Lesson 9: More and less than 1%

Goals

- Comprehend that percentages do not have to be a whole number.
- Recognise that 0.1% of a number is 1/10 of 1% of the number.
- Use reasoning about place value to calculate percentages that are not whole numbers, and explain (orally) the strategy.

Learning Targets

- I can find percentages of quantities like 12.5% and 0.4%.
- I understand that to find 0.1% of an amount I have to multiply by 0.001.

Lesson Narrative

Until now, students have been working with whole number percentages when they solve percentage increase and percentage decrease problems. As they move towards more complex contexts such as interest rates, taxes, tips and measurement error, they will encounter percentages that are not necessarily whole numbers. A percentage is a rate per 100, and now that students are working with ratios of fractions and their associated rates, they can work with fractional amounts per 100. In this lesson students consider situations where fractional percentages arise naturally. They also consider how to calculate a fractional percentage using a whole number percentage as a reference and dividing by 10 or 100. For example, if you know that 1% of 200 is 2, you can use the structure of the base-ten system to reason that 0.1% of 200 is 0.2 and 0.01% of 200 is 0.02.

This lesson gives students an opportunity to show that they can attend to precision by being careful about the difference between a fractional percentage and a fraction, for example understanding that 0.4% of a quantity is not the same as 0.4 times the quantity.

Building On

- Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, bar models, double number line diagrams, or equations.

Addressing

- Use proportional relationships to solve multistep ratio and percentage problems. Examples: simple interest, tax, markups and markdowns, gratuities and commissions, fees, percentage increase and decrease, percentage error.

Building Towards

- Use proportional relationships to solve multistep ratio and percentage problems. Examples: simple interest, tax, markups and markdowns, gratuities and commissions, fees, percentage increase and decrease, percentage error.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Co-Craft Questions
- Compare and Connect
- Discussion Supports
- Number Talk
- Think Pair Share

Student Learning Goals

Let's explore percentages smaller than 1%.

9.1 Number Talk: What Percentage?

Warm Up: 5 minutes

The purpose of this number talk is to reason about a progressive set of percentages from benchmark percentages to 1% to "unfriendly" percentages. The reasoning parallels the reasoning from earlier work where students are guided to find a unit rate and use the unit rate to solve generic percentage problems. In this activity, there are five problems, so in the interest of time it may not be possible to share all possible strategies for each problem. Instead, gather two different strategies for each.

Instructional Routines

- Compare and Connect
- Discussion Supports
- Number Talk

Launch

Display each problem one at a time. Give students 30 seconds of quiet think time per problem and ask them to give a signal when they have an answer and a strategy. Keep each problem displayed throughout the discussion. Follow with a whole-class discussion. Students may have difficulty understanding the wording of the question "10 is what percentage of 50?" so when discussing strategies with the whole class, use *Compare and Connect* to see different ways (e.g., words, equations, double number-lines, etc.) to represent and solve these problems. Ask students "What is similar and what is different?" in their approaches.

Representation: Internalise Comprehension. To support working memory, provide students with sticky notes or mini whiteboards.
Supports accessibility for: Memory; Organisation

Anticipated Misconceptions

Students might think the question is asking them to calculate 10% of 50. Ask students a variation of the question: What percentage of 50 is 10?

Student Task Statement

Determine the percentage mentally.

10 is what percentage of 50?

5 is what percentage of 50?

1 is what percentage of 50?

17 is what percentage of 50?

Student Response

- 20%, because $\frac{10}{50} = \frac{20}{100}$.
- 10%, because $\frac{5}{50} = \frac{10}{100}$.
- 2%, because $\frac{1}{50} = \frac{2}{100}$.
- 34%, because $\frac{17}{50} = \frac{34}{100}$.

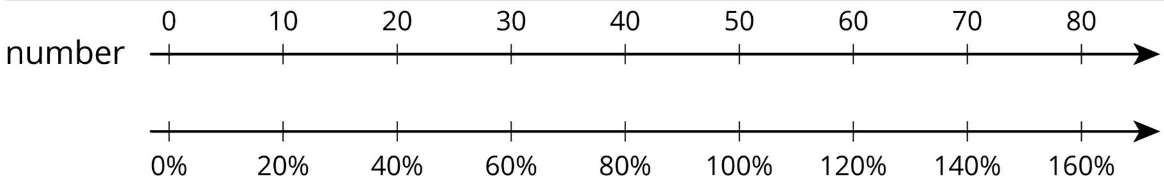
Activity Synthesis

Ask students to share their strategies for each problem. Record and display their explanations for all to see. To involve more students in the conversation, consider asking:

- "Who can restate ___'s reasoning in a different way?"
- "Did anyone solve the problem the same way but would explain it differently?"
- "Did anyone solve the problem in a different way?"
- "Does anyone want to add on to _____'s strategy?"
- "Do you agree or disagree? Why?"

Since students may not have encountered the idea of percentage rate recently, take the time to show any representations of the relationship that come up.

For example, a double number line:



A table:

number	percentage of 50
50	100
10	20
5	10
1	2
17	34
75	150

An equation: If x represents the number and y represents its percentage of 50, then $y = 2x$ since 1 is 2% of 50.

A shortcut that they learned previously: For example, $17 \div 50 = 0.34$, and 0.34 is $\frac{34}{100}$, so 17 is 34% of 50.

Speaking: Discussion Supports.: Display sentence frames to support students when they explain their strategy. For example, "First, I _____ because . . ." or "I noticed _____ so I . . . ." Some students may benefit from the opportunity to rehearse what they will say with a partner before they share with the whole class.
Design Principle(s): Optimise output (for explanation)

9.2 Waiting Tables

10 minutes (there is a digital version of this activity)

This activity gives students an opportunity to put into practice some things they already know about finding percentage rates. Additionally, the idea of a fraction of a percentage appears for the first time. Encourage students to use any representation they would like to calculate the percentage of appetisers, entrees and desserts. Monitor for students who used various representations and ask them to share during the discussion. The main focus should be on the fractional percentages they encounter in this problem for the first time.

Instructional Routines

- Co-Craft Questions
- Think Pair Share

Launch

Tell students they will be finding some more percentages. Encourage them to use any representation they understand, for example, a double number line or a table. Students in groups of 2. Give students 1–2 minutes of quiet work time, followed by partner then whole-class discussion.

If using the digital activity, students will use an applet to find and check percentages.

Representation: Internalise Comprehension. Activate or supply background knowledge about diagrams that can be used to represent percentage rates such as tables and double number line diagrams.
Supports accessibility for: Memory; Conceptual processing Writing and Listening: Co-Craft Questions. Before students begin work, display the waiter's situation without revealing the questions. Ask students to write down possible mathematical questions that might be asked about the situation. Invite pairs to compare their questions, and then ask for a few to be shared in a whole-class discussion. Reveal the actual questions about the waiter's situation that students will answer. This will help students make sense of the problem before attempting to solve it.
Design Principle(s): Optimise output for explanation

Anticipated Misconceptions

If students round to the nearest percentage, they will get that 33% of the dishes were appetisers and 43% of the dishes were entrées. Along with the 25% desserts, their percentages will sum to 101%. Point out that all of the dishes taken as percentages should sum to 100% and encourage them to be critical of their method and try to figure out where the extra 1% came from.

Student Task Statement

During one waiter's shift, he delivered 13 appetisers, 10 entrées, and 17 desserts.

1. What percentage of the dishes he delivered were:
   a. desserts?
   b. appetisers?
   c. entrées?
2. What do your percentages add up to?

Student Response

Entrées: 25%. There are 40 total dishes because $13 + 17 + 10 = 40$. There are 10 desserts, and $10 \div 40 = 0.25$.

Appetisers: 32.5%. The 13 appetisers are 32.5% of the dishes, because $13 \div 40 = 0.325$.

Desserts: 42.5%. There are 17 entrées, and $17 \div 40 = 0.425$.

The total sums to 100%, because $32.5 + 42.5 + 25 = 100$.

Activity Synthesis

Select students to share the percentages they calculated for each type of dish the waiter delivered. Depending on the outcome of the warm-up, it may be appropriate once again to display different representations of percentages as rates per 100.

Double number line:

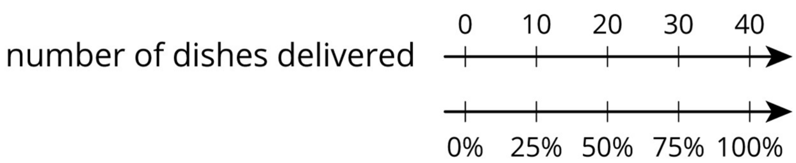


Table:

number of dishes	percentage
10	25
1	2.5
13	32.5
17	42.5

Equation: Students may have previously learned to represent relationships like this using an equation in a form $y = kx$. For example, to find what percentage 13 is of 40, they might write $13 = k \times 40$, and find that k is 0.325 by evaluating $13 \div 40$. 0.325 is the rate per 1, so 32.5 is the rate per 100.

Students may have never seen a percentage that was not a whole number. Spend a few minutes making sense of this. Ask students:

- What do you notice that is different about these percentages from the ones you have looked at before? (Some of these percentages are not whole numbers.)
- What do the percentages add up to? (Exactly 100.)
- What does 32.5% of 40 mean? (It's halfway between 32% and 33% of 40.)

9.3 Fractions of a Percentage

10 minutes

The purpose of this activity is to encourage students to look for efficient strategies while working with fractional percentages. Monitor for students using the following strategies:

- Using 1% to find 0.1%

- Making substitutions of known quantities to help calculate unknown quantities

Select students to share these strategies during discussion.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Discussion Supports

Launch

Give students 5 minutes of quiet work time, followed by whole-class discussion.

Action and Expression: Internalise Executive Functions. Chunk this task into more manageable parts to support students who benefit from support with organisation and problem solving. For example, present one question at a time and monitor students to ensure they are making progress throughout the activity.
Supports accessibility for: Organisation; Attention

Anticipated Misconceptions

When students calculate the various percentages of 60 they may make mistakes in the place value of the answers. Refer students to the previous activity's discussion. You may also want to ask students to calculate 10% of 60 and use that answer to calculate 30%.

If students get stuck calculating various percentages of 5 000, recommend they use the double number line provided. Ask them:

- What percentages are visible in the bottom number line?
- How much is that 1% in reference to the top number line?
- How can we use that 1% to figure out the other percentages?

Use these same questions if students get stuck calculating 15.1% and 15.7%.

Student Task Statement

1. Find each percentage of 60. What do you notice about your answers?

    30% of 60

    3% of 60

    0.3% of 60

    0.03% of 60

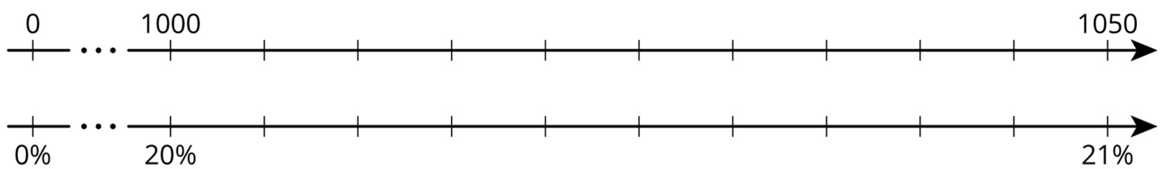
2. 20% of 5 000 is 1 000 and 21% of 5 000 is 1 050. Find each percentage of 5 000 and be prepared to explain your reasoning. If you get stuck, consider using the double number line diagram.

a. 1% of 5 000

b. 0.1% of 5 000

c. 20.1% of 5 000

d. 20.4% of 5 000



3. 15% of 80 is 12 and 16% of 80 is 12.8. Find each percentage of 80 and be prepared to explain your reasoning.

a. 15.1% of 80

b. 15.7% of 80

Student Response

1. Percentages of 60:

a. 18 since $0.3 \times 60 = 18$.

b. 1.8 since $0.03 \times 60 = 1.8$.

c. 0.18 since $0.003 \times 60 = 0.18$.

d. 0.018 since $0.0003 \times 60 = 0.018$.

I notice that each percentage is $\frac{1}{10}$ of the previous percentage.

2. Percentages of 5 000:

a. 50, because $21\% - 20\% = 1\%$ and $1\,050 - 1\,000 = 50$.

b. 5, because 5 is $\frac{1}{10}$ of 50.

c. 1 005, because $1\,000 + 5 = 1\,005$.

d. 1 020, because $1\,000 + 4 \times 5 = 1\,020$.

3. Percentages of 80:

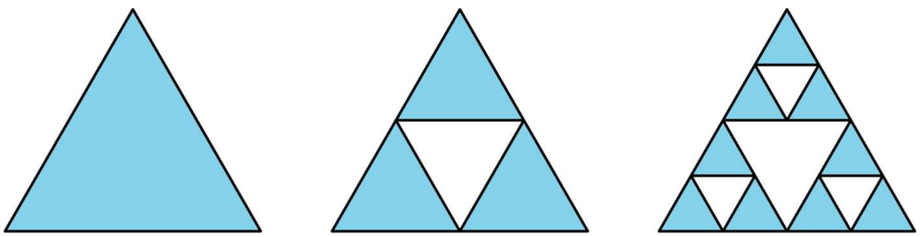
a. 12.08. One percentage of 80 is 0.8, because $16\% - 15\% = 1\%$ and $12.8 - 12 = 0.8$. So 0.1% of 80 is 0.08 because $\frac{1}{10} \times 0.8 = 0.08$. Then we add 15% of 80 to 0.1% of 80, which is $12 + 0.08$.

b. 12.56, because $12 + 7 \times 0.08 = 12.56$.

Are You Ready for More?

To make Sierpinski's triangle,

- Start with an equilateral triangle. This is step 1.
- Connect the midpoints of every side, and remove the middle triangle, leaving three smaller triangles. This is step 2.
- Do the same to each of the remaining triangles. This is step 3.
- Keep repeating this process.



1. What percentage of the area of the original triangle is left after step 2? Step 3? Step 10?
2. At which step does the percentage first fall below 1%?

Student Response

1. Step 2: 75%. Step 3: 56.25%. Step 10: about 5.63%
2. Step 17

Activity Synthesis

Select previously identified students to share the different strategies used to solve the problems. For the first problem, select students who use the answer to 30% of 60 to calculate the answers to the other problems.

- One likely strategy is one where you keep dividing by 10.
- Another is to make substitutions into an expression. For example, I know that 30% of 60 is 18 and I want to find 3% of 60. I also know that 3% is $\frac{1}{10}$ of 30%. 3% of 60 is $\frac{1}{10}$ of 30% of 60 therefore $\frac{1}{10}$ of $18 = 1.8$. Focus on the relationship between 30% and 3%. For the other problems, highlight strategies by students who recognised that they can use 1% of a number to calculate 0.1% of a number and make multiples of that to get, for example, 0.7% of a number.

Speaking: Discussion Supports. Use this routine to amplify mathematical uses of language to communicate about the relationship between quantities. As students share their strategies for the first question, revoice their statements to use appropriate mathematical language, such as, “10 times more” or “10 times less.” Invite students to use this language when describing their strategies.
Design Principle(s): Optimise output (for explanation)

9.4 Population Growth

Optional: 15 minutes

The purpose of this activity is for students to find a fractional percentage increase.

Look for students who calculate the percentage first and then add them together, and students who multiply by 1.08 and 1.008, respectively.

Instructional Routines

- Compare and Connect

Launch

Students in groups of 2. 4 minutes of quiet work time followed by partner and then whole-class discussion.

Representation: Internalize Comprehension. Allow students to use calculators to ensure inclusive participation in the activity.
Supports accessibility for: Memory; Conceptual processing

Anticipated Misconceptions

Students who want to multiply by $1 + \frac{p}{100}$ may have trouble determining the value of each digit. Have them think about the problem in steps. How can you find 8%? (Multiply by 0.08.) How can you find 0.8%? (Multiply by 0.008.)

Student Task Statement

1. The population of City A was approximately 243 000 people, and it increased by 8% in one year. What was the new population?

2. The population of city B was approximately 7 150 000, and it increased by 0.8% in one year. What was the new population?

Student Response

1. Approximately 262 000. $1.08 \times 243\,000 \approx 262\,000$.

2. Approximately 7 210 000. $1.008 \times 7\,150\,000 \approx 7\,210\,000$.

Activity Synthesis

Have selected students show solutions, starting with a solution where the percentage is found first and then added to the initial amount, then the approach where one multiplies by 1.08 or 1.008, respectively. Make sure everyone understands both methods. Help students see the connections between these strategies.

Representing, Speaking: Compare and Connect. Use this routine after students present their solutions for calculating the new population of each city. Ask students, "what is the same and what is different about these approaches?" Call students' attention to the different ways students represented the percentage increase in their strategies. These exchanges strengthen students' mathematical language use and reasoning based on percentage increases with and without fractional amounts.
Design Principle(s): Maximize meta-awareness

Lesson Synthesis

In this lesson, we worked with fractions of a percentage.

- "How are these percentages related to each other: 40%, 4%, 0.4%, 0.04%?" (Each is $\frac{1}{10}$ of the previous one.)
- "How can we use 40% to help calculate the other percentages?" (Use the fact that 4% is $\frac{1}{10}$ of 40% so if we know 40% of something we can reason to figure out 4%, 0.4% or others.)
- "If we know 1% of a number, how can we use that to help us calculate 0.5% of a number?" (Calculate 5% of that number (5 times 1%) and use same reasoning as above to figure out 0.5%. Alternatively, 0.5% of a number is half of 1% of that number.)

9.5 Percentages of 75

Cool Down: 5 minutes

Student Task Statement

Find each percentage of 75. Explain your reasoning.

1. What is 10% of 75?
2. What is 1% of 75?
3. What is 0.1% of 75?
4. What is 0.5% of 75?

Student Response

1. 7.5, because $0.1 \times 75 = 7.5$.

2. 0.75, because 1% is $\frac{1}{10}$ of 10%, and $\frac{1}{10} \times 7.5 = 0.75$.

3. 0.075, because 0.1% is $\frac{1}{10}$ of 1%, and $\frac{1}{10} \times 0.75 = 0.075$.

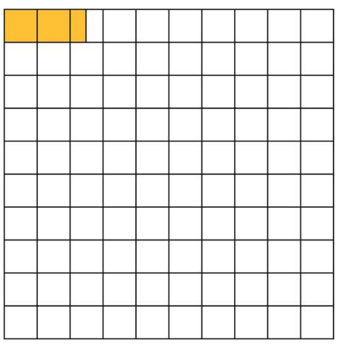
4. 0.375, because 0.5% is half of 1%, and $0.75 \div 2 = 0.375$.

Student Lesson Summary

A percentage, such as 30%, is a rate per 100. To find 30% of a quantity, we multiply it by $30 \div 100$, or 0.3.

The same method works for percentages that are not whole numbers, like 7.8% or 2.5%. In the square, 2.5% of the area is shaded.

To find 2.5% of a quantity, we multiply it by $2.5 \div 100$, or 0.025. For example, to calculate 2.5% interest on a bank balance of £80, we multiply $(0.025) \times 80 = 2$, so the interest is £2.



We can sometimes find percentages like 2.5% mentally by using convenient whole number percentages. For example, 25% of 80 is one fourth of 80, which is 20. Since 2.5 is one tenth of 25, we know that 2.5% of 80 is one tenth of 20, which is 2.

Lesson 9 Practice Problems

Problem 1 Statement

The student snack shop sold 32 items this week. For each snack type, what percentage of all snacks sold were of that type?

snack type	number of items sold
fruit cup	8
veggie sticks	6
crisps	14
water	4

Solution

Fruit cup: 25%, veggie sticks: 18.75%, crisps: 43.75%, water: 12.5%

Problem 2 Statement

Select **all** the options that have the same value as $3\frac{1}{2}\%$ of 20.

a. 3.5% of 20

b. $3\frac{1}{2} \times 20$

c. $(0.35) \times 20$

d. $(0.035) \times 20$

e. 7% of 10

Solution ["A", "D", "E"]

Problem 3 Statement

22% of 65 is 14.3. What is 22.6% of 65? Explain your reasoning.

Solution

14.69.
22.6% of 65 is 22% of 65 (or 14.3) and an additional 0.6% of 65. 1% of 65 is 0.65. 0.1% of 65 is 0.065. 0.6% of 65 is $6 \times (0.065) = 0.39$. So 22.6% of 65 is 14.69, because $14.3 + 0.39 = 14.69$.

Problem 4 Statement

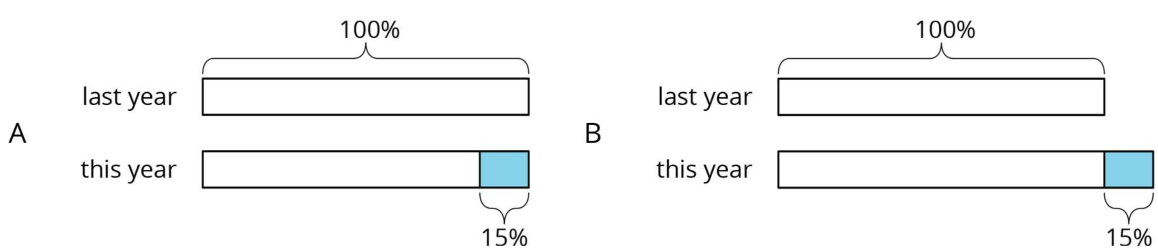
A bakery used 30% more sugar this month than last month. If the bakery used 560 kilograms of sugar last month, how much did it use this month?

Solution

728 kilograms

Problem 5 Statement

Match each situation to a diagram. The diagrams can be used more than once.



A. The amount of apples this year decreased by 15% compared with last year's amount.

B. The amount of pears this year is 85% of last year's amount.

C. The amount of cherries this year increased by 15% compared with last year's amount.

D. The amount of oranges this year is 115% of last year's amount.

1. Diagram A

2. Diagram B

Solution

- A: 1
- B: 1
- C: 2
- D: 2

Problem 6 Statement

A certain type of car has room for 4 passengers.

a. Write an equation relating the number of cars (n) to the number of passengers (p).

b. How many passengers could fit in 78 cars?

c. How many cars would be needed to fit 78 passengers?

Solution

a. $p = 4n$

b. 312 passengers, because $4 \times 78 = 312$

c. 20 cars, because $78 \div 4 = 19.5$ and you can't use half of a car.