Lesson 9: Multi-step experiments

Goals

- Choose a method for representing the sample space of a compound event, and justify (orally) the choice.
- Use the sample space to determine the probability of a compound event, and explain (orally, in writing, and using other representations) the reasoning.

Learning Targets

- I can use the sample space to calculate the probability of an event in a multi-step experiment.

Lesson Narrative

In this lesson, students continue writing out the sample spaces for chance experiments that have multiple steps and also begin using those sample spaces to calculate the probability of certain events. Students may start listing the sample space using one method and then decide to switch to a different method when they get stuck in the middle of the problem or they might recognise certain aspects of the situation that would lead them to choose a particular method from the beginning. For instance, a problem that involves two spinners would be easy to represent with a table, but a problem that involves three spinners may be easier to represent with a tree diagram.

Building On

- Use brackets in numerical expressions, and evaluate expressions with these symbols.
- Identify parts of an expression using mathematical terms (sum, term, product, factor, quotient, coefficient); view one or more parts of an expression as a single entity. For example, describe the expression $2(8 + 7)$ as a product of two factors; view $(8 + 7)$ as both a single entity and a sum of two terms.

Addressing

- Understand that, just as with simple events, the probability of a compound event is the fraction of outcomes in the sample space for which the compound event occurs.
- Represent sample spaces for compound events using methods such as organised lists, tables and tree diagrams. For an event described in everyday language (e.g., "rolling double sixes"), identify the outcomes in the sample space which compose the event.

Instructional Routines

- Collect and Display
- Co-Craft Questions
- Discussion Supports

- Notice and Wonder
- Poll the Class
- Think Pair Share
- True or False

Student Learning Goals

Let's look at probabilities of experiments that have multiple steps.

9.1 True or False?

Warm Up: 5 minutes

The purpose of this warm-up is to gather strategies and understandings students have for averaging numbers. Understanding these strategies will help students develop fluency and will be useful later in this unit when students will need to be able to calculate averages of values.

While 3 problems are given, it may not be possible to share every strategy for all the problems. Consider gathering only 2 or 3 different strategies per problem, saving most of the time for the final question.

Instructional Routines

- True or False

Launch

Reveal one problem at a time. Give students 30 seconds of quiet think time for each problem and ask them to give a signal when they have an answer and a strategy. Keep all previous problems displayed throughout the talk. Follow with a whole-class discussion.

Student Task Statement

Is each equation true or false? Explain your reasoning.

$8 = (8 + 8 + 8 + 8) \div 3$

$(10 + 10 + 10 + 10 + 10) \div 5 = 10$

$(6 + 4 + 6 + 4 + 6 + 4) \div 6 = 5$

Student Response

Explanations vary. Sample responses:

False, since $(8 + 8 + 8 + 8) = 4 \times 8$ and $4 \times 8 \div 3$ is not 8.

True, since $(10 + 10 + 10 + 10 + 10) = 5 \times 10$ and $5 \times 10 \div 5 = 10$.

True, since $(6 + 4 + 6 + 4 + 6 + 4) = (10 + 10 + 10) = 3 \times 10$ and $3 \times 10 \div 6 = 10 \div 2 = 5$.

Activity Synthesis

Ask students to share their strategies for each problem. Record and display their responses for all to see. To involve more students in the conversation, consider asking:

- "Who can restate ___'s reasoning in a different way?"
- "Did anyone have the same strategy but would explain it differently?"
- "Did anyone solve the problem in a different way?"
- "Does anyone want to add on to _____'s strategy?"
- "Do you agree or disagree? Why?"

9.2 Spinning a Colour and Number

10 minutes

In this activity, students are reminded how to calculate probability based on the number of outcomes in the sample space, then apply that to multi-step experiments. The events are described in everyday language, so students need to reason abstractly to identify the outcomes described. This lesson begins with students returning to a problem they have previously seen when writing out the sample space. This will save students some time if they can recall or refer back to the initial problem. In the following activities, students will work with situations for which they have not written out the sample space to practise finding probabilities using all the necessary steps.

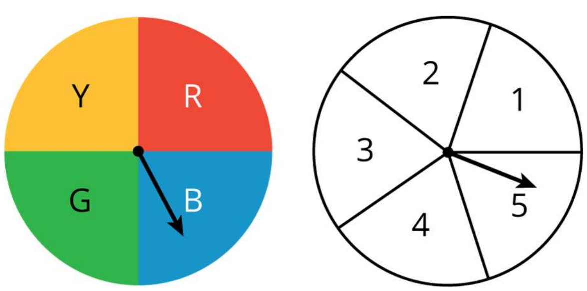
Instructional Routines

- Collect and Display
- Notice and Wonder

Launch

Arrange students in groups of 2.

Display the two spinners for all to see. Ask students, "What do you notice? What do you wonder?"



Give students 1 minute to think about the image. Record their responses for all to see.

Students may notice:

- The number of sections in each spinner.
- The labels for the two spinners.
- Within each spinner, the sections are equally sized.

Students may wonder:

- Do you choose which one to spin or do you spin both?
- If you spin both, how many different outcomes will there be?
- Is this part of a game? If so, what is a "good" spin?

Tell students: "For sample spaces where each outcome is equally likely, recall that the probability of an event can be calculated by counting the number of outcomes in the event and dividing that number by the total number of outcomes in the sample space." For example, in the previous lesson, students found that there were 12 possible outcomes when flipping a coin and rolling a dice. If we wanted the probability of getting heads and rolling an even number, we count that there are 3 ways to do this (H2, H4, and H6) out of the 12 outcomes in the sample space. So, the probability of getting heads and an even number should be $\frac{3}{12}$ or $\frac{1}{4}$ or 0.25.

Remind students that they have already drawn out the sample space for this chance experiment in a previous activity, and they may use that to help answer the questions.

Give students of 5 minutes quiet work time followed by partner and whole-class discussion.

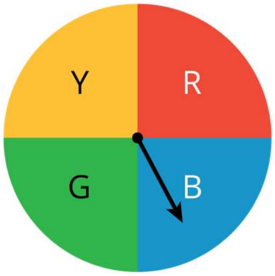
Representation: Access for Perception. Provide access to concrete manipulatives. Provide spinners for students to view or manipulate. These hands-on models will help students identify characteristics or features, and support finding outcomes for calculating probabilities.
Supports accessibility for: Visual-spatial processing; Conceptual processing

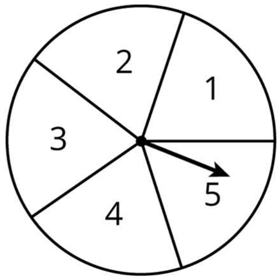
Student Task Statement

The other day, you wrote the sample space for spinning each of these spinners once.

What is the probability of getting:

1. Green and 3?
2. Blue and any odd number?
3. Any colour other than red and any number other than 2?





Student Response

1. $\frac{1}{20}$, since there is only 1 outcome in the event and there are 20 equally likely outcomes in the sample space.

2. $\frac{3}{20}$, since there are 3 outcomes that have blue and an odd number.

3. $\frac{12}{20}$, since there are 12 outcomes that have any colour besides red and any number besides 2.

Activity Synthesis

The purpose of the discussion is for students to explain their interpretations of the questions and share methods for solving.

Some questions for discussion:

- "How did you calculate the number of outcomes in the sample space?" (Counting the items in the tree, table, or list, or using the multiplication idea from an earlier lesson.)
- "Although we had the sample space for this situation in a previous problem, how could you find the sample space if you did not know it already?" (Draw a tree, table, or list.)
- "For each problem, how many outcomes were in the event that was described? How did you count them?"

Representing, Speaking, Listening: Collect and Display. As pairs discuss strategies for calculating the probability of each outcome, circulate and write down the words and phrases students use to explain their reasoning. Listen for students who reference their representation of the sample space (e.g., list, table, or tree) to determine the probability of each outcome. As students review the language collected in the visual display, encourage students to revise and improve how ideas are communicated. For example, a phrase such as: "The probability is $\frac{1}{20}$, because there are 20 outcomes" can be improved with the phrase "The probability is $\frac{1}{20}$, because there is only 1 outcome in the event, and there are 20 equally likely outcomes in the sample space." This routine will provide feedback to students in a way that supports sense-making while simultaneously increasing meta-

awareness of language.
Design Principle(s): Support sense-making; Maximise meta-awareness

9.3 Dice and Coins

20 minutes

In this activity, students continue to calculate probabilities for multi-step experiments using the number of outcomes in the sample space. The first problem involves a situation for which students have already seen the sample space. Following this problem, the class will discuss the merits of the different representations for writing out the sample space. The next two problems involve situations in which students may need to write out the sample space on their own. Students are also reminded that some events have a probability of 0, which represents an event that is impossible. In the discussion following the activity, students are asked to think about the probabilities of two events that make up the entire sample space and have no outcomes common to both events.

Instructional Routines

- Discussion Supports
- Poll the Class

Launch

Keep students in groups of 2.

Assign each group a representation for writing out the sample space: a tree, a table, or a list. Tell students that they should write out the sample space for the first problem using the representation they were assigned. (This was done for them in a previous lesson and they are allowed to use those as a guide if they wish.)

Tell students that they should work on the first problem only and then pause for a discussion before proceeding to the next problems.

Give students 2 minutes of partner work time for the first problem followed by a pause for a whole-class discussion centred around the different representations for sample space.

After all groups have completed the first question, select at least one group for each representation and have them explain how they arrived at their answer. As the groups explain, display the appropriate representations for all to see. Ask each of the groups how they counted the number of outcomes in the sample space as well as the number of outcomes in the event using their representation.

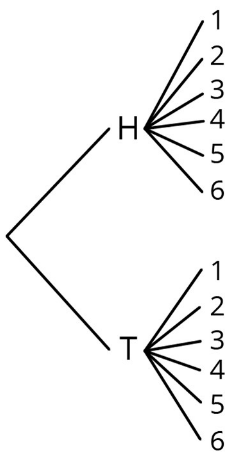
List:

Heads 1, heads 2, heads 3, heads 4, heads 5, heads 6, tails 1, tails 2, tails 3, tails 4, tails 5, tails 6

Table:

1	2	3	4	5	6
H1	H2	H3	H4	H5	H6
T1	T2	T3	T4	T5	T6

Tree:



After students have had a chance to explain how they used the representations, ask students to give some pros and cons for using each of the representations. For example, the list method may be easy to write out and interpret, but could be very long and is not the easiest method for keeping track of which outcomes have been written and which still need to be included.

Allow the groups to continue with the remaining problems, telling them they may use any method they choose to work with the sample space for these problems. Give students 10 minutes of partner work time followed by a whole-class discussion about the activity as a whole.

Anticipated Misconceptions

Some students may not recognise that rolling a 2 then a 3 is different from rolling a 3 then a 2. Ask students to imagine the dice are different colours to help see that there are actually 2 different ways to get these results.

Similarly, some students may think that HHT counts the same as HTH and THH. Ask the student to think about the coins being flipped one at a time rather than all tossed at once. Drawing an entire tree and seeing all the branches may further help.

Student Task Statement

The other day you looked at a list, a table, and a tree that showed the sample space for rolling a dice and flipping a coin.

1. Your teacher will assign you one of these three structures to use to answer these questions. Be prepared to explain your reasoning.

a. What is the probability of getting tails and a 6?

b. What is the probability of getting heads and an odd number?

Pause here so your teacher can review your work.

2. Suppose you roll two dice. What is the probability of getting:

a. Both dice showing the same number?

b. *Exactly* one dice showing an even number?

c. *At least* one dice showing an even number?

d. Two values that have a sum of 8?

e. Two values that have a sum of 13?

3. Jada flips three coins. What is the probability that all three will land showing the same side?

Student Response

1.

a. $\frac{1}{12}$, since there is only 1 outcome in the sample space that matches the criteria.

b. $\frac{3}{12}$, since there are 3 outcomes in the sample space that have heads and an odd number.

2.

a. $\frac{6}{36}$, since there are 6 outcomes where the same number is showing and the sample space contains 36 equally likely outcomes.

b. $\frac{18}{36}$, since there are 18 outcomes where exactly one of the dice shows an even number.

c. $\frac{27}{36}$, since there are 27 outcomes where at least one of the dice shows an even number.

d. $\frac{5}{36}$, since there are 8 outcomes where the sum is 8 (2 and 6, 3 and 5, 4 and 4, 5 and 3, 6 and 2).

e. 0, since it is impossible to get two values whose sum is 13.

3. $\frac{2}{8}$, since there are 2 ways to get the coins showing the same side (all heads or all tails) and 8 outcomes in the sample space.

Activity Synthesis

The purpose of the discussion is for students to explain their methods for solving the problems and to discuss how writing out the sample space aided in their solutions.

Poll the class on how they calculated the number of outcomes in the sample space and the number of outcomes in the event for the second set of questions given these options: List, Table, Tree, Calculated Outcomes Without Writing Them All Out, Another Method.

Consider these questions for discussion:

- "Which representation did you use for each of the problems?"
  - "Do you think you will always try to use the same representation, or can you think of situations when one representation might be better than another?"
- "Did you have a method for finding the number of outcomes in the sample space or event that was more efficient than just counting them?" (The number of outcomes in the sample space for the dice could be found using $6 \times 6 = 36$. To find the number of outcomes with at least 1 even number, I knew there would be 6 for each time an even was rolled first and only 3 for each time an odd number was rolled first, so I found the number of outcomes by $3 \times 6 + 3 \times 3 = 27$.)
- "One of the events had a probability of zero. What does this mean?"
  (It is impossible.)
- "What would be the probability of an event that was certain?" (1)
- "Jada was concerned with having all the coins show the same side. What would be the probability of having at least 1 coin *not* match the others?" ($\frac{6}{8}$, since there are 6 outcomes where at least 1 coin does not match: HHT, TTH, HTH, THT, HTT, THH.)
  - "How do the answers to Jada's question and the one we just answered relate to one another?" (Since every outcome in the sample space has either "at least one heads" or "all tails," and there is no outcome that applies to both events, together the sum of their probabilities must be 100% or 1.)

Speaking: Discussion Supports. Use this routine to support whole-class discussion. For each response or observation that is shared, ask students to restate and/or revoice what they heard using precise mathematical language. Consider providing students time to restate what they hear to a partner, before selecting one or two students to share with the class. Ask the original speaker if their peer was accurately able to restate their thinking. Call students' attention to any words or phrases that helped to clarify the original statement. This will provide more students with an opportunity to produce language as they interpret the reasoning of others.
Design Principle(s): Support sense-making

9.4 Pick a Card

Optional: 15 minutes

The activity provides further practice in finding probabilities of events.

In this activity, students see an experiment that has two steps where the result of the first step influences the possibilities for the second step. Often this process is referred to as doing something “without replacement.” At this stage, students should approach these experiments in a very similar way to all of the other probability questions they have encountered, but they must be very careful about the number of outcomes in the sample space.

Instructional Routines

- Co-Craft Questions
- Think Pair Share

Launch

Keep students in groups of 2. Give students 5–7 minutes of quiet work time followed by partner and whole-class discussion.

Identify students who are not noticing that it is impossible to draw the same colour twice based on the instructions. Refocus these students by asking them to imagine drawing a red card on the first pick and thinking about what’s possible to get for the second card.

Representation: Access for Perception. Provide access to concrete manipulatives. Provide five different coloured cards for students to view or manipulate. These hands-on models will help students identify characteristics or features, and support finding outcomes for calculating probabilities.
Supports accessibility for: Visual-spatial processing; Conceptual processing Conversing: Co-Craft Questions. Display the initial task statement that begins, “Imagine there are five cards...", before revealing the questions that follow. Ask pairs to write down possible mathematical questions that can be answered about the situation. As pairs share their questions with the class, listen for and highlight questions that ask how the “without putting the card back” part of the scenario would change the outcome of the sample space. This will help students consider the differences between this problem and previous problems and how that impacts the sample space.
Design Principle(s): Optimise output; Maximise meta-awareness

Anticipated Misconceptions

Students may misread the problem and think that they replace the card before picking the next one. Ask these students to read the problem more carefully and ask the student, “What is possible to get when you draw the second card while you already have a red card in your hand?”

Student Task Statement

Imagine there are 5 cards. They are coloured red, yellow, green, white, and black. You mix up the cards and select one of them without looking. Then, without putting that card back, you mix up the remaining cards and select another one.

1. Write the sample space and tell how many possible outcomes there are.
2. What structure did you use to write all of the outcomes (list, table, tree, something else)? Explain why you chose that structure.
3. What is the probability that:
   a. You get a white card and a red card (in either order)?
   b. You get a black card (either time)?
   c. You do not get a black card (either time)?
   d. You get a blue card?
   e. You get 2 cards of the same colour?
   f. You get 2 cards of different colours?

Student Response

1. Sample space: RY, RG, RW, RB, YR, YG, YW, YB, GR, GY, GW, GB, WR, WY, WG, WB, BR, BY, BG, BW. There are 20 different outcomes.
2. Answers vary. Sample response. I used a tree, since it was easier to keep track of how the first card selected would affect what was possible for the second card.
   a. $\frac{2}{20} = \frac{1}{10}$, because there are 2 outcomes that have those 2 cards (RW and WR), and the outcomes in the sample space are equally likely.
   b. $\frac{8}{20} = \frac{2}{5}$, because there are 8 outcomes that have a black card (RB, YB, GB, WB, BR, BY, BG, BW).
   c. $\frac{12}{20} = \frac{3}{5}$, because there are 12 outcomes that do not have a black card.
   d. 0, because there are no blue cards in the deck.
   e. 0, because there is only 1 card of each colour and you cannot pick the same card twice if you don't put the first one back.
   f. 1, because all of the possible outcomes have two different colours.

Are You Ready for More?

In a game using five cards numbered 1, 2, 3, 4, and 5, you take two cards and add the values together. If the sum is 8, you win. Would you rather pick a card and put it back before picking the second card, or keep the card in your hand while you pick the second card? Explain your reasoning.

Student Response

You are more likely to win if you put the card back.

If you put it back, you can win with these outcomes: 35, 44, or 53. Since this way has 25 equally likely outcomes in the sample space, the probability of winning is $\frac{3}{25} = 0.12$. If you do not put it back, you can win with these outcomes: 35 or 53. Since this way has 20 equally likely outcomes in the sample space, the probability of winning is $\frac{2}{20} = 0.1$.

Activity Synthesis

The purpose of the discussion is for students to compare the same context with replacement and without replacement.

Consider asking these questions for discussion:

- "What would change about your calculations if the experiment required replacing the first card before picking a second card?" (There would be 25 outcomes in the sample space. The probability of getting the same colour twice would be $\frac{5}{25}$. The probability of getting different colours would be $\frac{20}{25}$. The probability of getting red and white would be $\frac{2}{25}$. The probability of getting a black card would be $\frac{9}{25}$ and not getting a black card would be $\frac{16}{25}$. It would still be impossible to get a blue card, so its probability would be 0.)
- "What do you notice about the sum of the probability of getting a black card and the probability of not getting a black card?" (They have a sum of 1.)
  - "Explain why these outcomes might have probabilities with this relationship." (Since you either get a black card or not, together their probabilities should be 1 or 100%.)

Lesson Synthesis

These discussion questions will help students reflect on their learning:

- "When the outcomes in the sample space are equally likely, how is the size of the sample space used to calculate the probability of an event?"
- "Now that you've have plenty of practice, do you have a favourite method for writing out the sample space?"

– "Are there times that one strategy for writing out the sample space makes more sense than others?"

9.5 A Dice and 10 Cards

Cool Down: 5 minutes

Student Task Statement

Lin plays a game that involves a standard dice and a deck of ten cards numbered 1 through 10. If both the dice and card have the same number, Lin gets another turn. Otherwise, play continues with the next player.

What is the probability that Lin gets another turn?

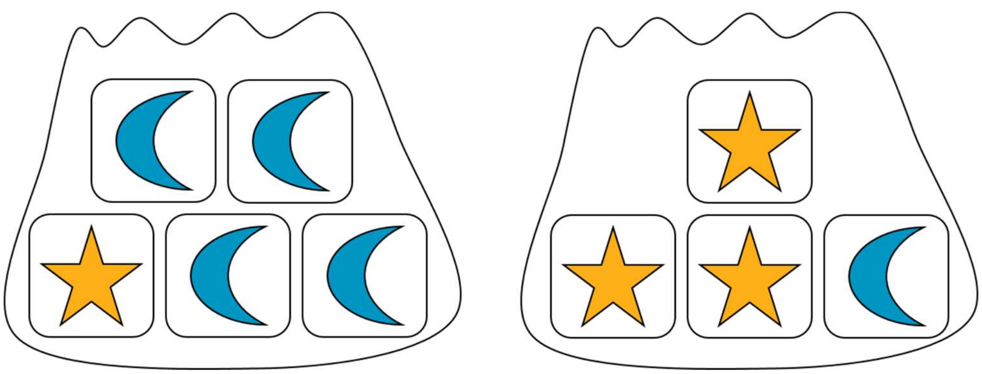
Student Response

$\frac{6}{60}$ (or equivalent), since there are 6 outcomes for which the numbers match and 60 equally likely outcomes in the sample space ($6 \times 10 = 60$).

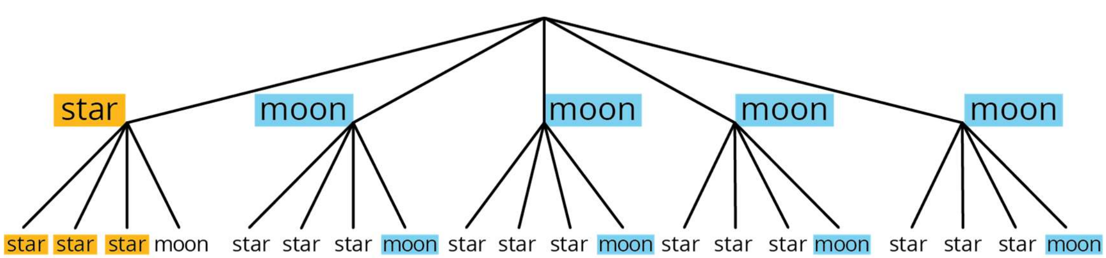
Student Lesson Summary

Suppose we have two bags. One contains 1 star block and 4 moon blocks. The other contains 3 star blocks and 1 moon block.

If we select one block at random from each, what is the probability that we will get two star blocks or two moon blocks?



To answer this question, we can draw a tree diagram to see all of the possible outcomes.



There are $5 \times 4 = 20$ possible outcomes. Of these, 3 of them are both stars, and 4 are both moons. So the probability of getting 2 star blocks or 2 moon blocks is $\frac{7}{20}$.

In general, if all outcomes in an experiment are equally likely, then the probability of an event is the fraction of outcomes in the sample space for which the event occurs.

Lesson 9 Practice Problems

Problem 1 Statement

A vending machine has 5 colours (white, red, green, blue, and yellow) of gumballs and an equal chance of dispensing each. A second machine has 4 different animal-shaped rubber bands (lion, elephant, horse, and alligator) and an equal chance of dispensing each. If you buy one item from each machine, what is the probability of getting a yellow gumball and a lion band?

Solution

$\frac{1}{20}$

Problem 2 Statement

The numbers 1 through 10 are put in one bag. The numbers 5 through 14 are put in another bag. When you pick one number from each bag, what is the probability you get the same number?

Solution

$\frac{6}{100}$, since you can get 5s, 6s, 7s, 8s, 9s, or 10s, and there are 100 possible outcomes in the sample space (10×10).

Problem 3 Statement

When rolling 3 standard dice, the probability of getting all three numbers to match is $\frac{6}{216}$. What is the probability that the three numbers *do not* all match? Explain your reasoning.

Solution

$\frac{210}{216}$, since the numbers either all match or do not match, so the rest of the options must be of this kind.

Problem 4 Statement

For each event, write the sample space and tell how many outcomes there are.

a. Roll a standard dice. Then flip a coin.

b. Select a month. Then select 2020 or 2025.

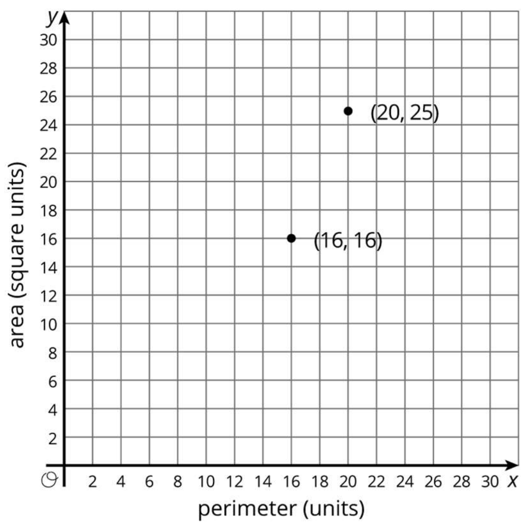
Solution

a. 12 outcomes: 1h, 1t, 2h, 2t, 3h, 3t, 4h, 4t, 5h, 5t, 6h, 6t

b. 24 outcomes: Jan 2020, Jan 2025, Feb 2020, Feb 2025, Mar 2020, Mar 2025, Apr 2020, Apr 2025, May 2020, May 2025, June 2020, June 2025, July 2020, July 2025, Aug 2020, Aug 2025, Sep 2020, Sep 2025, Oct 2020, Oct 2025, Nov 2020, Nov 2025, Dec 2020, Dec 2025.

Problem 5 Statement

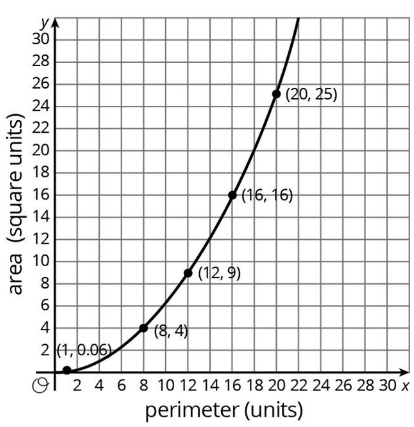
On a graph of the area of a square vs. its perimeter, a few points are plotted.



a. Add some more ordered pairs to the graph.

b. Is there a proportional relationship between the area and perimeter of a square? Explain how you know.

Solution

a. 

b. There is a not a proportional relationship between area and perimeter. When graphed, the ordered pairs do not lie on a straight line that passes through the origin.