Grades 5-8 (AS), 9-12 (AS)

Duration: 45 min

Tools: 1-2 Logifaces Set / group

Group work

Keywords: Mathematical Induction, Proof, Algebraic identities, Arithmetic sequence

311 - Growing Pine Trees

MATHS / SEQUENCES

LOGIFACES METHODOLOGY

Erasmus+

TEACHER

Logifaces

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DESCRIPTION

LEVEL 1 Students work in teams, they can combine multiple Logifaces sets. They build a pine tree using the blocks.

LEVEL 2 Students try to build larger and larger pine trees. If there are not enough Logifaces pieces to build it, they make a drawing. They make rules to describe the building of the trees.

LEVEL 3 How many more pieces do we need to add to the 2nd tree to get the 3rd tree? What about the 3rd and the 4th tree? Students' task is to say the pattern, rather than the final answer.

LEVEL 4 How many more pieces do we need to add to the 99th tree to get to the 100th tree?

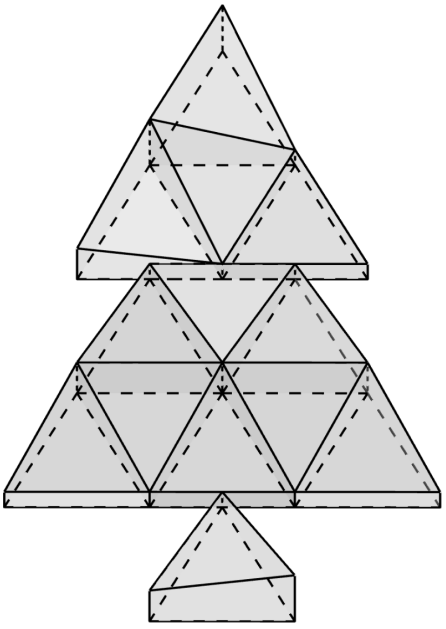
LEVEL 5 How many more pieces do we need to add to the (n-1)th tree to get to the nth tree?

LEVEL 6 How many pieces are there in the nth tree?

SOLUTIONS / EXAMPLES

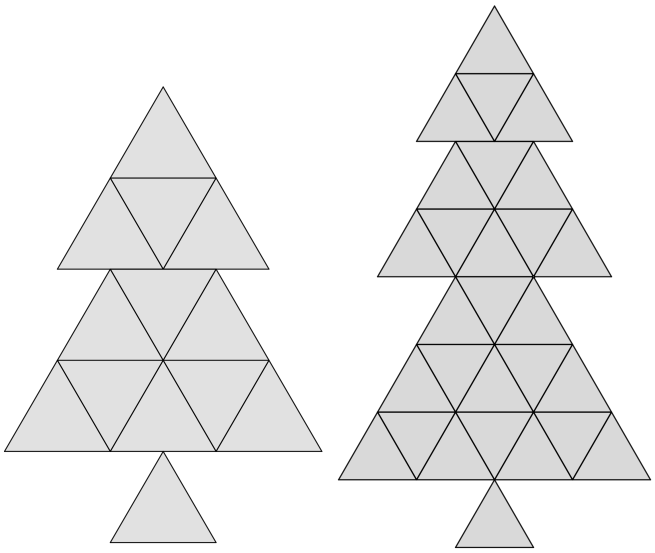
LEVEL 1 Students can build various different constructions, and the teacher should accept all of them as valid.

Two examples are shown in the Figures.



LEVEL 2 Students can continue with their own construction, or the class can choose one specific construction to work on. An example is shown in the figures.

Here we show answers for the example shown in the pictures. Note that the answers will be different for different constructions.



An example of a rule describing the construction: To build the new tree from the previous one, take away the bottom piece (trunk), copy the lower block (this had 3 rows for the 2nd tree), and add one more row (which is two pieces longer than the previous one), then add the trunk back.

LEVEL 3 3+5+7+9 and 3+5+7+9+11. Observe that $9 = 2 \times 3 + 3$ and $11 = 2 \times 4 + 3$.

LEVEL 4 3+5+7+ ... +203, where $203 = 2 \times 100 + 3$, similarly to the solution of Level 3.

LEVEL 5 In general, the nth pine tree is created from the (n-1)th by adding n+1 new rows above the trunk. The new rows consist of 3, 5, …, 2n+3 small triangles. Hence the number of pieces added to the (n-1)th tree is $3 + 5 + 7 + \ldots + (2n + 3)$. This is the sum of an arithmetic sequence, which can be calculated by the formula $a_1 + \ldots + a_n = \frac{(a_1+a_n)\times n}{2}$ that gives
$3 + 5 + 7 + \ldots + (2n + 3) = \frac{(2n+3+3)\times(n+1)}{2} = (n + 3)(n + 1)$.

Another way to calculate this sum is to use the result of exercise 309 - Triangular Square Number: a large triangle with n small triangles on each side consists of n^2 small triangles. Here the rows added to the (n-1)th tree form a large triangle with (n+2) small triangles on each side but without the top small triangle. Hence the number of pieces added is $(n + 2)^2 - 1 = (n + 3)(n + 1)$.

LEVEL 6 The top part has 4 pieces, and the bottom part 1 piece. For the rest of the pieces we need to calculate the following sum:

3+5+
3+5+7+
...
3+5+...+(2n+3)

First let's calculate the sum of the following (the numbers in bold are extra compared to the sum above)

1+
1+**3**+
1+3+5+
...
1+3+5+ ... +(2n+3)

The sum of each row is a square number (see exercise 309 - Triangular Square Number), and the sum of all the rows is the sum of square numbers from 1 to $(n+2)^2$.

Using the formula that the sum of the first n square numbers is $\frac{n(n+1)(2n+1)}{6}$ we get that the sum of all rows is $\frac{(n+2)(n+3)(2n+5)}{6}$.

Now we need to subtract the extra numbers, which is a 1 in each row, and a 3 in the 2nd row, altogether $(n+2)\times 1+3=n+5$, so we get $\frac{(n+2)(n+3)(2n+5)}{6}-n-5$.

At last we need to add the top part of the pine tree (4 pieces) and the trunk (1 piece), so we get $\frac{(n+2)(n+3)(2n+5)}{6}-n$ for the total number of pieces in the nth tree.

PRIOR KNOWLEDGE

LEVELS 1-4	None
LEVEL 5	Sum of an arithmetic sequence
LEVEL 6	Sum of the first n square numbers

RECOMMENDATIONS / COMMENTS

The levels are recommended for different age groups: Levels 1-5 for grades 5-12, Level 6 for grades 9-12. The level 5-6 exercises are challenging questions.
Exercise 309 - Triangular Square Number is recommended before the Level 5-6 questions.