Mathematics Education Innovation

Lesson 15: Writing systems of equations

Goals

- Categorise (in writing) systems of equations, including systems with infinitely many or no solutions, and calculate the solution for a system using a variety of strategies.
- Create a system of equations that represents a situation and interpret (orally and in writing) the solution in context.

Learning Targets

- I can write a system of equations from a real-world situation.

Lesson Narrative

Previously, students have been given systems of equations to interpret and solve. In this lesson, they learn to write their own systems representing different contexts, and to interpret the solutions for those systems. Different contexts can lead to systems in different forms, so students also continue to practise looking at different systems and thinking ahead about how to solve them. When students represent a real-world problem with a system, they develop an important skill for mathematical modelling.

Addressing

- Analyse and solve pairs of simultaneous linear equations.
- Solve systems of two linear equations in two variables algebraically, and estimate solutions by graphing the equations. Solve simple cases by inspection. For example, $3x + 2y = 5$ and $3x + 2y = 6$ have no solution because $3x + 2y$ cannot simultaneously be 5 and 6.
- Solve real-world and mathematical problems leading to two linear equations in two variables. For example, given coordinates for two pairs of points, determine whether the line through the first pair of points intersects the line through the second pair.

Building Towards

- Solve real-world and mathematical problems leading to two linear equations in two variables. For example, given coordinates for two pairs of points, determine whether the line through the first pair of points intersects the line through the second pair.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Clarify, Critique, Correct
- Information Gap Cards
- Discussion Supports

- Think Pair Share

Required Materials

Pre-printed slips, cut from copies of the blackline master

Info Gap: Racing and Play Tickets Problem Card 1 Priya and Lin are having a race. The equation $y = 9.5x$ represents one person's progress. If one of them had a head start, how long is it until the other person catches up?	Info Gap: Racing and Play Tickets Data Card 1 • The equation $y = 9.5x$ represents Lin's progress, where y is her distance in feet, from the starting line, and x is the time, in seconds, that she has been running. • Priya had the head start. She was 18 feet in front of the starting line when Lin started. • Priya runs at a constant 8 feet per second.
Info Gap: Racing and Play Tickets Problem Card 2 A school sells adult tickets and student tickets for the drama play. One equation that represents the situation is $x + y = 115$. How many of each type of ticket did they sell?	Info Gap: Racing and Play Tickets Data Card 2 • The equation $x + y = 115$ represents how many tickets were sold, where x is the number of student tickets and y is the number of adult tickets. This equation is equivalent to $x = 115 - y$. • Adult tickets cost £8 each. • Student tickets cost £3 each. • The school made £720 in total from ticket sales.

Required Preparation

You will need the Info Gap: Racing and Play Tickets blackline master for this lesson. Make 1 copy for every 4 students, and cut them up ahead of time.

Student Learning Goals

Let's write systems of equations from real-world situations.

15.1 How Many Solutions? Matching

Warm Up: 5 minutes

This warm-up asks students to connect the algebraic representations of systems of equations to the number of solutions. Efficient students will recognise that this can be done without solving the system, but rather using gradient, y-intercept, or other methods for recognising the number of solutions.

Monitor for students who use these methods:

1. Solve the systems to find the number of solutions.
2. Use the gradient and y-intercept to determine the number of solutions.
3. Manipulate the equations into another form, then compare the equations.
4. Notice that the left side of the second equation in system C is double the left side of the first equation, but the right side is not.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect

Launch

Arrange students in groups of 2. Tell students that each option can be used more than once or not at all. Allow students 2 minutes of work time followed by a whole-class discussion.

Student Task Statement

Match each system of equations with the number of solutions the system has.

1. $\begin{cases} y = -\frac{4}{3}x + 4 \\ y = -\frac{4}{3}x - 1 \end{cases}$
2. $\begin{cases} y = 4x - 5 \\ y = -2x + 7 \end{cases}$
3. $\begin{cases} 2x + 3y = 8 \\ 4x + 6y = 17 \end{cases}$
4. $\begin{cases} y = 5x - 15 \\ y = 5(x - 3) \end{cases}$

1. No solutions
2. One solution
3. Infinitely many solutions

Student Response

1. 1
2. 2
3. 1
4. 3

Activity Synthesis

The purpose of the discussion is to bring out any methods students used to find the number of solutions for the systems.

Select previously identified students to share their methods for finding the number of solutions in the sequence given in the narrative. After each student shares their method, ask the class which method they preferred to answer the given question. Connect each problem to the concepts learned in the previous lesson by asking students to describe how the graphs of the lines for each system might intersect.

15.2 Situations and Systems

10 minutes

In this activity, students are presented with a number of scenarios that could be solved using a system of equations. Students are not asked to solve the systems of equations, since the focus at this time is for students to understand how to set up the equations for the system and to understand what the solution represents in context.

Instructional Routines

- Clarify, Critique, Correct
- Think Pair Share

Launch

Arrange students in groups of 2. Suggest that groups split up the problems so that one person works on the first and third problem while their partner works on the second and fourth. Students may work with their partners to get help when they are stuck, but are encouraged to try to set up the equations on their own first. Partners should discuss their systems and interpretation of the solution after each has had a chance to work on their own.

Allow students 5–7 minutes of partner work time followed by a whole-class discussion.

Engagement: Provide Access by Recruiting Interest. Leverage choice around perceived challenge. Invite students to select 2–3 of the situations to complete.
Supports accessibility for: Organisation; Attention; Social-emotional skills

Student Task Statement

For each situation:

- Create a system of equations.
- Then, without solving, interpret what the solution to the system would tell you about the situation.

1. Lin's family is out for a bike ride when her dad stops to take a picture of the scenery. He tells the rest of the family to keep going and that he'll catch up. Lin's dad spends 5 minutes taking the photo and then rides at 0.24 miles per minute until he meets up with the rest of the family further along the bike path. Lin and the rest were riding at 0.18 miles per minute.

2. Noah is planning a kayaking trip. Kayak Rental A charges a base fee of £15 plus £4.50 per hour. Kayak Rental B charges a base fee of £12.50 plus £5 per hour.

3. Diego is making a large batch of pastries. The recipe calls for 3 strawberries for every apple. Diego used 52 fruits all together.

4. Flour costs £0.80 per pound and sugar costs £0.50 per pound. An order of flour and sugar weighs 15 pounds and costs £9.00.

Student Response

1. $$\begin{cases} d = 0.24t \\ d = 0.18t + 5 \times 0.18 \end{cases}$$

   The solution would represent the time (t) it would take for Lin's dad to catch up with the rest of the family.

2. $$\begin{cases} y = 15 + 4.5x \\ y = 12.5 + 5x \end{cases}$$

   The solution would represent the amount of time (x) spent with the kayak so that the cost (y) would be the same from each rental company.

3. $$\begin{cases} s = 3a \\ a + s = 52 \end{cases}$$

   The solution would represent the number of apples (a) and the number of strawberries (s) that Diego used to make the large batch of pastries.

4. $$\begin{cases} 0.8f + 0.5s = 9 \\ f + s = 15 \end{cases}$$

   The solution would represent the number of pounds of flour (f) and the number of pounds of sugar (s) purchased in this order.

Activity Synthesis

The focus of the discussion should be on making sense of the context and interpreting the solutions within the context of the problems.

Invite groups to share their systems of equations and interpretation of the solution for each problem. As groups share, record their systems of equations for all to see. When necessary, ask students to explain the meaning of the variables they used. For example, t represents the number of minutes the family rides after Lin's dad starts riding again after taking the picture.

To highlight the connections between the situations and the equations that represent them, ask:

- "How many solutions will each of these systems of equations have?" (Each system has exactly one solution. I can tell this because the gradients of each pair of equations are different.)
- "If Lin's dad biked 0.17 miles per minute instead of 0.24 miles per minute, how would that change the system of equations?" (The first equation would be $d = 0.17t$.)
    - "How many solutions would there be for this new system where Lin's dad rides slower?" (Based on the equations there should still be one solution.)
    - "Would Lin's dad ever catch up with the family?" (He would not. He started farther back and rides slower than the family. The solution to the system would have a negative value for time which does not make sense in the context of the problem.)

If students disagree that there is a solution to the modified first problem in which Lin's dad rides slower than the family, you may display the graph of the modified system and point out the point where the lines intersect. So, although the system has a solution, it is disregarded in this context since it does not make sense.

Representing, Conversing: Clarify, Critique, Correct. To help students make sense of the solution to a system of equations, offer an incorrect or ambiguous response for the problem about Noah's kayaking trip. After revealing the correct system of equations, display this ambiguous statement: "This solution represents the same amount of money for both rental places." Ask pairs of students to clarify the meaning of this response and then critique it. Invite pairs to offer a correct response by asking "What language might you add or change to make this statement more accurate?" Improved responses should include a reference to both variables (e.g. the amount of time and cost). This will help students interpret the solution to a system of equations in a specific context.
Design Principle(s): Maximise meta-awareness; Support sense-making

15.3 Info Gap: Racing and Play Tickets

20 minutes

In this activity students have an opportunity to apply what they know about systems of linear equations to solve a problem about a real-world situation. One equation for each situation is given. Students may choose to write another equation to create a system that represents the constraints in the problem, and then solve the system algebraically or by graphing. Another possible strategy would be to pull quantities out of the given equation and solve the problem arithmetically. Monitor for students who use each of these strategies to share during the whole-class discussion.

The info gap structure requires students to make sense of problems by determining what information is necessary, and then to ask for information they need to solve it. This may take several rounds of discussion if their first requests do not yield the information they need. It also allows them to refine the language they use and ask increasingly more precise questions until they get the information they need.

Here is the text of the cards for reference and planning:

Info Gap: Racing and Play Tickets Problem Card 1 Priya and Lin are having a race. The equation $y = 9.5x$ represents one person's progress. If one of them had a head start, how long is it until the other person catches up?	Info Gap: Racing and Play Tickets Data Card 1 • The equation $y = 9.5x$ represents Lin's progress, where y is her distance in feet, from the starting line, and x is the time, in seconds, that she has been running. • Priya had the head start. She was 18 feet in front of the starting line when Lin started. • Priya runs at a constant 8 feet per second.
Info Gap: Racing and Play Tickets Problem Card 2 A school sells adult tickets and student tickets for the drama play. One equation that represents the situation is $x + y = 115$. How many of each type of ticket did they sell?	Info Gap: Racing and Play Tickets Data Card 2 • The equation $x + y = 115$ represents how many tickets were sold, where x is the number of student tickets and y is the number of adult tickets. This equation is equivalent to $x = 115 - y$. • Adult tickets cost £8 each. • Student tickets cost £3 each. • The school made £720 in total from ticket sales.

Instructional Routines

- Information Gap Cards

Launch

Arrange students in groups of 2. Provide access to geometry toolkits. In each group, distribute the first problem card to one student and a data card to the other student. After debriefing on the first problem, distribute the cards for the second problem, in which students switch roles.

Engagement: Develop Effort and Persistence. Display or provide students with a physical copy of the written directions. Check for understanding by inviting students to rephrase directions in their own words. Keep the display of directions visible throughout the activity.

Supports accessibility for: Memory; Organisation Conversing: This activity uses Information Gap to give students a purpose for discussing information necessary to solve a problem about a real-world situation by applying what they know about systems of linear equations. Display questions or question starters for students who need a starting point such as: "Can you tell me . . . (specific piece of information)", and "Why do you need to know . . . (that piece of information)?"
Design Principle(s): Cultivate Conversation

Student Task Statement

Your teacher will give you either a *problem card* or a *data card*. Do not show or read your card to your partner.

If your teacher gives you the *problem card*:

1. Silently read your card and think about what information you need to be able to answer the question.

2. Ask your partner for the specific information that you need.

3. Explain how you are using the information to solve the problem.

   Continue to ask questions until you have enough information to solve the problem.

4. Share the *problem card* and solve the problem independently.

5. Read the *data card* and discuss your reasoning.

If your teacher gives you the *data card*:

1. Silently read your card.

2. Ask your partner *"What specific information do you need?"* and wait for them to *ask* for information.

   If your partner asks for information that is not on the card, do not do the calculations for them. Tell them you don't have that information.

3. Before sharing the information, ask "*Why do you need that information?*" Listen to your partner's reasoning and ask clarifying questions.

4. Read the *problem card* and solve the problem independently.

5. Share the *data card* and discuss your reasoning.

Student Response

1. 12 seconds

2. 40 student tickets and 75 adult tickets

Activity Synthesis

Select students with different strategies to share their approaches to each question, starting with less efficient methods and ending with more efficient methods.

15.4 Solving Systems Practice

Optional: 10 minutes

In this activity, students solve a variety of systems of equations, some involving fractions, some involving substitution, and some involving inspection. This gives students a chance to practice using the methods they have learned in this section for solving systems of equations to solidify that learning. Some of the systems listed are ones students could have used in an earlier activity in this lesson, to describe the situations there. In the discussion, students compare the systems here to the ones they wrote in that activity and interpret the answer in that context.

Instructional Routines

- Discussion Supports

Launch

Keep students in groups of 2. Allow students 5–7 minutes of partner work time followed by a whole-class discussion.

Representation: Internalise Comprehension. Activate or supply background knowledge. Allow students to use calculators to ensure inclusive participation in the activity.
Supports accessibility for: Memory; Conceptual processing

Student Task Statement

Here are a lot of systems of equations:

- $\begin{cases} y = -2x + 6 \\ y = x - 3 \end{cases}$
- $\begin{cases} y = 5x - 4 \\ y = 4x + 12 \end{cases}$
- $\begin{cases} y = \frac{2}{3}x - 4 \\ y = -\frac{4}{3}x + 9 \end{cases}$
- $\begin{cases} 4y + 7x = 6 \\ 4y + 7x = -5 \end{cases}$
- $\begin{cases} y = x - 6 \\ x = 6 + y \end{cases}$

- $\begin{cases} y = 0.24x \\ y = 0.18x + 0.9 \end{cases}$

- $\begin{cases} y = 4.5x + 15 \\ y = 5x + 12.5 \end{cases}$

- $\begin{cases} y = 3x \\ x + y = 52 \end{cases}$

1. Without solving, identify 3 systems that you think would be the least difficult for you to solve and 3 systems you think would be the most difficult. Be prepared to explain your reasoning.

2. Choose 4 systems to solve. At least one should be from your "least difficult" list and one should be from your "most difficult" list.

Student Response

1. Answers vary.

2. Possible solutions.

   a. $x = 3, y = 0$

   b. $x = 16, y = 76$

   c. $x = 6\frac{1}{2}, y = \frac{1}{3}$

   d. No solution

   e. Infinitely many solutions

   f. $x = 15, y = 3.6$

   g. $x = 5, y = 37.5$

   h. $x = 13, y = 39$

Activity Synthesis

There are two key takeaways from this discussion. The first is to reinforce that some systems can be solved by reasoning whether it's possible for a solution to one equation to also be a solution to another. The second takeaway is that there are some systems that students will only be able to solve after learning techniques in future years.

For each problem, ask students to indicate if they identified the system as least or most difficult. Record the responses for all to see.

Bring students' attention to this system:

$$\begin{cases} 4y + 7x = 6 \\ 4y + 7x = -5 \end{cases}$$

Ask students what the two equations in the system have in common to each other and to think about whether a solution to the first equation could also be a solution to the second. One can reason there is no solution because $4y + 7x$ cannot be equal to both 6 and -5.

Ask students to return to the earlier activity and see if they can find any of those systems in these problems. (Lin's family ride is the sixth system. Noah's kayaking trip is the seventh system. Diego's baking is the eighth system.) After students notice the connection, invite students who chose those systems to solve to interpret the numerical solutions in the contexts from the earlier activity.

Students may be tempted to develop the false impression that all systems where both equations are given as linear combinations can be solved by inspection. Conclude the discussion by displaying this system that students defined in the last activity about sugar and flour:

$$\begin{cases} 0.8x + 0.5y = 9 \\ x + y = 15 \end{cases}$$

Tell students that this system has one solution and they will learn more sophisticated techniques for solving systems of equations like this in future grades.

Speaking, Listening, Conversing: Discussion Supports. Before the whole-class discussion, use this routine to give students an opportunity to discuss their reasons for labelling systems of equations as "least difficult" or "most difficult." Provide the following sentence frame: "_____ is the least/most difficult to solve because _____." Encourage listeners to press for detail by asking questions such as "What would make that problem easier to solve?"; "What do your least/most difficult problems have in common?" This will help students make connections between the structures of given systems of equations, and possible strategies they can use to solve them.
Design Principle(s): Cultivate conversation; Maximise meta-awareness

Lesson Synthesis

To wrap up the lessons on solving systems of equations, consider displaying the three systems of equations and asking students how they might begin to solve the systems.

$\begin{cases} y = 2x + 1 \\ y = \frac{1}{2}x + 10 \end{cases}$ (Both graphing and substitution methods work well)

$\begin{cases} x = 5 - 2y \\ 2x + 6y = 16 \end{cases}$ (Substitution works best)

$\begin{cases} 5x + 4y = 20 \\ 10x + 8y = 60 \end{cases}$ (Inspection may work best)

If there is time, consider assigning each system to small groups for them to solve, then share their solutions with the class.

15.5 Solve This

Cool Down: 5 minutes

This cool-down asks students to solve a system of equations that has rational numbers in the system of equations as well as in the solution. Watch for students who use the substitution method correctly even if they get stuck working with fractions.

Student Task Statement

Solve

$$\begin{cases} y = \frac{3}{4}x \\ \frac{5}{2}x + 2y = 5 \end{cases}$$

Student Response

$x = \frac{5}{4}, y = \frac{15}{16}$

Student Lesson Summary

We have learned how to solve many kinds of systems of equations using algebra that would be difficult to solve by graphing. For example, look at

$$\begin{cases} y = 2x - 3 \\ x + 2y = 7 \end{cases}$$

The first equation says that $y = 2x - 3$, so wherever we see y, we can substitute the expression $2x - 3$ instead. So the second equation becomes $x + 2(2x - 3) = 7$.

We can solve for x:

$x + 4x - 6$	$= 7$	distributive property
$5x - 6$	$= 7$	combine like terms
$5x$	$= 13$	add 6 to each side
x	$= \frac{13}{5}$	multiply each side by $\frac{1}{5}$

We know that the y value for the solution is the same for either equation, so we can use either equation to solve for it. Using the first equation, we get:

$$y = 2\left(\frac{13}{5}\right) - 3 \qquad \text{substitute } x = \frac{13}{5} \text{ into the equation}$$

$$y = \frac{26}{5} - 3 \qquad \text{multiply } 2\left(\frac{13}{5}\right) \text{ to make } \frac{26}{5}$$

$$y = \frac{26}{5} - \frac{15}{5} \qquad \text{rewrite 3 as } \frac{15}{5}$$

$$y = \frac{11}{5}$$

If we substitute $x = \frac{13}{5}$ into the other equation, $x + 2y = 7$, we get the same y value. So the solution to the system is $\left(\frac{13}{5}, \frac{11}{5}\right)$.

There are many kinds of systems of equations that we will learn how to solve in future years, like $\begin{cases} 2x + 3y = 6 \\ -x + 2y = 3 \end{cases}$

Or even $\begin{cases} y = x^2 + 1 \\ y = 2x + 3 \end{cases}$

Lesson 15 Practice Problems

1. **Problem 1 Statement**

Kiran and his cousin work during the summer for a landscaping company. Kiran's cousin has been working for the company longer, so his pay is 30% more than Kiran's. Last week his cousin worked 27 hours, and Kiran worked 23 hours. Together, they earned £493.85. What is Kiran's hourly pay? Explain or show your reasoning.

Solution

£8.50. Explanations vary. Sample response: n = Kiran's hourly wage and c = Kiran's cousin's hourly wage. $c = 1.3n$ and $27c + 23n = 493.85$. Substituting $1.3n$ for c yields the equation $27(1.3n) + 23n = 493.85$.

2. **Problem 2 Statement**

Decide which story can be represented by the system of equations $y = x + 6$ and $x + y = 100$. Explain your reasoning.

a. Diego’s teacher writes a test worth 100 points. There are 6 more multiple choice questions than short answer questions.

b. Lin and her younger cousin measure their heights. They notice that Lin is 6 inches taller, and their heights add up to exactly 100 inches.

Solution

The second story. Explanations vary. Sample response: In the first story, $y = x + 6$ can be written where x and y represent the number of questions of each type, but the other fact is about points, so $x + y = 100$ does not make sense. In the second story, Lin's height can be represented by y, and her younger cousin's height can be represented by x.

3. **Problem 3 Statement**

Clare and Noah play a game in which they earn the same number of points for each goal and lose the same number of points for each penalty. Clare makes 6 goals and 3 penalties, ending the game with 6 points. Noah earns 8 goals and 9 penalties and ends the game with -22 points.

a. Write a system of equations that describes Clare and Noah's outcomes. Use x to represent the number of points for a goal and y to represent the number of points for a penalty.

b. Solve the system. What does your solution mean?

Solution

a. Clare: $6x + 3y = 6$, Noah: $8x + 9y = \text{-}22$

b. (4,-6). A goal earns 4 points and a penalty earns -6 points.

4. **Problem 4 Statement**

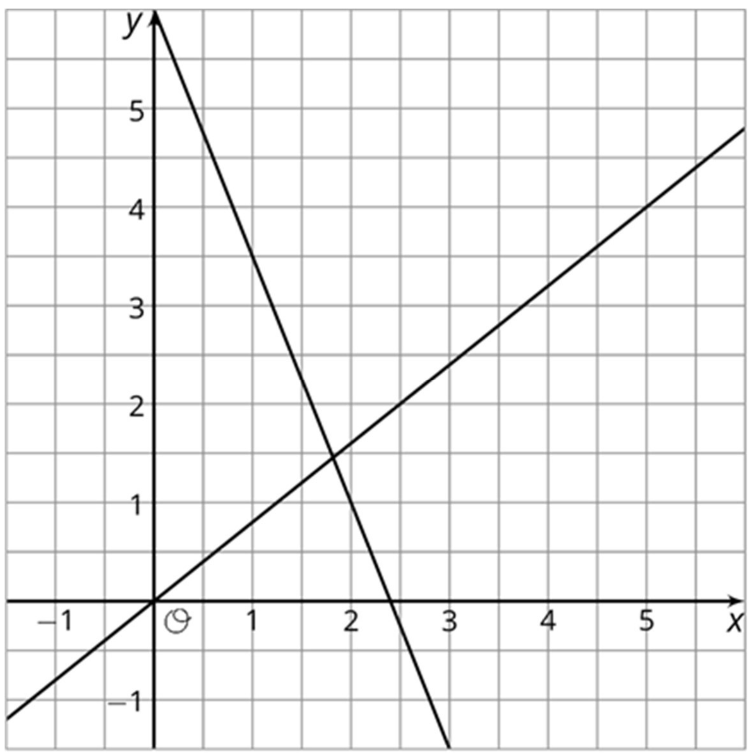
Solve: $\begin{cases} y = 6x - 8 \\ y = \text{-}3x + 10 \end{cases}$

Solution

(2,4). First solve $6x - 8 = \text{-}3x + 10$ for x and substitute that value into either of the original equations to solve for y.

5. **Problem 5 Statement**

a. Estimate the coordinates of the point where the two lines meet.



b. Choose two equations that make up the system represented by the graph.

i. $y = \frac{5}{4}x$

ii. $y = 6 - 2.5x$

iii. $y = 2.5x + 6$

iv. $y = 6 - 3x$

v. $y = 0.8x$

c. Solve the system of equations and confirm the accuracy of your estimate.

Solution

a. Answers vary. Sample response: (1.8,1.4)

b. ii and v

c. $x \approx 1.82$, $y \approx 1.46$ (the exact values are $x = \frac{20}{11}$ and $y = \frac{16}{11}$). Find the x coordinate of the intersection point by solving $6 - 2.5x = 0.8x$. To find the y coordinate, substitute this value of x into either equation.