Lesson 5: Introduction to linear relationships

Goals

- Compare and contrast (orally and in writing) proportional and non-proportional linear relationships.

- Interpret (orally and in writing) features of the graph (i.e., gradient and y-intercept) of a non-proportional linear relationship.

Learning Targets

- I can find the rate of change of a linear relationship by figuring out the gradient of the line representing the relationship.

Lesson Narrative

After revisiting examples of proportional relationships in the previous lessons, this lesson is the first of four lessons that moves from proportional relationships to **linear relationships** with positive rates of change. The two activities use a situation where the height of a stack of styrofoam cups is not proportional to the number of cups in the stack. Students use the same tools they learned to represent proportional relationships in this situation—graphs, tables, and equations. They see that each cup increases the height of the stack by the same amount (unit rate becomes rate of change) and that they can use this to answer questions about the height for an unknown number of cups. They investigate and describe similarities and differences between linear relationships and proportional relationships in this context. They make connections between the **rate of change** of the relationship and the gradient of a line representing the relationship.

In this lesson, the focus is proportionality vs. linear relationships and rate of change. The meaning of the vertical intercept of the graph comes up briefly but will be revisited more fully in the next lesson.

The first two activities in this lesson use a particular type of cup. Photos are included of all measurements needed, so this lesson can be used without any additional preparation. However, if desired, the lesson could be modified so that students measure stacks of actual cups.

Building On

- Apply and extend previous understandings of multiplication and division to divide fractions by fractions.

- Decide whether two quantities are in a proportional relationship, e.g., by testing for equivalent ratios in a table or graphing on a coordinate plane and observing whether the graph is a straight line through the origin.

Addressing

- Understand the connections between proportional relationships, lines, and linear equations.

Instructional Routines

- Collect and Display

- Discussion Supports

- Number Talk

Required Materials
Graph paper
Rulers

Required Preparation

If students will use actual cups, gather cups and perform activities ahead of time so that you know measurements for your particular type of cup.

Student Learning Goals

Let's explore some relationships between two variables.

5.1 Number Talk: Fraction Division

Warm Up: 5 minutes

This number talk encourages students to think about the numbers in a calculation problem and rely on what they know about structure, patterns, fractions, and division to mentally solve a problem. Only one problem is presented to allow students to share a variety of strategies for division. Encourage the students who solve the problem procedurally to think about the meaning of division and how that supports what they did.

Instructional Routines

- Discussion Supports

- Number Talk

Launch

Display the problem for all to see. Give students 2 minutes of quiet think time and ask them to give a signal when they have an answer and a strategy. Follow with a whole-class discussion.

Representation: Internalise Comprehension. To support working memory, provide students with sticky notes or mini whiteboards.
Supports accessibility for: Memory; Organisation

Student Task Statement

Find the value of $2\frac{5}{8} \div \frac{1}{2}$.

Student Response

$5\frac{1}{4}$ or equivalent

Possible strategies:

- Multiply by the reciprocal: $\frac{21}{8} \times 2$

- Distributive property: $\left(2 \div \frac{1}{2}\right) + \left(\frac{5}{8} \div \frac{1}{2}\right)$

- A visual solution that shows how many $\frac{1}{2}$'s are in $2\frac{5}{8}$

- Long division: $2.625 \div 0.5$

Activity Synthesis

Invite students to share their strategies. Record and display student explanations for all to see. Ask students to explain how they thought about the division problem and, if solved procedurally, why that strategy works. To involve more students in the conversation, consider asking:

- "Who can restate ___'s reasoning in a different way?"

- "Did anyone solve the problem the same way but would explain it differently?"

- "Did anyone solve the problem in a different way?"

- "Does anyone want to add on to ____'s strategy?"

- "Do you agree or disagree? Why?" At the end of the discussion, if time permits, ask a few students to share a story problem for which the expression $2\frac{5}{8} \div \frac{1}{2}$ represents the answer.

Speaking: Discussion Supports.: Display sentence frames to support students when they explain their strategy. For example, "First, I ____ because . . ." or "I noticed ____ so I . . . ." Some students may benefit from the opportunity to rehearse what they will say with a partner before they share with the whole class.

Design Principle(s): Optimise output (for explanation)

5.2 Stacking Cups

10 minutes

In this task, students are presented with a situation that leads to a linear relationship that is not proportional because there is a non-zero starting amount. By trying to answer the question, "How many cups are needed to get to a height of 50 cm?" the students explore the rate of change, which is the increase per cup after the first cup. The rate of change can be seen in the graph as the gradient. Students use representations and ideas from previous lessons on proportional relationships. They use tables and graphs to represent the situation and make deductions by generalising from repeated reasoning, arguing that each additional cup increases the height of the stack by the same amount. As students are working, suggest that students make a graph or a table if they are stuck or if they have trouble explaining their reasoning. Students should be prepared to share their strategies with the class.

Students can attack this problem using any method. If they get stuck, here are some things it would be useful to figure out:

• How much height does each cup add?

• Is the first cup different from the others?

• Create a graph or table to help you reason about this problem.

As students work, identify students who use different strategies, i.e. graphs, tables, equations.

This activity and the next were written using a particular type of cup. Photos are included of all measurements needed, so these activities can be used without any additional preparation. However, if desired, the lesson could be modified so that students are measuring stacks of actual cups. The numbers then may have to be adjusted since the cups come in different sizes. Teachers who want to use real cups should measure the height of two stacks ahead of time, find the rate of change and make sure that it is approximately constant. Note that rounding error will likely play a role unless the number of cups in the stacks is chosen carefully, so some flexibility in answers may be necessary if students measure actual stacks of cups.

Instructional Routines
• Collect and Display

Launch

Before students look at the picture with all of the given information, display just this part of the picture.

Ask students how many cups they see, and how tall the stack is. (There are 6 cups, and the stack is 15 cm tall.) Then, ask students to quietly make a prediction about the height of 12 cups and give you a silent signal when they have a response. When everyone has come up with an answer, ask a few students to share their responses and write these on the board. It is likely that some students will say 30 centimetres, and some might have an argument for why the height of 12 cups must be less than 30 centimetres. This question will be resolved as soon as students look at the photo in the task. Tell students that their job is to figure out how many cups would be needed in order to stack them to a height of 50 centimetres.

Arrange students in groups of 2–4. Give access to rulers and graph paper. Allow individual think time of 3 minutes before students work together in groups.

Representation: Internalise Comprehension. Begin the activity with concrete or familiar contexts. Set up a display of foam cups next to a ruler, as depicted. Encourage students to consider the rate of change and whether or not there is a proportional relationship. *Supports accessibility for: Conceptual processing; Memory Conversing, Representing, Writing: Collect and Display.* As students discuss, listen for and collect the language students use to describe whether they think the relationship is proportional. Write the students' words and phrases on a visual display and update it throughout the remainder of the lesson. Remind students to borrow language from the display as needed. This will help students read and use mathematical language during their paired and whole-group discussions. *Design Principle(s): Optimise output (for explanation); Maximise meta-awareness*

Anticipated Misconceptions

Students compute $\frac{15}{6} = 2.5$ or $\frac{23}{12} = 1.9$ and use this as the increase per cup. Measurement is approximate; students may be looking for integer values. Make sure values agree approximately between students, but reassure them that they are close enough.

Student Task Statement

We have two stacks of styrofoam cups.

- One stack has 6 cups, and its height is 15 cm.

- The other stack has 12 cups, and its height is 23 cm.

How many cups are needed for a stack with a height of 50 cm?

Student Response

33 cups (32.25 cups get us to 50 cm height). Possible strategy:

number of cups	height in centimetres
6	15
12	23
18	31
24	39

30	47
33	51

Activity Synthesis

Poll the class about the number of cups they came up with to reach a total height of 50 cm.

If the class used real cups, stack enough of them to get to a height of 50 cm, and let students check if they were correct. Otherwise display this photo:
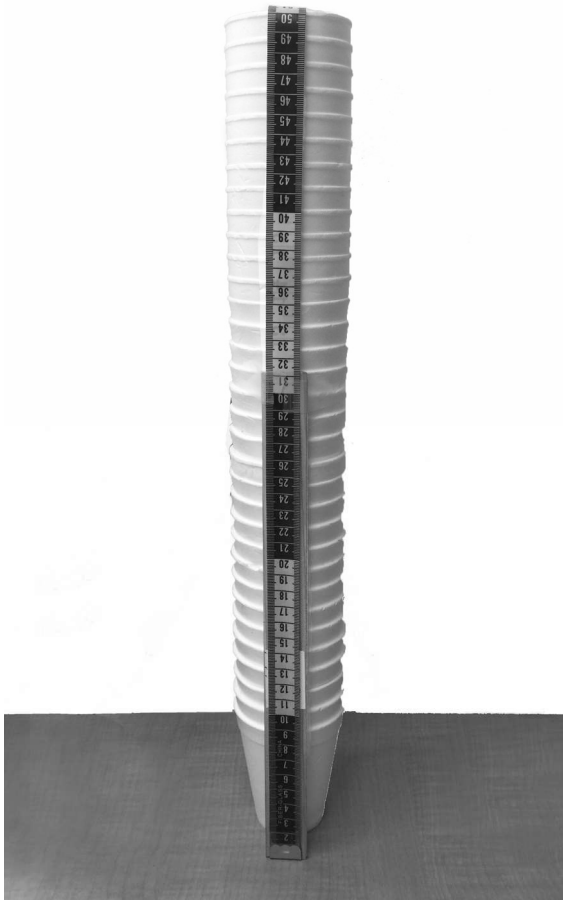
Ask students who used different strategies to share their reasoning with the class.

Ask students if they think the relationship is proportional. If there is a disagreement, ask one person per opinion to share their reasons. Encourage them to come to the conclusion that it is not proportional since doubling the number of cups from 6 to 12 did not double the height. Students can see that when stacking the cups only the rim adds to the height. The first cup adds more height since there are additionally 7 cm from the bottom of the cup to the bottom of the rim.

Tell students that even though the relationship is not proportional, this relationship has things in common with proportional relationships, and that they will explore this in the next activity.

5.3 Connecting Gradient to Rate of Change

15 minutes

The previous task asks students to estimate how many cups it will take to create a 50 cm stack. Students observe that the height of the stack grows regularly with the number of cups, but it is *not* a proportional relationship. In this task, they examine the relationship more closely, graph the relationship, and then interpret the graph.

Each successive cup, after the first, adds the same amount of height to the stack. This amount of height added per cup is the rate of change for the relationship between number of cups in a stack and height of the stack. It is also the gradient of the line that represents this relationship. This makes sense because the gradient of a line can be calculated by finding the amount of change in y (the height of the stack) when x (the number of cups in the stack) increases by 1.

Students also find where their graph intersects the y-axis and interpret this value in terms of the situation: it is the height of the bottom part of the first cup, below the rim. If this part of the cups was removed, then the graph would go through (0,0), and the relationship would be proportional because the stacks would just be stacks of the cup rims.

Monitor for students who recall work with gradient triangles from previous learning when they evaluate the gradient of the line. A natural choice for a gradient triangle in this case would be one with a horizontal side length that is a multiple of 3 and a vertical side length that is a multiple of 4 (the given data points would encourage a choice where the horizontal side length is 6 and the vertical side length is 8). As they interpret the meaning of the gradient in the context, monitor for correct language, i.e. "the height added to the stack, in cm, *per* cup."

Instructional Routines
- Discussion Supports

Launch

Keep students in the same groups. Remind students that the number of cups and the height of the stack is not a proportional relationship. Ask students how they determined the number of cups needed to make a stack 50 cm in height. A table is a likely choice and the table had some nice structure, structure which allowed them to effectively make calculations and predictions, even though it was not a ratio table. Tell them that in this task, they are to examine that structure more carefully using a graph.

Give students 8–10 minutes of group work time and follow with a whole-class discussion.

Anticipated Misconceptions

Students may need a reminder of the meaning of gradient for a line. Encourage them to begin by drawing a gradient triangle.

Students may struggle with the question about the meaning of the y-intercept for the graph. Ask them to think about the cups. Ask them how much height is added with each cup after the first? What happens if you *subtract* this amount from the first cup?

Student Task Statement

1. If you didn't create your own graph of the situation before, do so now.
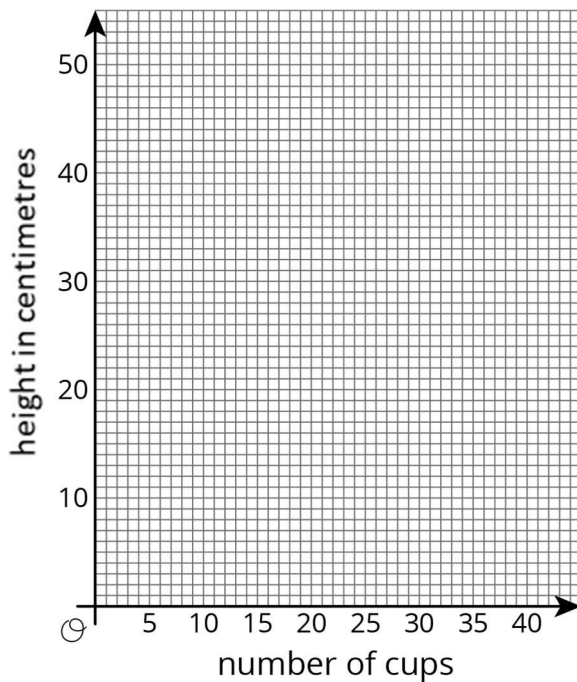
2. What are some ways you can tell that the number of cups is not proportional to the height of the stack?

3. What is the **gradient** of the line in your graph? What does the gradient mean in this situation?

4. At what point does your line intersect the vertical axis? What do the coordinates of this point tell you about the cups?

5. How much height does each cup after the first add to the stack?

Student Response
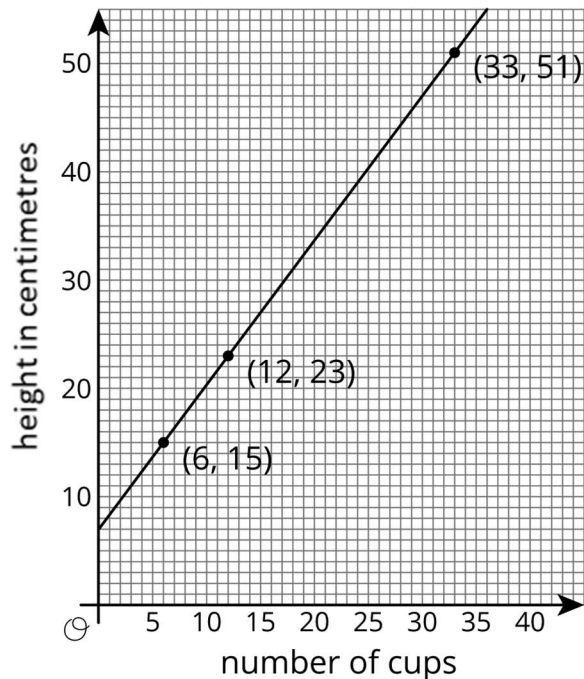
1. See graph.

2. If the relationship were proportional, then the line would pass through the origin. Since the line does not pass through the origin, the relationship is not proportional. Proportional relationships have constant quotients of coordinate pairs: $\frac{15}{6}$ is not equal to $\frac{23}{12}$. Or another way of looking at it, we can't scale: $2 \times 6 = 12$ but $2 \times 15 \neq 23$.

3. The gradient is $\frac{4}{3}$. Every cup (except for the first one) adds $\frac{4}{3}$ cm to the height. Or every rim, including the first one, adds $\frac{4}{3}$ cm to the height.

4. The line intersects the vertical axis at the point $(0,7)$. If we have 0 cups, the height shows at 7 cm. This does not make sense if we think about the number of cups in the stack. If however, we notice that each cup has a rim, then 7 cm is the height from the bottom of the first cup to its rim.

5. $\frac{4}{3}$ cm

Activity Synthesis

Begin by displaying a correct graph to support the discussion. Select students to use the graph and share:

- how they can tell the number of cups is not proportional to the height of the stack.

- how they calculated the gradient of the line (on the grid, draw in any triangles used).

- what the point on the graph (0,7) means. (It tells us the distance from the bottom to the rim of the first cup.)

Display the prompts "How can you see on the graph the amount that one cup adds to the height?" and give students 2–3 minutes of quiet work time to write a brief response. Select 3–4 students to share their responses. If not suggested by students, draw a gradient triangle with a horizontal distance of 1 on the graph or display this image:
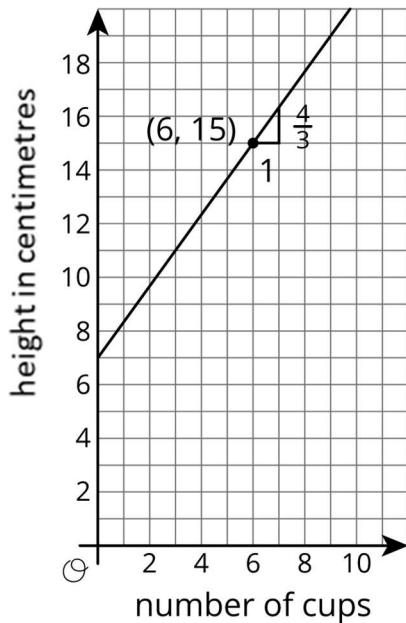
Draw a right triangle to see that each cup increases the height by $\frac{4}{3}$ cm.

It is important to bring out here that the rate per 1, $\frac{4}{3}$, is not the constant of proportionality, since this is not a proportional relationship! This value is how much each cup adds to the height of the stack, and it is called the "rate of change." The **rate of change** of y in a linear relationship between x and y is the increase in y when x increases by 1. Note that the rate of change of the relationship has the same value as the **gradient** of the line representing the relationship. So asking "what is the gradient?" is the same as asking "how much height does each cup after the first add to the stack?"

Lastly, students' graphs may consist either of discrete points corresponding to coordinate pairs (number of cups, height) or of the entire line as shown in the solution. It is understood that only the points that represent a whole number of cups have a valid interpretation in the context. This continuous graphical representation of a linear relationship, whether the context is continuous or discrete, is very common and will be seen throughout this unit.

Speaking, Listening, Conversing: Discussion Supports. During the discussion, invite groups to use these sentence frames: "The number of cups is/is not proportional to the height of the stack because ___." As partners share their thinking, press for details by asking questions such as: "How does the graph support your thinking?" and "Where do you see a constant

rate of change?" while pressing for mathematical language. This will help students use the graph to make sense of the data and interpret the gradient in context.
Design Principle(s): Support sense-making

Lesson Synthesis

The main focus of this lesson is the transition from proportional relationships to **linear relationships** that are not proportional:

• Understand that there are linear relationships that are not proportional.

• Note that the **rate of change** of the linear relationship is the same value as the **gradient** of a line representing the relationship.

• Interpret the rate of change in the context of the situation.

In order to highlight this focus, ask students:

• "How can we tell if a linear relationship is proportional or not? From the graph? From a table? From the context?" (Check that when both variables are 0, this makes sense: on the graph, in the table, or in the context.)

• "What does the rate of change of a linear relationship tell us?" (The gradient of the graph.)

5.4 Stacking More Cups

Cool Down: 5 minutes

Students apply what they have learned about graphing a non-proportional relationship to another stack of cups that are different in size from those analysed in the activities.

Student Task Statement

A shorter style of cup is stacked tall. The graph displays the height of the stack in centimetres for different numbers of cups. How much does each cup after the first add to the height of the stack? Explain how you know.
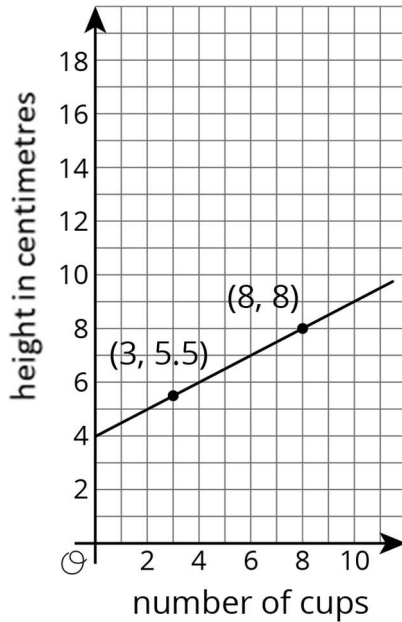
Student Response

0.5 cm (or equivalent). Since 5 cups add 2.5 cm to the height of the stack, each cup adds 0.5 cm.

Student Lesson Summary

Andre starts babysitting and charges £10 for travelling to and from the job, and £15 per hour. For every additional hour he works he charges another £15. If we graph Andre's earnings based on how long he works, we have a line that starts at £10 on the vertical axis and then increases by £15 each hour. A **linear relationship** is any relationship between two quantities where one quantity has a constant **rate of change** with respect to the other.
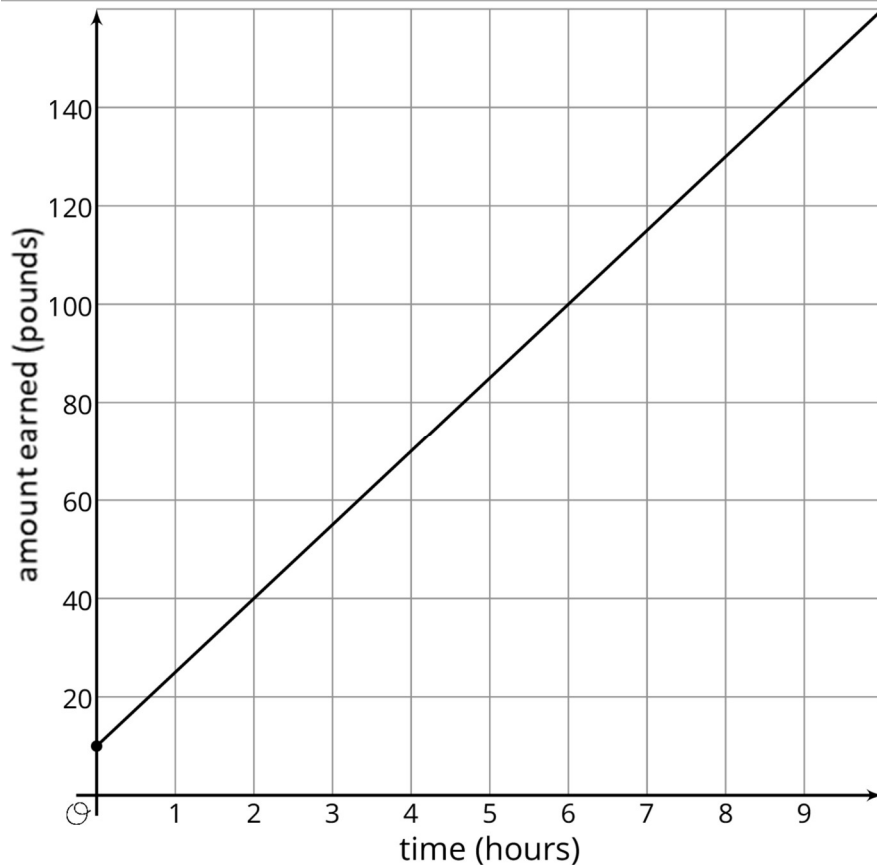
We can figure out the rate of change using the graph. Because the rate of change is constant, we can take any two points on the graph and divide the amount of vertical change by the amount of horizontal change. For example, take the points (2,40) and (6,100). They mean that Andre earns £40 for working 2 hours and £100 for working 6 hours. The rate of change is $\frac{100}{6-2} = 15$ pounds per hour. Andre's earnings go up £15 for each hour of babysitting.

Notice that this is the same way we calculate the **gradient** of the line. That's why the graph is a line, and why we call this a linear relationship. The rate of change of a linear relationship is the same as the gradient of its graph.

With proportional relationships we are used to graphs that contain the point (0,0). But proportional relationships are just one type of linear relationship. In the following lessons, we will continue to explore the other type of linear relationship where the quantities are not both 0 at the same time.

Glossary
- linear relationship

Lesson 5 Practice Problems

1. **Problem 1 Statement**

   A restaurant offers delivery for their pizzas. The total cost is a delivery fee added to the price of the pizzas. One customer pays £25 to have 2 pizzas delivered. Another customer pays £58 for 5 pizzas. How many pizzas are delivered to a customer who pays £80?
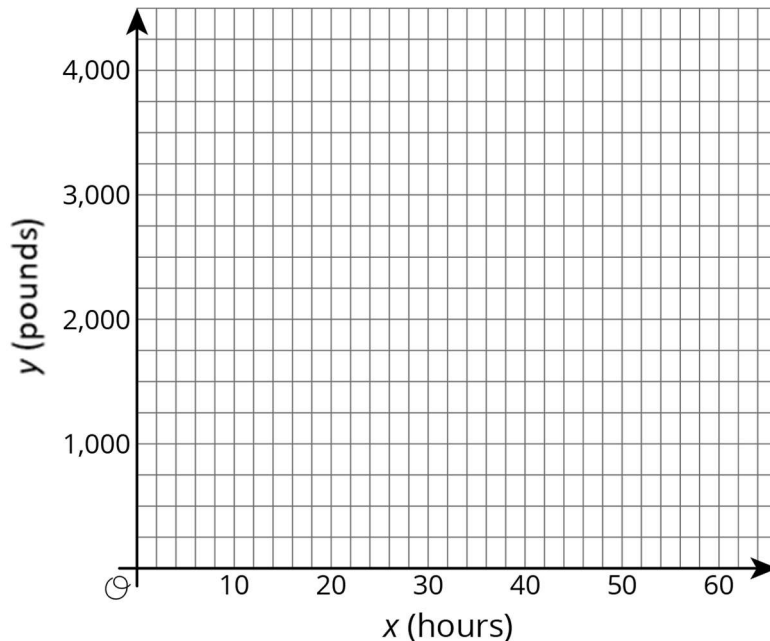
   Solution

   7 pizzas

2. **Problem 2 Statement**

   To paint a house, a painting company charges a flat rate of £500 for supplies, plus £50 for each hour of labour.

   a. How much would the painting company charge to paint a house that needs 20 hours of labour? A house that needs 50 hours?

   b. Draw a line representing the relationship between x, the number of hours it takes the painting company to finish the house, and y, the total cost of painting the house. Label the two points from the earlier question on your graph.

   c. Find the gradient of the line. What is the meaning of the gradient in this context?

   Solution

   a. £1 500, £3 000
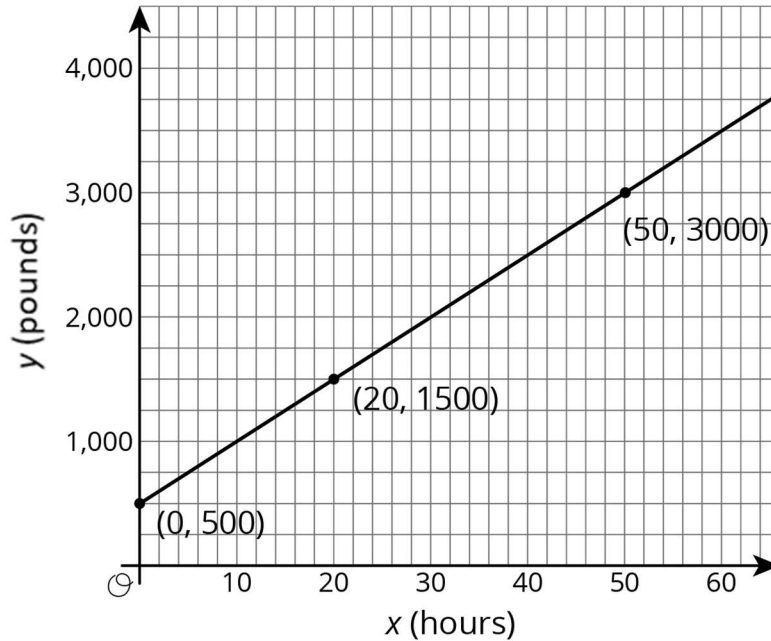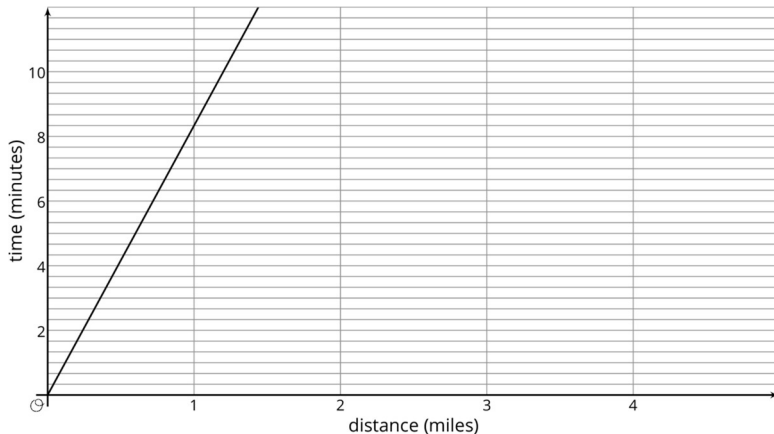
b.

c. The gradient of the line is 50, which is the same as the price per hour, in pounds, that the painting company charges for labour.

3. **Problem 3 Statement**

Tyler and Elena are on the cross country team.

Tyler's distances and times for a training run are shown on the graph.

Elena's distances and times for a training run are given by the equation $y = 8.5x$, where x represents distance in miles and y represents time in minutes.

a. Who ran farther in 10 minutes? How much farther? Explain how you know.

b. Calculate each runner's pace in minutes per mile.

c. Who ran faster during the training run? Explain or show your reasoning.

Solution

    a.   Tyler, who ran 2 hundredths of a mile farther.

    b.   Tyler had a pace of $8\frac{1}{3}$ minutes per mile, and Elena had a pace of 8.5 minutes per mile.

    c.   Tyler ran faster because it took him fewer minutes to run a mile.

4. **Problem 4 Statement**

Write an equation for the line that passes through (2,5) and (6,7).

Solution

$\frac{y-5}{x-2} = \frac{1}{2}$ (or $y = \frac{1}{2}x + 4$, or equivalent)