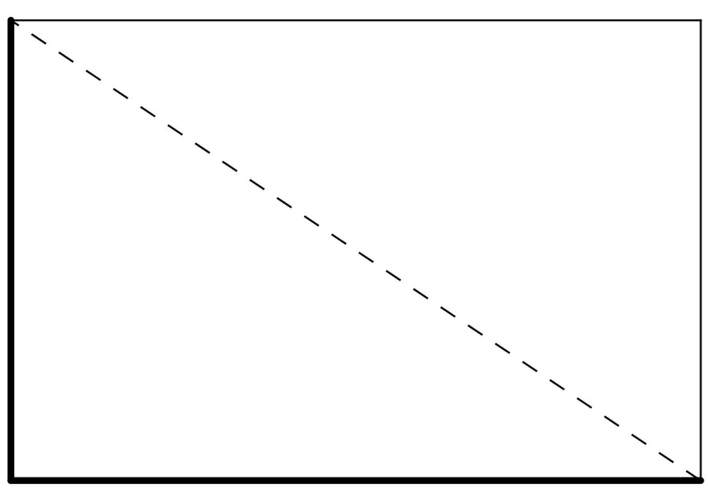
Representation: Access for Perception. Read the situation aloud. Students who both listen to and read the information will benefit from extra processing time.
Supports accessibility for: Language Reading: Three Reads. Use this routine to support reading comprehension of this problem without solving it for students. In the first read, students read the problem with the goal of comprehending the situation (e.g., Mai and Tyler are racing from one corner of the rectangular field to the opposite corner.). In the second read, ask students to identify important quantities that can be counted or measured (e.g., the length and width of the field; Mai's speed on her bike; Tyler's running speed). After the third read, reveal the question: "Who do you think will win? By how much?" Ask students to brainstorm possible solution strategies to answer the question (e.g., Use Pythagoras' theorem to calculate the length of the diagonal across the field; Use Mai's and Tyler's speeds to calculate how long it will take them to reach the opposite corner.). This will help students concentrate on making sense of the situation before rushing to a solution or method.
Design Principle(s): Support sense-making

Anticipated Misconceptions

Some students may rush through reading the problem and mix up who is travelling which path at what speed. Encourage these students to label the relevant distances and information on the diagram.

Student Task Statement

Mai and Tyler were standing at one corner of a large rectangular field and decided to race to the opposite corner. Since Mai had a bike and Tyler did not, they thought it would be a fairer race if Mai rode along the path that surrounds the field while Tyler ran the shorter distance directly across the field. The field is 100 metres long and 80 metres wide. Tyler can run at around 5 metres per second, and Mai can ride her bike at around 7.5 metres per second.



1. Before making any calculations, who do you think will win? By how much? Explain your thinking.

2. Who wins? Show your reasoning.

Student Response

1. Answers vary. Sample response: Mai is going faster, but she also has a longer distance to travel. I think that she might be fast enough to beat Tyler even if she is going a little farther.

2. Mai wins. Mai has 180 metres to travel going 7.5 metres per second, which will take her 24 seconds since $\frac{180}{7.5} = 24$. Using Pythagoras' theorem, Tyler travels $\sqrt{16\,400}$ metres going 5 metres per second, which will take him approximately 25.6 seconds since $\frac{\sqrt{16\,400}}{5} \approx 25.6$. Mai beats Tyler to the opposite corner by about 1.6 seconds.

Are You Ready for More?

A calculator may be necessary to answer the following questions. Round answers to the nearest hundredth.

1. If you could give the loser of the race a head start, how much time would they need in order for both people to arrive at the same time?

2. If you could make the winner go slower, how slow would they need to go in order for both people to arrive at the same time?

Student Response

1. Tyler needs roughly 1.6 seconds of head start to beat Mai. He travels approximately 128.06 metres going 5 metres per second, which will take him 25.6 seconds. Mai finishes in 24 seconds.

2. About 7.03 s per second. If Mai goes 7.03 metres per second, then she will finish the race in $\frac{180}{7.03} \approx 25.6$ seconds. She will beat Tyler by a fraction of a second.

Activity Synthesis

The purpose of this discussion is for students to share how they organised their work for the problem and see how accurate students' predictions at the start of the activity were. Select 1–2 previously identified students to share their work. Directly draw attention to how careful labelling of provided diagrams and organisation of calculations when problems have multiple steps helps to both solve problems and identify errors. For example, in problems where multiple calculations are completed using a calculator, it is easy to copy an incorrect number, such as writing 14 600 instead of 16 400 for the sum $100^2 + 80^2$.

10.3 Internal Dimensions

15 minutes

The purpose of this task is for students to use Pythagoras' theorem to calculate which cuboid has the longer diagonal length. To complete the activity, students will need to picture or sketch the right-angled triangles necessary to calculate the diagonal length.

Identify groups using well-organised strategies to calculate the diagonals. For example, for Cuboid K some groups may draw in the diagonal on the bottom face of the cube and label it with an unknown variable, such as s, to set up the calculation for the diagonal as $\sqrt{6^2 + s^2}$. Students would then know that to finish this calculation, they would need to find the value of s, which is can be done using the edge lengths of the face of the cuboid s is drawn on.

The idea that Pythagoras' theorem might be applied several times in order to find a missing length is taken even further in the extension, where students can apply it over a dozen times, hopefully noticing some repeated reasoning along the way, in order to find the last length.

Instructional Routines

- Co-Craft Questions
- Think Pair Share

Launch

Arrange students in groups of 2. Provide access to calculators. Ask students to read and consider the first problem and then give a signal when they have done so. If possible, use an actual cuboid, such as a small box, to help students understand what length the diagonal shows in the image. Once students understand what the first problem is asking, poll the class to see which cuboid they think has the longer diagonal and post the results of the poll for all to see. Select 2–3 students to share why they chose the cuboid they did. The conversation should illuminate a few key points: Cuboid K has one edge with length 6 units, which is longer than any of the edges of Cuboid L, but it also has a side of length 4 units, which is shorter than any of the edges of Cuboid L. Cuboid K also has a smaller volume than Cuboid L.

Give 1 minute of quiet think time for students to brainstorm how they will calculate the lengths of each diagonal. Ask partners to discuss their strategies before starting their calculations. Follow with a whole-class discussion.

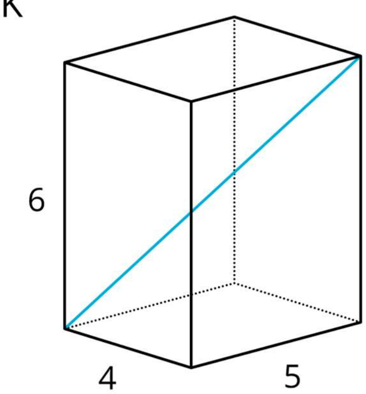
Representation: Internalise Comprehension. Provide access to 3-D models of cuboids (for example, a tissue box) for students to view or manipulate. Use colour coding to highlight and make connections between corresponding parts.
Supports accessibility for: Visual-spatial processing; Conceptual processing Conversing, Writing: Co-Craft Questions. Before revealing the questions in this activity, display only the image of the cuboids. Invite students to work with a partner to write mathematical questions that could be asked about the cuboids. Select 1–2 groups to share their questions

with the class. Listen for and amplify questions about the lengths of the diagonals. For example, "What is the length of the diagonal of Cuboid K?" and "Which cuboid has the longer diagonal length?" If no student asks these questions, ask students to adapt a question to align with the learning goals of this lesson. Then reveal and ask students to work on the actual questions of the task. This routine will help develop students' meta-awareness of language as they generate questions about calculating the diagonal length of a cuboid.
Design Principle(s): Maximise meta-awareness

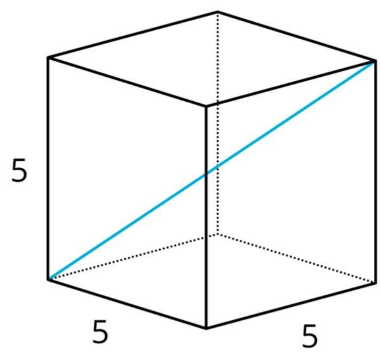
Student Task Statement

Here are two cuboids:

K



L



1. Which shape do you think has the longer diagonal? Note that the shapes are not drawn to scale.

2. Calculate the lengths of both diagonals. Which one is actually longer?

Student Response

1. Answers vary. Sample Response: I think Cuboid L has a longer diagonal since it has a larger volume than Cuboid K and similar dimensions.

2. Cuboid K has a longer diagonal of $\sqrt{77}$ units. Sample response: for Cuboid K, the length of the diagonal, d, is $\sqrt{6^2 + s^2}$, where s is the length of the diagonal of the bottom face of the cuboid, since $6^2 + s^2 = d^2$ by Pythagoras' theorem. We can calculate s by first recognising it as the hypotenuse of the triangle with shorter sides of length 4 and 5, which means $s = \sqrt{41}$ since $4^2 + 5^2 = 41$. Putting these together, $d = \sqrt{77}$ since $\sqrt{6^2 + s^2} = \sqrt{6^2 + (\sqrt{41})^2} = \sqrt{36 + 41} = \sqrt{77}$. Noticing that this is the same as $\sqrt{6^2 + 4^2 + 5^2}$, which is the square root of the sum of the squares of the three edge lengths of the cuboid, we can conclude that the diagonal of Cuboid L has length $\sqrt{75}$ since $\sqrt{5^2 + 5^2 + 5^2} = \sqrt{75}$.

Activity Synthesis

Ask groups to share how they calculated the diagonal of one of the cuboids. Display the figures in the activity for all to see and label the them while groups share.

If time allows and no groups pointed out how the diagonal length of the cuboids are the square root of the sum of the squares of the three edge lengths, use Cuboid K and the calculations students shared to do so. For example, to calculate the diagonal of Cuboid K, students would have to calculate $\sqrt{6^2+\left(\sqrt{41}\right)^2}$, but $\sqrt{41}$ is from $\sqrt{5^2+4^2}$ which means the diagonal length is really $\sqrt{6^2+5^2+4^2}$ since $\sqrt{6^2+\left(\sqrt{41}\right)^2}=\sqrt{6^2+\left(\sqrt{5^2+4^2}\right)^2}=\sqrt{6^2+5^2+4^2}$. So, for a cuboid with sides d, e, and f, the length of the diagonal of the cuboid is just $\sqrt{d^2+e^2+f^2}$.

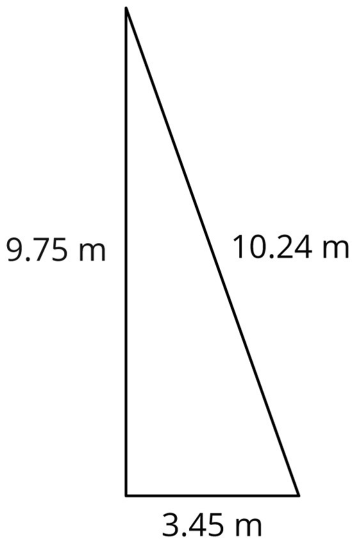
Lesson Synthesis

Tell students that there are many situations in the world where we can use Pythagoras' theorem to solve problems. Ask, "What situations that you can think of involve right-angled triangles?" Give brief quiet think time, then invite students to share their ideas. For example, thick wires (called guy-wires) are used to keep telephone poles upright and their length depends on how high up the pole they attach and how far away from the pole they hook into the ground.

10.4 Jib Sail

Cool Down: 5 minutes

Student Task Statement



Sails come in many shapes and sizes. The sail on the right is a triangle. Is it a right-angled triangle? Explain your reasoning.

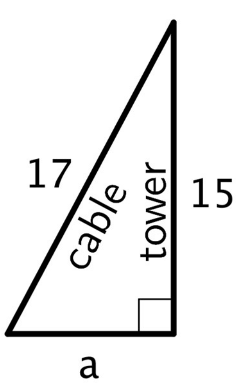
Student Response

No. The sum of the squares of the two shorter sides is 106.965 square metres, and the square of the longest side is 104.8576 square metres. So by the converse of Pythagoras' theorem, it is not a right-angled triangle.

Student Lesson Summary

Pythagoras' theorem can be used to solve any problem that can be modelled with a right-angled triangle where the lengths of two sides are known and the length of the other side needs to be found. For example, let's say a cable is being placed on level ground to support a tower. It's a 17-foot cable, and the cable should be connected 15 feet up the tower. How far away from the bottom of the tower should the other end of the cable connect to the ground?

It is often very helpful to draw a diagram of a situation, such as the one shown here:

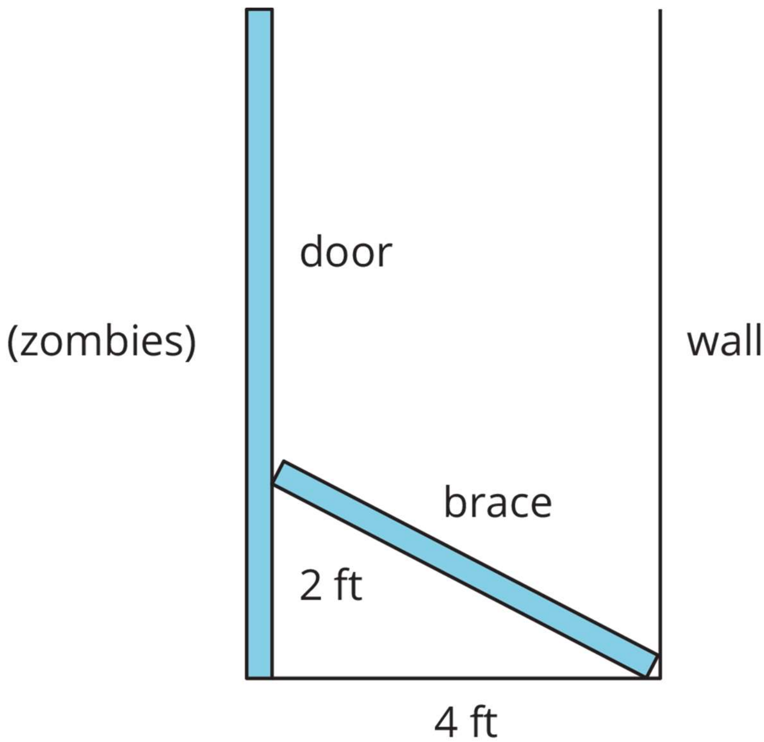


It's assumed that the tower makes a right angle with the ground. Since this is a right-angled triangle, the relationship between its sides is $a^2 + b^2 = c^2$, where c represents the length of the hypotenuse and a and b represent the lengths of the other two sides. The hypotenuse is the side opposite the right angle. Making substitutions gives $a^2 + 15^2 = 17^2$. Solving this for a gives $a = 8$. So, the other end of the cable should connect to the ground 8 feet away from the bottom of the tower.

Lesson 10 Practice Problems

1. **Problem 1 Statement**

   A man is trying to zombie-proof his house. He wants to cut a length of wood that will brace a door against a wall. The wall is 4 feet away from the door, and he wants the brace to rest 2 feet up the door. About how long should he cut the brace?



   Solution

   Around 4.5 feet. Solving $2^2 + 4^2 = c^2$, we get $c = \sqrt{20}$, which is approximately 4.5.

2. **Problem 2 Statement**

   At a restaurant, a rubbish bin's opening is rectangular and measures 7 inches by 9 inches. The restaurant serves food on trays that measure 12 inches by 16 inches. Jada says it is impossible for the tray to accidentally fall through the rubbish bin opening because the shortest side of the tray is longer than either edge of the opening. Do you agree or disagree with Jada's explanation? Explain your reasoning.

Solution

I disagree. Explanations vary. Sample explanation: It is impossible for the tray to fall through the opening, but not for the reason Jada gives. The longest dimension of the rubbish bin opening is the diagonal. The diagonal is $\sqrt{130}$ inches long, because $7^2 + 9^2 = 130$. The diagonal is between 11 and 12 inches long, because $11^2 < 130 < 12^2$. The tray cannot fall through the opening because the diagonal is a little shorter than the shortest dimension of the tray.

3. **Problem 3 Statement**

   Select **all** the sets that are the three side lengths of right-angled triangles.

   a. 8, 7, 15

   b. 4, 10, $\sqrt{84}$

   c. $\sqrt{8}$, 11, $\sqrt{129}$

   d. $\sqrt{1}$, 2, $\sqrt{3}$

   Solution ["B", "C", "D"]

4. **Problem 4 Statement**

   For each pair of numbers, which of the two numbers is larger? How many times larger?

   a. 12×10^9 and 4×10^9

   b. 1.5×10^{12} and 3×10^{12}

   c. 20×10^4 and 6×10^5

   Solution

   a. 12×10^9, 3 times larger

   b. 3×10^{12}, 2 times larger

   c. 6×10^5, 3 times larger

5. **Problem 5 Statement**

   A line contains the point (3,5). If the line has negative gradient, which of these points could also be on the line?

a. (2,0)

b. (4,7)

c. (5,4)

d. (6,5)

Solution C

6. **Problem 6 Statement**

Noah and Han are preparing for a skipping rope contest. Noah can jump 40 times in 0.5 minutes. Han can jump y times in x minutes, where $y = 78x$. If they both jump for 2 minutes, who jumps more times? How many more?

Solution

Noah jumps 160 times and Han jumps 156 times, so Han jumps 4 more times.