Lesson 1: Fermi problems

Goals

- Estimate quantities in a real-world situation and explain (orally and in writing) the estimation strategy.
- Justify (orally) why it is unreasonable to have an exact answer for a situation that involves estimation, and critique (orally) different estimates.
- Make simplifying assumptions and determine what information is needed to solve a Fermi problem about distance, volume, or surface area.

Lesson Narrative

This lesson is optional. The activities in this lesson plan are sometimes called "Fermi problems" after the famous physicist Enrico Fermi. A Fermi problem requires students to make a rough estimate for quantities that are difficult or impossible to measure directly. Often, they use rates and require several calculations with fractions and decimals, making them well-aligned to Year 7 work. Fermi problems are examples of mathematical modelling, because one must make simplifying assumptions, estimates, research, and decisions about which quantities are important and what mathematics to use. They also encourage students to attend to precision, because one must think carefully about how to appropriately report estimates and choose words carefully to describe the quantities.

Each of these activities can stand on its own. If students do the first before the second, the second will take less time. It is very likely that it would take more than a single day to do all of the activities in this lesson. One option is to let students choose an activity that interests them. If you choose to conduct the lesson in this manner, begin by posing these scenarios, one for each activity in this lesson, to students:

1. "Imagine that an ant ran from Camborne to John O'Groats."
2. "Imagine a warehouse that has a rectangular floor and contains all of the boxes of breakfast cereal bought in the United Kingdom every year."
3. "Imagine that the Washington Monument had to be completely retiled."

Then ask:

1. "Which one interests you the most? Why?"
2. "What questions could we ask about each situation?"
3. "What information would you want to know in order to investigate that particular situation?"

Each student or group can explore the problem that interests them the most and share their findings.

As with all lessons in this unit, all related standards have been addressed in prior units; this lesson provides an *optional* opportunity to go more deeply and make connections between domains.

Addressing

- Solve real-world and mathematical problems involving area, surface area, and volume.
- Compute fluently with multi-digit numbers and find common factors and multiples.
- Understand ratio concepts and use ratio reasoning to solve problems.

Instructional Routines

- Group Presentations
- Co-Craft Questions
- Discussion Supports
- Poll the Class

Required Materials

Four-function calculators
Internet-enabled device
Tools for creating a visual display
Any way for students to create work that can be easily displayed to the class. Examples: chart paper and markers, whiteboard space and markers, shared online drawing tool, access to a document camera.

Required Preparation

Internet-enabled devices are only necessary if students will conduct research to find quantities that they need to know. As an alternative, you can supply the information when they ask for it.

Tools for creating a visual display are only needed if you would like students to present their work in an organised way and have the option of conducting a gallery walk.

Student Learning Goals

Let’s make some estimates.

1.1 Ant Trek

Optional: 20 minutes

In this Fermi problem, students estimate the time it would take an ant to run from Camborne to John O’Groats. Monitor for different approaches to solving the problem, and select students to share during the discussion. In particular, look for:

- A range of estimates

- Different levels of precision in reporting the final estimate
- Different representations (for example, double number lines and tables)

Instructional Routines

- Group Presentations
- Co-Craft Questions
- Poll the Class

Launch

Arrange students in groups of 1–4. Provide access to four-function calculators. If students are doing their own research, provide access to internet-enabled devices. If conducting a gallery walk at the end, provide access to tools for making a visual display.

Before starting, ask students to come up with a guess about the answer and poll the class. Record all guesses.

Ask students to brainstorm the information they need to answer the question. Give the information provided when students ask for it, or provide access to internet-enabled devices so that students can find the information they need.

Vital information to have on hand includes:

- The distance between Camborne and John O'Groats is about 815 miles (1 312 km).
- An ant can run about 18 mm per second.

Also of interest is the fact that most ants do not live long enough to complete this trip. Many ants live for only a couple months. If students realise this, ask them how many ant lifetimes it would take for an ant to make this journey.

Conversing: Co-Craft Questions. To use this routine to support student understanding of the context of the problem. Display a map that shows a route between Camborne and John O'Groats. Invite students to work with a partner to create a list of possible mathematical questions that could be asked about the situation. Invite students to share their questions with the class and call students' attention to the different ways students integrate measurements (distance travelled, speed) in their questions.
Design Principle(s): Maximise meta-awareness; Support sense-making

Anticipated Misconceptions

Students may struggle to get started with no given information. Ask these students what they need to know in order to estimate how long it takes the ant to travel across the country. They may respond in a variety of ways, for example:

- How far does the ant travel in a day (or a different period of time)? Give them the information that the ant travels 18 mm per second.

- How far is it from Camborne to John O'Groats? (They may ask for this distance in miles.) Give them the information that it is about 1 312 km.

Students may struggle to convert mm to km. Ask these students how many mm are in a cm? in a metre? How many metres are in a km?

Students may be distracted by other concerns, like what the ant will eat, whether or not the ant will travel throughout the winter, whether or not the ant can cross motorways. Help these students understand that it is okay to make simplifying assumptions so that the calculations are reasonable.

Student Task Statement

How long would it take an ant to run from Camborne to John O'Groats?

Student Response

Answers vary. Sample response:

It will take the ant about 2 years (and 4 months) to run from Camborne to John O'Groats without stopping. The ant can run 18 mm per second. There are 60 seconds per minute and 60 minutes per hour, so this is $18 \times 60 \times 60 = 64\,800$, or 64 800 mm per hour. 64 800 mm per hour is 64.8 metres per hour, or 0.0648 km per hour. So to travel 1 312 km, it would take $1\,312 \div 0.0648$ or approximately 20 247 hours. There are 24 hours per day and 365 days per year, so dividing by 24 and then 365 tells us that it will take the ant about 2 years (and 4 months) to make this trip without stopping.

Activity Synthesis

Invite students or groups to share different solution approaches. Consider asking students to create a visual display and conducting a gallery walk. Highlight strategies that include keeping careful track of the information used including:

- How far the ant has to travel
- How fast the ant travels
- A step-by-step analysis changing mm per second to km per day and eventually km per year

Make sure to discuss why student estimates are not all identical. For these open-ended problems that require many assumptions, there is not a single valid approach or answer. Some reasons for a variety of answers in this case include:

- Students may take into account that the ant will need to rest or sleep.
- Students may choose different speeds for the ant.
- Students may round their answers differently.

If students report their answers in a variety of ways (to the nearest year, nearest month, nearest day, nearest hour, etc.), ask them what is most appropriate. Reporting to the nearest year is perhaps most appropriate, however, because there are so many hidden variables that influence the answer, including:

- The speed is only approximate and would not remain constant for years.
- The distance is also only approximate as the ant will need to circumnavigate many obstacles.
- An answer to the nearest year communicates the general idea of how long it will take very effectively.

1.2 Stacks and Stacks of Cereal Boxes

Optional: 20 minutes

In this Fermi problem, students estimate the total volume occupied by all of the breakfast cereal purchased in a year in the United Kingdom.

Monitor for different approaches to solving the problem, and select students to share during the discussion. In particular, look for:

- A range of estimates
- Different levels of precision in reporting the final estimate
- Different arrangements of boxes

Instructional Routines

- Group Presentations

Launch

Arrange students in groups of 1–4. Provide access to four-function calculators. If students are doing their own research, provide access to internet-enabled devices. If conducting a gallery walk at the end, provide access to tools for making a visual display.

Before starting, ask students to come up with a guess about the answer and poll the class. Record all guesses.

Ask students to brainstorm the information they need to answer the question. Give the information provided when students ask for it, or provide access to internet-enabled devices so that students can find the information they need.

Vital information to have on hand includes:

- Every year, people in the U.K. buy 5 billion boxes of breakfast cereal.
- A “typical” cereal box has dimensions of 6 cm by 20 cm by 30 cm.

Representation: Internalise Comprehension. Represent the same information through different modalities by using physical objects to represent abstract concepts. Some students may benefit by starting with an image, video, or a typical size cereal box to support visualisation.
Supports accessibility for: Conceptual processing; Visual-spatial processing

Anticipated Misconceptions

- Students may not think about *how* the boxes are packed. That is, they may make a volume calculation and then perform division. (Ask these students to think about how the boxes are placed in the warehouse.)
- Students may struggle with unit conversions. (Encourage them to go step by step: How many centimetres are in a metre? How many square centimetres are in a square metre? How many cubic centimetres are in a cubic metre?)

Student Task Statement

Imagine a warehouse that has a rectangular floor and that contains all of the boxes of breakfast cereal bought in the United Kingdom in one year.

If the warehouse is 3 metres tall, what could the side lengths of the floor be?

Student Response

Answers vary. Sample response:

The warehouse would be a little more than 2.5 km wide on each side. Boxes of cereal come in many different sizes, but for our purpose, let's assume that the dimensions are 6 cm by 20 cm by 30 cm. So the volume of a single box is about 3 600 cubic centimetres. There are 100^3 or 1 000 000 cubic centimetres per cubic metre, so a box of cereal is about 0.0036 cubic metres. Since there are 5 billion boxes, the total volume is about 18 000 000 cubic metres. If the warehouse is 3 metres tall, then the area of the floor needs to be at least $18\,000\,000 \div 3$ or 6 000 000 square metres. So if the warehouse floor were a square with side length 2 500 metres, it could hold all of the boxes. Note that there are 1 000 metres in a kilometre, so the warehouse would be around 2.5 km wide on each side.

Activity Synthesis

Invite students or groups to share different solution approaches. Consider asking students to create a visual display and conducting a gallery walk. Highlight strategies that include keeping careful track of the information used including:

- What is the volume of each box?
- How are the boxes arranged for storage?
- How are the dimensions of the warehouse floor determined?

Make sure to discuss why different people have different estimates.

- Some students may just calculate volume and not think of how the boxes are packed. (This is OK as the amount of empty space will not be large unless the boxes are placed haphazardly.)
- Students may place the boxes in a different way.
- Students may or may not take account of the small amount of space in the warehouse that is not filled by the boxes.

There are an infinite number of rectangles with suitable area to serve as the floor of the warehouse. Note, however, that having an appropriate area (e.g., 6 000 000 square metres) is *not* sufficient. For example, we could not have a warehouse that is 0.01 metre wide and 600 000 000 metres long. In general, if one of the dimensions is too small, it increases the amount of wasted space.

1.3 Covering the Washington Monument

Optional: 20 minutes

In this Fermi problem, students estimate the total number of tiles needed to cover the Washington Monument.

Monitor for different approaches to solving the problem, and select students to share during the discussion. In particular, look for:

- A range of estimates
- Different levels of precision in reporting the final estimate
- Different choices made to simplify the situation

Instructional Routines

- Group Presentations
- Discussion Supports

Launch

Arrange students in groups of 1–4. Provide access to four-function calculators. If students are doing their own research, provide access to internet-enabled devices. If conducting a gallery walk at the end, provide access to tools for making a visual display.

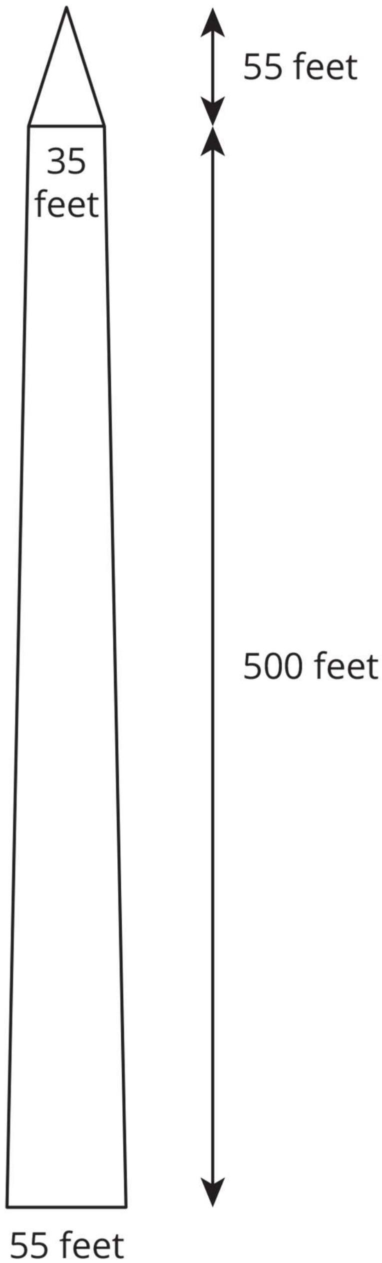
Show a picture of the Washington Monument.

Before starting, ask students to come up with a guess how many tiles it would take to cover the Washington Monument and poll the class. Record all guesses.

Ask students to brainstorm the information they need to answer the question. Give the information provided when students ask for it, or provide access to internet-enabled devices so that students can find the information they need.

Vital information to have on hand includes:

- Dimensions of the monument:



- Standard sizes for square tiles: side lengths of 1 inch, 6 inches, 8 inches, 1 foot, and $1\frac{1}{2}$ feet.

Note that 55 feet is the height of the pyramid at the top, not the height of each triangular face. Students would need the Pythagoras' Theorem to find the height of these faces, so either tell them it is close to 57.5 feet or they can use the 55 feet for their estimate. This does not have a significant impact on the area calculation, as the pyramid at the top does not account for most of the surface area.

Action and Expression: Internalise Executive Functions. Provide students with a graphic organiser to capture information about the different assumptions they made when calculating the areas of the trapezium shaped faces of the Washington Monument building. *Supports accessibility for: Language; Organisation*

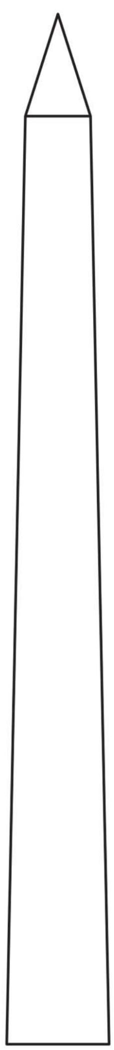
Anticipated Misconceptions

Students may not account for the faces of the monument that you cannot see in the picture. Remind these students that some faces are hidden. If three-dimensional building blocks are available, consider having students who have difficulty visualising the monument build a small model. It is difficult to model the exact shape of the Washington Monument, but the physical model may help students make the needed calculations.

Students may have difficulty converting units, especially if their tiles are measured in inches. Ask these students how many inches are in a foot and how many square inches are in a square foot.

Student Task Statement

How many tiles would it take to cover the Washington Monument?



Student Response

Answers vary. Sample response:

First, decide how big the tiles are. Let's assume they are square tiles with side length 1 inch. If two of the trapezium shaped sides of the Washington Monument are put together (one right side up, the other upside down), the result is a parallelogram with base 90 feet and height 500 feet. So the area of those two sides would be 45 000 square feet, as would the area of the other two sides. Using 57.5 feet for the height of the triangular faces at the top, their total area is $2 \times 35 \times 57.5$ or 4 025 square feet. So the total surface area of the monument is about 94 025 square feet. Since there are 12 inches per foot, there are 144 square inches per square foot. So the surface area of the monument is about $94\,025 \times 144$ or 13 539 600 square inches. So it would require about 13 540 000 square tiles of side length 1 inch to cover the Washington Monument.

Activity Synthesis

Invite students or groups to share different solution approaches. Consider asking students to create a visual display and conducting a gallery walk. Highlight different methods of calculating the area of the trapezium shaped faces of the Washington Monument including:

- Putting two trapezium faces (or all 4 faces) together to make a parallelogram
- Decomposing each trapezium face into a rectangle and two triangles and then rearranging

Discuss why different people have different estimates. Some reasons to highlight include:

- Using different-sized tiles to cover the Washington Monument
- Thinking about how some tile is wasted, namely the tiles that go on the edge of each face
- Rounding

This problem has a different flavour than the two preceding ones. The only place estimation comes into play is with how to cover the top triangles and perhaps thinking about what happens to tiles at the edge of a face. What all the problems share, however, is that we cannot give an *exact* answer. To provide a reasoned estimate, keeping track of all assumptions being made is vital. This tells us not only how accurately we should report our answer but also helps us communicate clearly to others who may have made different assumptions.

Speaking: Discussion Supports. Give students additional time to make sure that everyone in their group can explain their strategy for estimating the number of tiles using appropriate mathematical language (e.g., face, height, width, surface area, trapezium, etc.). Be sure to vary who is called on to represent the ideas of each group. This routine will prepare students for the role of group representative and to support each other to take on that role. *Design Principle(s): Optimise output (for explanation)*

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