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ERROR BOUNDS FOR INTERPOLATING CUBIC SPLINES UNDER VARIOUS END CONDITIONS*

THOMAS R. LUCAS†

Abstract. Conditions are given when some finite difference type operators acting on the second derivative of a C^2-cubic spline interpolate of a function f over a locally uniform partition approximate f'', f''' and f^{iv} at selected knots by orders up to $O(h^4)$. Points are identified where f', f'' and f''' are approximated by s', s'' and s''' to the order h^4, h^3 and h^2 respectively. End conditions are analyzed which give these results globally over uniform partitions for sufficiently smooth functions.

1. Introduction. Let $\pi_m = \{a = x_0 < x_1 < \cdots < x_m = b\}$ be a partition of $[a, b]$. Then s is said to be a cubic spline over π_m if $s \in C^2[a, b]$ and s restricted to $[x_{i-1}, x_i]$ is a cubic polynomial for $1 \leqq i \leqq m$. The space of all such cubic splines is denoted by $\mathrm{Sp}(\pi_m, 3)$. If in addition $s(x_i) = f(x_i)$ for $0 \leqq i \leqq m$, s is said to be an $\mathrm{Sp}(\pi_m, 3)$-interpolate of f. Since $\dim \mathrm{Sp}(\pi_m, 3) = m + 3$, two additional linearly independent conditions, usually taken near the endpoints, are required to uniquely determine an $\mathrm{Sp}(\pi_m, 3)$-interpolate of a function.

The following notation will be used. Superscripts of several types denote derivatives. Thus f'', $f^{(2)}$ and f^{ii} all denote the second derivative of f. Define $f_i^{(j)} = f^{(j)}(x_i)$, $0 \leqq i \leqq m$, $h_i = x_i - x_{i-1}$, $1 \leqq i \leqq m$, $\bar{h} = \max(h_i)$, $\underline{h} = \min(h_i)$. For any $\sigma \geqq 1$, $P_\sigma[a, b]$ is the collection of all partitions for which $\bar{h} \leqq \sigma \underline{h}$; if $\sigma = 1$ the partitions are uniform and $\underline{h} = \bar{h} = h$.

For uniform partitions the following identity is well known [1, p. 12]:

$$s'_{i-1} + 4s'_i + s'_{i+1} = 3h^{-1}[s_{i+1} - s_{i-1}], \qquad 1 \leqq i \leqq m - 1. \tag{1.1}$$

Curtis and Powell [4] have found potential quantitative relations between f and an interpolating cubic spline s (the end conditions do not enter into their calculations) by the formal use of the calculus of difference operators. For example from (1.1), letting $E = e^{hD}$ be the forward difference operator, they get

$$(E^{-1} + 4I + E)s'_i = 3h^{-1}[E - E^{-1}]f_i$$

and thus

$$s'_i = 3h^{-1}(e^{-hD} + 4I + e^{hD})^{-1}(e^{hD} - e^{-hD})f_i.$$

Formally expanding the power series and dividing (assuming f to be sufficiently smooth) gives

$$s'_i = f'_i - \frac{h^4}{180} f^{\text{v}}_i + O(h^6). \tag{1.2}$$

In a similar fashion, using other spline identities, they have developed the formal expressions

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† Department of Mathematics, The University of North Carolina at Charlotte, Charlotte, North Carolina 28213. This research was supported in part by a grant from the Foundation of the University of North Carolina.

(1.3) $$s_i'' = f_i'' - \frac{h^2}{12} f_i^{\text{iv}} + O(h^4),$$

and

(1.4) $$\frac{s'''(x_i^+) - s'''(x_i^-)}{h} = f_i^{\text{iv}} + O(h^4).$$

Since $s''(x)$ is a linear function between the partition points, (1.4) is equivalent to

(1.5) $$f_i^{\text{iv}} = \frac{s_{i+1}'' - 2s_i'' + s_{i-1}''}{h^2} + O(h^4).$$

Combining (1.3) and (1.5) gives

(1.6) $$f_i'' = \frac{s_{i+1}'' + 10s_i'' + s_{i-1}''}{12} + O(h^4).$$

The formal expressions (1.2), (1.6) and (1.5) give approximations to f_i', f_i'' and f_i^{iv} of $O(h^4)$ accuracy in terms of simple linear functionals on an interpolating cubic spline s.

With the exception of (1.2) there seems to have been no published exploration of the existence of higher order estimates of this general type. That $\max |f_i' - s_i'| = O(h^4)$ for $f \in C^5[a, b]$ and uniform partitions with the end conditions

(1.7) $$s'(a) = f'(a), \qquad s'(b) = f'(b)$$

has been established by Birkhoff and deBoor [2] and Hall [6]. Kershaw [10] has shown that for $f \in C^5[a, b]$ and nonuniform partitions with either the end conditions (1.7) or for periodic f the end conditions

(1.8) $$s'(a) = s'(b), \qquad s''(a) = s''(b),$$

that the error bound

(1.9) $$\max |f_i' - s_i'| \leqq \tfrac{1}{24}\bar{h}^2 \max_j |h_j - h_{j-1}| \, \|f^{(4)}\|_\infty + \tfrac{1}{60}\bar{h}^4 \|f^{(5)}\|_\infty$$

is valid. Using an earlier result [8], Kershaw [10] has also shown that for the "inferior" end conditions

(1.10) $$s''(a) = f''(a), \qquad s''(b) = f''(b),$$

or

(1.11) $$s''(a) = 0, \qquad s''(b) = 0$$

local results similar to (1.9) hold in intervals bounded away from the end points, in particular with uniform partitions $|f_i' - s_i'| = O(h^4)$ at a number of interior knots.

The purpose of this study is to determine conditions which insure the validity of a number of error bounds which include (1.2)–(1.6). Our approach is to show that the local validity of the expression (1.3) or its higher order generalization

$$(1.3)' \qquad s_i'' = f_i'' - \frac{h^2}{12} f_i^{\mathrm{iv}} + \frac{h^4}{360} f_i^{\mathrm{vi}} + O(h^6)$$

for a family of functions $\mathfrak{F}$ over a fixed subinterval where the partitions are uniform implies that a whole class of other error bounds are also satisfied inside the subinterval. These include the ones mentioned above and in addition

$$s'(x_i + 0.5h) = f'(x_i + 0.5h) + O(h^4),$$
$$s''(z_i) = f''(z_i) + O(h^3),$$

where z_i is either of the roots of the second degree Legendre polynomial $3z^2 - 1 = 0$, normalized to $[x_i, x_{i+1}]$, and

$$s'''(x_i + 0.5h) = f'''(x_i + 0.5h) + O(h^2).$$

This program is carried out in § 2. Also a lemma is given which shows that if $|f_i'' - s_i''| \leqq Kh^2$ over a subinterval $[\alpha, \beta]$ in which f is in C^4 then

$$\|D^j(f(x) - s(x))\|_{L^\infty[x_{i_0}, x_{i_1}]} \leqq Kh^{4-j}, \qquad 0 \leqq j \leqq 2,$$

where $x_{i_0}, x_{i_1} \in \pi$, $\alpha \leqq x_{i_0} < x_{i_1} \leqq \beta$, showing that the usual spline error bounds are included in this theory.

In § 3 we develop global results for uniform partitions with $f \in C^n[a, b]$ for some n, $4 \leqq n \leqq 8$, by giving conditions when variations of (1.3) or (1.3)′ are globably valid. As might be expected from Kershaw's results, the choice of end conditions plays a critical role in the quality of these error bounds if the function is sufficiently smooth. A number of end conditions are analyzed which come progressively closer to the full order of (1.3)′, one of them achieving it.

Recently Kammerer and Reddien [7] have developed local error bounds for Sp $(\pi, 3)$-interpolates of a function which is only locally in C^4, which are similar to the usual global $L^\infty[a, b]$ bounds. In § 4 we extend this approach to give general conditions under which high order local smoothness of f over a locally uniform partition implies that bounds of the type (1.3) and (1.3)′ hold in such local regions.

The applications of these results would include: (i) a significantly increased choice of high quality computationally feasible end conditions with a means for judging their utility; (ii) knowledge of where the evaluation of derivatives of spline interpolates will be of higher order; (iii) the means to compute $O(h^4)$ order estimates of f_i', f_i'', f_i''' and f_i^{iv} from just the values of an interpolating cubic spline and its derivatives at the knots, for sufficiently smooth f; (iv) the suggestion of new modified collocation schemes of high order for nonlinear boundary value problems.

The following cubic spline identities will be needed [1]:

$$(1.12) \qquad \begin{aligned} &\frac{h_i}{h_i + h_{i+1}} s_{i-1}'' + 2s_i'' + \frac{h_{i+1}}{h_i + h_{i+1}} s_{i+1}'' \\ &= \frac{6}{h_i + h_{i+1}} \left[\frac{s_{i+1} - s_i}{h_{i+1}} - \frac{s_i - s_{i-1}}{h_i} \right], \qquad 1 \leqq i \leqq m - 1; \end{aligned}$$

$$(1.13) \quad s_i' = \frac{h_i}{6} s_{i-1}'' + \frac{h_i}{3} s_i'' + \frac{s_i - s_{i-1}}{h_i}, \qquad 1 \leqq i \leqq m;$$

$$(1.13)' \quad s_{i-1}' = -\frac{h_i}{3} s_{i-1}'' - \frac{h_i}{6} s_i'' + \frac{s_i - s_{i-1}}{h_i}, \qquad 1 \leqq i \leqq m;$$

$$(1.14) \quad s'(x_i + 0.5h_{i+1}) = \frac{s_{i+1} - s_i}{h_{i+1}} - \frac{h_{i+1}}{24}(s_{i+1}'' - s_i''), \qquad 0 \leqq i \leqq m-1;$$

$$(1.15) \quad s(x_i + 0.5h) = \frac{s_i + s_{i+1}}{2} - \frac{h^2}{16}(s_i'' + s_{i+1}''), \qquad 0 \leqq i \leqq m-1;$$

$$(1.15)' \quad s(x_i + 0.25h) = \frac{3s_i + s_{i+1}}{4} - \frac{h^2}{128}(7s_i'' + 5s_{i+1}''), \qquad 0 \leqq i \leqq m-1;$$

$$(1.15)'' \quad s(x_i + 0.75h) = \frac{s_i + 3s_{i+1}}{4} - \frac{h^2}{128}(5s_i'' + 7s_{i+1}''), \qquad 0 \leqq i \leqq m-1;$$

$$(1.16) \quad \frac{s_{i+1}' - s_i'}{h} = \frac{s_{i+1}'' + s_i''}{2}, \qquad 0 \leqq i \leqq m-1.$$

For uniform partitions (1.12) becomes

$$(1.12)' \quad s_{i-1}'' + 4s_i'' + s_{i+1}'' = 6h^{-2}[s_{i-1} - 2s_i + s_{i+1}], \qquad 1 \leqq i \leqq m-1.$$

We denote any generic constant which is independent of the maximum partition spacing by the general symbol K. It may take on different values in any two usages.

2. Locally induced error bounds.

THEOREM 1. *Let* $[a', b']$ *and* $[\alpha, \beta]$ *be subintervals of* $[a, b]$ *related by* $a \leqq a' \leqq \alpha < \beta \leqq b' \leqq b$. *Let* $\{\pi_m\}$ *be a sequence of partitions of* $[a, b]$ *such that each* π_m *restricted to* $[a', b']$ *is uniform. Denote the first and last knots of* π_m *in* $[\alpha, \beta]$ *by* x_{i_0} *and* x_{i_1}, *and the partition size over* $[a', b']$ *by* h. *Suppose* $\mathfrak{F}$ *is a family of functions over* $[a, b]$ *and* (E) *is a pair of interpolating spline end conditions such that for each* $f \in \mathfrak{F}$ *and* $\pi_m \in \{\pi_m\}$ *there corresponds a unique* $\mathrm{Sp}(\pi_m, 3)$*-interpolate,* s, *satisfying the end conditions* (E), *and there exist constants* $\bar{r} \in \{4, 5, 6\}$, $K_4, K_5, \cdots, K_{\bar{r}}$ *dependent on* f *but independent of* $\pi_m \in \{\pi_m\}$ *such that if* $f \in C^r[a', b'] \cap \mathfrak{F}$ *where* $4 \leqq r \leqq \bar{r}$, r *an integer, then*

$$(2.1) \qquad \max_{i_0 \leqq i \leqq i_1} \left| f_i'' - \frac{h^2}{12} f_i^{\mathrm{iv}} - s_i'' \right| \leqq K_r h^{r-2}.$$

Then if $f \in C^r[a', b'] \cap \mathfrak{F}$ *with* $4 \leqq r \leqq \min(5, \bar{r})$, *there exist constants* $\{K_l^r\}_{l=1}^5$ *independent of* $\pi_m \in \{\pi_m\}$ *such that*

(i) $\max\limits_{i_0 \leqq i \leqq i_1} |f_i' - s_i'| \leqq K_1^r h^{r-1}$,

(ii) $\max\limits_{i_0 \leqq i \leqq i_1 - 1} |f'(x_i + 0.5h) - s'(x_i + 0.5h)| \leqq K_2^r h^{r-1}$,

(iii) $\max\limits_{i_0 \leqq i \leqq i_1 - 1} |f''(x_i + \lambda h) - s''(x_i + \lambda h)| \leqq K_3^r h^{r-2}$,

where $\lambda = (3 \pm \sqrt{3})/6 \doteq 0.211$ *or* 0.789,

(iv) $\max\limits_{i_0 \leqq i \leqq i_1 - 1} |f'''(x_i + 0.5h) - s'''(x_i + 0.5h)| \leqq K_4^r h^{r-3}$,

$$\text{(v)} \quad \max_{i_0+1 \leqq i \leqq i_1-1} \left| f_i''' - \frac{1}{2h}(s_{i+1}'' - s_{i-1}'') \right| \leqq K_5^r h^{r-3}.$$

In addition if $f \in C^r[a', b'] \cap \mathfrak{F}$ with $4 \leqq r \leqq \bar{r}$, there exist constants $\{K_l^r\}_{l=6}^8$ independent of $\pi_m \in \{\pi_m\}$ such that

$$\text{(vi)} \quad \max_{i_0+1 \leqq i \leqq i_1-1} \left| f_i'' - \frac{1}{12}(s_{i+1}'' + 10s_i'' + s_{i-1}'') \right| \leqq K_6^r h^{r-2},$$

$$\text{(vii)} \quad \max_{i_0+1 \leqq i \leqq i_1-1} \left| f_i^{\text{iv}} - \frac{1}{h^2}(s_{i+1}'' - 2s_i'' + s_{i-1}'') \right| \leqq K_7^r h^{r-4},$$

$$\text{(viii)} \quad \left| f_{i_0}'' - \frac{1}{12}(14s_{i_0}'' - 5s_{i_0+1}'' + 4s_{i_0+2}'' - s_{i_0+3}'') \right| \leqq K_8^r h^{r-2},$$

$$\text{(viii)}' \quad \left| f_{i_1}'' - \frac{1}{12}(14s_{i_1}'' - 5s_{i_1-1}'' + 4s_{i_1-2}'' - s_{i_1-3}'') \right| \leqq K_8^r h^{r-2}.$$

Finally, if there exist constants $\hat{K}_r$ such that $K_r \leqq \hat{K}_r \|f^{(r)}\|_{L^\infty[a',b']}$ for all $f \in C^r[a', b'] \cap \mathfrak{F}$, $4 \leqq r \leqq \bar{r}$, then there are constants $\{\hat{K}_l^r\}_{l=1}^8$ independent of f such that

$$K_l^r \leqq \hat{K}_l^r \|f^{(r)}\|_{L^\infty[a',b']}, \qquad 1 \leqq l \leqq 8.$$

Proof. The final claim may be seen by inspection of the individual proofs. We begin with bound (i). Let $i_0 + 1 \leqq i \leqq i_1$. Recall for comparison the spline identity (1.13). By a direct expansion about x_i using Taylor's theorem, if $f \in C^n[a', b']$, $n = 4$ or 5, then

$$f_i' = \frac{h}{6}\left(f_{i-1}'' - \frac{h^2}{12} f_{i-1}^{\text{iv}}\right) + \frac{h}{3}\left(f_i'' - \frac{h^2}{12} f_i^{\text{iv}}\right) + \frac{f_i - f_{i-1}}{h} + R_i^n,$$

where $|R_i^4| \leqq 6^{-1} h^3 \|f^{\text{iv}}\|_{L^\infty[x_{i-1},x_i]}$ and $|R_i^5| \leqq 20^{-1} h^4 \|f^{\text{v}}\|_{L^\infty[x_{i-1},x_i]}$. Subtracting (1.13) from this expression and recalling that s interpolates f over π_m gives

$$f_i' - s_i' = \frac{h}{6}\left(f_{i-1}'' - \frac{h^2}{12} f_{i-1}^{\text{iv}} - s_{i-1}''\right) + \frac{h}{3}\left(f_i'' - \frac{h^2}{12} f_i^{\text{iv}} - s_i''\right) + R_i^n.$$

So for $4 \leqq r \leqq 5$ and by the use of (2.1),

$$|f_i' - s_i'| \leqq \frac{K_r}{2} h^{r-1} + |R_i^r|.$$

A similar relation for $i = i_0$ can be established using (1.13)$'$. Thus (i) is valid with $K_1^4 = 0.5K_4 + 6^{-1}\|f^{\text{iv}}\|_{L^\infty[a',b']}$ and $K_1^5 = 0.5K_5 + 20^{-1}\|f^{\text{v}}\|_{L^\infty[a',b']}$. Note that the use of Peano's theorem [5, p. 69] would have given a sharper bound on R_i^4 and R_i^5, but as our interest is in the asymptotic rates for a wide variety of expressions, we prefer the simplicity of Taylor's theorem.

By a routine calculation using Taylor's expansion, for $f \in C^n[a', b']$, $n = 4$ or 5 and $i_0 \leqq i \leqq i_1 - 1$,

$$\text{(2.2)} \quad \begin{aligned} f'(x_i + 0.5h) = h^{-1}(f_{i+1} - f_i) - (h/24)[f_{i+1}'' - (h^2/12)f_{i+1}^{\text{iv}} \\ - (f_i'' - (h^2/12)f_i^{\text{iv}})] + R_i^n, \end{aligned}$$

where $|R_i^n| \leqq \bar{K}_2^n h^{n-1} \|f^{(n)}\|_{L^\infty[a',b']}$. Hence (ii) follows by subtracting (1.14) from (2.2) and using (2.1) with $K_2^r = 12^{-1}K_r + \bar{K}_2^r \|f^{(r)}\|_{L^\infty[a',b']}$, $4 \leqq r \leqq 5$.

For bound (iii), if $\bar{r} = 4$ the proof is straightforward. Suppose $\bar{r} = 5$ and $f \in C^5[a', b']$. Let $i_0 \leqq i \leqq i_1 - 1$. Since s'' is linear between x_i and x_{i+1}, for any $\lambda \in [0, 1]$,

$$(2.3) \qquad (1 - \lambda)s_i'' + \lambda s_{i+1}'' = s''(x_i + \lambda h).$$

By a Taylor expansion about $\bar{x} = x_i + \lambda h$,

$$(2.4) \qquad \begin{aligned} &(1 - \lambda)(f_i'' - (h^2/12)f_i^{\text{iv}}) + \lambda(f_{i+1}'' - (h^2/12)f_{i+1}^{\text{iv}}) \\ &\quad = f''(\bar{x}) + \tfrac{1}{12}(6\lambda(1 - \lambda) - 1)h^2 f^{\text{iv}}(\bar{x}) + R_i^5, \end{aligned}$$

where $|R_i^5| \leqq Kh^3 \|f^{(5)}\|_{L^\infty[a',b']}$. But $6\lambda(1 - \lambda) - 1 = 0$ if and only if $\lambda = (3 \pm \sqrt{3})/6$. Assigning either of these values to λ, and subtracting (2.3) from (2.4) it follows from (2.1) that

$$|f''(\bar{x}) - s''(\bar{x})| \leqq h^3(K_5 + K\|f^{(5)}\|_{L^\infty[a',b']}) \equiv h^3 K_3^5.$$

The remaining error bounds follow by similar arguments involving the use of appropriate spline identities and Taylor expansions. For example bound (iv) uses $s'''(x_i + 0.5h) = h^{-1}(s_{i+1}'' - s_i'')$ while bound (vi) may be derived from $f_i'' = 12^{-1}[f_{i+1}'' + 10f_i'' + f_{i-1}'' - (h^2/12)(f_{i+1}^{\text{iv}} + 10f_i^{\text{iv}} + f_{i-1}^{\text{iv}})] + R_i^n$, where $|R_i^n| \leqq \bar{K}_6^n h^{n-2} \|f^{(n)}\|_{L^\infty[a',b']}$ when $f \in C^n[a', b']$, $n = 4$, 5 or 6.

The formal expression (1.5) of Curtis and Powell suggests that the error bound (vii) of Theorem 1 might be strengthened for smoother f. In another direction, a better approximation for f_i''' is given.

THEOREM 2. *Suppose $\{\pi_m\}$, $\mathfrak{F}$ and* (E) *are as in Theorem* 1 *and in addition there exist constants $\hat{r} \in \{6, 7, 8\}$, $L_6, L_7, \cdots, L_{\hat{r}}$ independent of $\pi_m \in \{\pi_m\}$ such that if $f \in C^r[a', b'] \cap \mathfrak{F}$ where $6 \leqq r \leqq \hat{r}$, then*

$$\max_{i_0 \leqq i \leqq i_1} \left| f_i'' - \frac{h^2}{12} f_i^{\text{iv}} + \frac{h^4}{360} f_i^{\text{vi}} - s_i'' \right| \leqq L_r h^{r-2}.$$

Then there exist constants C_1^r and C_2^r independent of $\pi_m \in \{\pi_m\}$ such that for $f \in C^r[a', b'] \cap \mathfrak{F}$ with $6 \leqq r \leqq \hat{r}$,

$$\text{(i)} \quad \max_{i_0+1 \leqq i \leqq i_1-1} \left| f_i^{\text{iv}} - \frac{1}{h^2}(s_{i+1}'' - 2s_i'' + s_{i-1}'') \right| \leqq C_1^r h^{r-4}.$$

If $6 \leqq r \leqq \min(7, \hat{r})$,

$$\text{(ii)} \quad \max_{i_0+2 \leqq i \leqq i_1-2} \left| f_i''' - \frac{1}{24h}(-s_{i+2}'' + 14s_{i+1}'' - 14s_{i-1}'' + s_{i-2}'') \right| \leqq C_2^r h^{r-3}.$$

Moreover, if there exist constants $\hat{L}_r$ such that $L_r \leqq \hat{L}_r \|f^{(r)}\|_{L^\infty[a',b']}$ for all $f \in C^r[a', b'] \cap \mathfrak{F}$, $6 \leqq r \leqq \hat{r}$, then there exist constants $\hat{C}_1^r$ and $\hat{C}_2^r$ independent of f such that

$$(2.5) \qquad C_1^r \leqq \hat{C}_1^r \|f^{(r)}\|_{L^\infty[a',b']} \quad \text{and} \quad C_2^r \leqq \hat{C}_2^r \|f^{(r)}\|_{L^\infty[a',b']}.$$

The proof is similar to that of Theorem 1 and is left to the reader.

We end this section with a lemma which demonstrates that the bound (2.1) implies that the optimum spline error bounds for locally C^4-smooth functions are valid in this setting. The locally uniform partition requirement on $\{\pi_m\}$ is dropped.

LEMMA 1. *Let* $a \leqq a' \leqq \alpha < \beta \leqq b' \leqq b$, $\{\pi_m\}$ *be a sequence of partitions of* $[a, b]$ *with the first and last knots of* π_m *in* $[\alpha, \beta]$ *being* x_{i_0} *and* x_{i_1}, $\mathfrak{F}$ *be a family of functions over* $[a, b]$ *and* (E) *be a pair of interpolating spline end conditions. Suppose for each* $f \in \mathfrak{F}$ *and* $\pi_m \in \{\pi_m\}$ *there is a unique* Sp $(\pi_m, 3)$*-interpolate s satisfying* (E), *and a constant* K_4 *independent of* $\pi_m \in \{\pi_m\}$ *such that if* $f \in C^4[a', b'] \cap \mathfrak{F}$ *then*

$$(2.6) \qquad \max_{i_0 \leqq i \leqq i_1} |f_i'' - s_i''| \leqq K_4 \bar{h}^2.$$

Then there exist constants $\{M_j\}_{j=0}^2$ *independent of* $\pi_m \in \{\pi_m\}$ *such that if* $f \in C^4[a, b] \cap \mathfrak{F}$,

$$(2.7) \qquad \|D^j(f(x) - s(x))\|_{L^\infty[x_{i_0}, x_{i_1}]} \leqq M_j \bar{h}^{4-j}, \qquad 0 \leqq j \leqq 2.$$

If $K_4 \leqq \hat{K}_4 \|f^{(4)}\|_{L^\infty[a', b']}$ *for all* $f \in C^4[a', b'] \cap \mathfrak{F}$ *then there exist* $\hat{M}_j$ *with* $M_j \leqq \hat{M}_j \|f^{(4)}\|_{L^\infty[a', b']}$.

Proof. By Taylor's theorem if $f \in C^4[a', b']$ and $0 \leqq \lambda \leqq 1$,

$$(2.8) \qquad (1 - \lambda) f_i'' + \lambda f_{i+1}'' = f''(x_i + \lambda h_{i+1}) + R_i^4,$$

where $|R_i^4| \leqq (1/8)\bar{h}^2 \|f^{(4)}\|_{L^\infty[a', b']}$. Subtracting (2.3) from (2.8) and using (2.6) establishes (2.7) for $j = 2$ with $M_2 = K_4 + (1/8)\|f^{(4)}\|_{L^\infty[a', b']}$. Since $f_i - s_i = 0$, $i_0 \leqq i \leqq i_1$, and $f - s \in C^2[x_{i_0}, x_{i_1}]$, Lagrange's theorem gives

$$\|f - s\|_{L^\infty[x_{i_0}, x_{i_1}]} \leqq (1/8)\bar{h}^2 \|D^2(f - s)\|_{L^\infty[x_{i_0}, x_{i_1}]},$$

so (2.7) is valid for $j = 0$ with $M_0 = M_2/8$. The $j = 1$ case follows by Lemma 1 of Kershaw [9].

3. End conditions yielding high order error bounds for uniform partitions. In this section we shall identify some end conditions which lead to the results of Theorems 1 and 2 for sufficiently smooth functions. All partitions π_m of $[a, b]$ in this section are assumed to be uniform with partition norm h. We begin with three lemmas.

LEMMA 2. *Let* $f \in C^n[a, b]$ *for some* n, $4 \leqq n \leqq 6$. *Let* π_m *be a uniform partition of* $[a, b]$ *and s some* Sp $(\pi_m, 3)$*-interpolate of* f. *Denote the expression* $f_i'' - (h^2/12) f_i^{\mathrm{iv}} - s_i''$ *by* e_i, $0 \leqq i \leqq m$. *Then*

$$(3.1) \qquad e_{i-1} + 4e_i + e_{i+1} = R_i^n, \qquad 1 \leqq i \leqq m - 1,$$

where $|R_i^n| \leqq K h^{n-2} \|f^{(n)}\|_{L^\infty[a,b]}$ *and* K *is independent of* f *and* π_m.

Proof. By Taylor's theorem, if $4 \leqq n \leqq 6$,

$$(3.2) \qquad \begin{aligned} &f_{i-1}'' - \frac{h^2}{12} f_{i-1}^{\mathrm{iv}} + 4\left(f_i'' - \frac{h^2}{12} f_i^{\mathrm{iv}}\right) + f_{i+1}'' - \frac{h^2}{12} f_{i+1}^{\mathrm{iv}} \\ &\quad = 6h^{-2}(f_{i-1} - 2f_i + f_{i+1}) + R_i^n, \qquad 1 \leqq i \leqq m - 1, \end{aligned}$$

where $|R_i^n| \leqq K h^{n-2} \|f^{(n)}\|_{L^\infty[a,b]}$. The result follows by subtracting (1.12)′ from (3.2), where s is an arbitrary Sp $(\pi_m, 3)$-interpolate of f. If desired a sharp K could be determined explicitly as a function of n by use of Peano's theorem. The next result follows by a similar argument.

LEMMA 3. *Let $f \in C^n[a, b]$ for some n, $6 \leqq n \leqq 8$. Let π_m be a uniform partition of $[a, b]$ and s some* $\mathrm{Sp}(\pi_m, 3)$*-interpolate of f. Denote the expression*

$$f_i'' - (h^2/12)f_i^{\mathrm{iv}} + (h^4/360)f_i^{\mathrm{vi}} - s_i''$$

by $\bar{e}_i$, $0 \leqq i \leqq m$. Then

$$\bar{e}_{i-1} + 4\bar{e}_i + \bar{e}_{i+1} = \bar{R}_i^n, \qquad 1 \leqq i \leqq m-1,$$

where $|R_i^n| \leqq Kh^{n-2}\|f^{(n)}\|_{L^\infty[a,b]}$.

LEMMA 4. *If $A = \{a_{ij}\}$ is an $m \times m$ matrix and $a_{ii} \geqq \sum_{j=1,j\neq i}^m |a_{ij}| + \delta$, $1 \leqq i \leqq m$, where $\delta > 0$, then $\|A^{-1}\|_\infty \leqq \delta^{-1}$.*

Proof. Let $y \in R^m$ and $x = A^{-1}y$. Then $Ax = y$ and if $\|x\|_\infty = |x_i|$

$$|a_{ii}x_i| = \left| y_i - \sum_{j=1,j\neq i}^m a_{ij}x_j \right| \leqq \|y\|_\infty + \sum_{j=1,j\neq i}^m |a_{ij}|\,\|x\|_\infty,$$

so $\delta\|x\|_\infty \leqq \|y\|_\infty$ and $\|A^{-1}y\|_\infty \leqq \delta^{-1}\|y\|_\infty$.

We first consider the model problem: periodic end conditions for periodic functions.

THEOREM 3. *Let $f \in C^n(-\infty, \infty) \cap C_p[a, b]$ for some n, $4 \leqq n \leqq 8$, and let s be the* $\mathrm{Sp}(\pi_m, 3)$*-interpolate of f satisfying the periodic end conditions* (1.8). *If $4 \leqq n \leqq 6$, then*

$$\max_{0\leqq i\leqq m}\left| f_i'' - \frac{h^2}{12}f_i^{\mathrm{iv}} - s_i'' \right| \leqq K_r h^{r-2}\|f^{(r)}\|_{L^\infty[a,b]}, \qquad 4 \leqq r \leqq n. \tag{3.3}$$

If $6 \leqq n \leqq 8$, then

$$\max_{0\leqq i\leqq m}\left| f_i'' - \frac{h^2}{12}f_i^{\mathrm{iv}} + \frac{h^4}{360}f_i^{\mathrm{vi}} - s_i'' \right| \leqq K_r h^{r-2}\|f^{(r)}\|_{L^\infty[a,b]}, \qquad 6 \leqq r \leqq n. \tag{3.4}$$

The constants K_r are independent of f and π_m.

Proof. Suppose $4 \leqq n \leqq 6$. Extending s periodically to $(-\infty, \infty)$ we see from Lemma 2 that besides (3.1), we have $e_{m-1} + 4e_0 + e_1 = R_0^n$ where $|R_0^n| \leqq Kh^{n-2}\|f^{(n)}\|_\infty$ and $e_m = e_0$, giving the matrix system

$$\begin{bmatrix} 4 & 1 & 0 & 0 & \cdots & & & 1 \\ 1 & 4 & 1 & 0 & \cdots & & & 0 \\ 0 & 1 & 4 & 1 & \cdots & & & 0 \\ & & & \cdots & & & & \\ 0 & & \cdots & & & 1 & 4 & 1 \\ 1 & & \cdots & & & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} e_0 \\ e_1 \\ \vdots \\ \\ e_{m-2} \\ e_{m-1} \end{bmatrix} = \begin{bmatrix} R_0^n \\ R_1^n \\ \vdots \\ \\ \\ R_{m-1}^n \end{bmatrix}.$$

By Lemma 4, $\|e_i\|_\infty \leqq 0.5\|R_i^n\|_\infty \leqq 0.5Kh^{n-2}\|f^{(n)}\|_\infty$, giving (3.3). A similar application of Lemma 3 gives (3.4).

There is a "naturalness" about the periodic end conditions (1.8) that has no ready generalization to nonperiodic functions giving error bounds of the quality of Theorem 3. The most frequently appearing end conditions in the literature other than (1.8) are (1.7), (1.10) and (1.11), but even the best of these, (1.7), fails to give an $O(h^4)$ bound on (3.3) for $f \in C^6[a, b]$. This leads us to consider a number of new (as far as a literature search has been able to determine) end conditions, some of which seem to offer distinct advantages over the above ones. We also include several other choices that have been previously considered. (See Table 1.)

TABLE 1

Some cubic spline end conditions

Ref.	Label	Left end condition	$O(h^n)$
[12]	H1	$s'_0 = (6h)^{-1}(-11s_0 + 18s_1 - 9s_2 + 2s_3)$	4
[3]	H2	$s''_0 - 2s''_1 + s''_2 \equiv \Delta^2 s''_0 = 0$	4
*	H3	$\Delta^3 s''_0 = 0$	5
*	H4	$\Delta^4 s''_0 = 0$	6
[3]	F1	$s(a + 0.5h) = f(a + 0.5h)$	4
[4]	F2	$\Delta s(a + 0.5h) = \Delta f(a + 0.5h)$	5
*	F3	$8s(a + 0.25h) - 9s(a + 0.5h) + 8s(a + 0.75h)$ $= 8f(a + 0.25h) - 9f(a + 0.5h) + 8f(a + 0.75h)$	6
[1]	D1	$s'_0 = f'_0$	5
*	D2	$\Delta s'_0 = \Delta f'_0$	6
[1]	DD1	$s''_0 = f''_0$	4
*	DD2	$s''_0 + 10s''_1 + s''_2 = 12f''_1$	6
*	DD3	$14s''_0 - 5s''_1 + 4s''_2 - s''_3 = 12f''_0$	6
*	DD4	$12s''_1 = 14f''_1 - f''_0 - f''_2$	6
*	DD5	$7s''_0 + 46s''_1 + 7s''_2 = 2f''_0 + 56f''_1 + 2f''_2$	8

These end conditions are classified as being homogeneous (H), dealing with function values (F), first derivative values (D) or second derivative values (DD). For brevity the conditions are stated for the left endpoint $x_0 = a$ only, and a similar condition is assumed to hold at $x_m = b$. The left-hand column either references an early investigator of the end condition or indicates it is new by marking it with an "*". The right column gives the order of $\max |f''_i - (h^2/12)f^{\text{iv}}_i + (h^4/360)f^{\text{vi}}_i - s''_i|$ for $f \in C^8[a, b]$ which is to be determined shortly, and thus gives an indicator of the accuracy of the end condition. Except for F3 and D2 these new end conditions should only be considered without modification when the first and last few partition points are uniformly spaced.

The remainder of this section concerns the rate at which

$$\max |f''_i - (h^2/12)f^{\text{iv}}_i - s''_i| \quad \text{or} \quad \max |f''_i - (h^2/12)f^{\text{iv}}_i + (h^4/360)f^{\text{vi}}_i - s''_i|$$

goes to zero for sufficiently smooth f where s is the Sp $(\pi_m, 3)$-interpolate of f for a set of end conditions from Table 1. Any generic constants K used are independent of both the function being interpolated and the mesh norm.

THEOREM 4. *Let s be the* Sp $(\pi_m, 3)$*-interpolate of f with end conditions* H1, H2, F1 or DD1. *If $f \in C^4[a, b]$ then*

$$\max_{0 \leqq i \leqq m} \left| f''_i - \frac{h^2}{12} f^{\text{iv}}_i - s''_i \right| \leqq K h^2 \| f^{(4)} \|_{L^\infty[a,b]}.$$

Proof. Clearly it suffices to show that $\max |f''_i - s''_i| \leqq Kh^2 \|f^{(4)}\|_\infty$. But for the end conditions H1 and DD1 this has been established by Swartz and Varga [12] and for H2 and F1 by de Boor [3].

THEOREM 5. *Let s be the* Sp$(\pi_m, 3)$*-interpolate of f with end conditions* H3, F2 *or* D1. *If* $f \in C^n[a,b]$ *for* $n = 4$ *or* 5 *then*

$$\max_{0 \leqq i \leqq m} \left| f_i'' - \frac{h^2}{12} f_i^{\text{iv}} - s_i'' \right| \leqq Kh^{n-2} \|f^{(n)}\|_{L^\infty[a,b]}.$$

Proof. For each of the above end conditions in turn, let $e_i = f_i'' - (h^2/12)f_i^{\text{iv}} - s_i''$, $0 \leqq i \leqq m$. Consider first H3. By an application of Taylor's theorem, if $f \in C^n[a,b]$ with $n = 4$ or 5 then $\Delta^3[f_0'' - (h^2/12)f_0^{\text{iv}}] = R_0^n$ with $|R_0^n| \leqq Kh^{n-2}\|f^{(n)}\|_\infty$. Subtracting the left end condition H3 gives $\Delta^3 e_0 = R_0^n$. A similar argument at b gives $\Delta^3 e_{m-3} = R_m^n$, where $|R_m^n| \leqq Kh^{n-2}\|f^{(n)}\|_\infty$. These two equations together with the results of Lemma 2 give the matrix system

$$\begin{bmatrix} 1 & -3 & 3 & -1 & & \cdots & & 0 \\ 1 & 4 & 1 & 0 & & \cdots & & 0 \\ 0 & 1 & 4 & 1 & & \cdots & & 0 \\ & & & \cdots & & & & \\ 0 & & \cdots & & 1 & 4 & 1 & 0 \\ 0 & & \cdots & & 0 & 1 & 4 & 1 \\ 0 & & \cdots & & 1 & -3 & 3 & -1 \end{bmatrix} \begin{bmatrix} e_0 \\ e_1 \\ e_2 \\ \vdots \\ e_{m-2} \\ e_{m-1} \\ e_m \end{bmatrix} = \begin{bmatrix} R_0^n \\ R_1^n \\ R_2^n \\ \vdots \\ R_{m-2}^n \\ R_{m-1}^n \\ R_m^n \end{bmatrix} \tag{3.5}$$

Subtracting row 1 from row 2 and adding row $m + 1$ to row m in the above system gives a uniformly diagonally dominant $(m - 1) \times (m - 1)$ reduced system $\bar{A}\bar{E} = \bar{R}$. Hence by Lemma 4, with $\delta = 2$, $\|\bar{E}\|_\infty \leqq 0.5\|\bar{R}\|_\infty \leqq Kh^{n-2}\|f^{(n)}\|_\infty$. Since $|e_0| \leqq 5\|\bar{E}\|_\infty + |R_1^n|$ by (3.1) with a similar bound for $|e_m|$ the result for H3 follows.

Now consider F2. By a double use of the spline identity (1.15) it follows that

$$s(a + 1.5h) - s(a + 0.5h) = \frac{s_2 - s_0}{2} - \frac{h^2}{16}(s_2'' - s_0''). \tag{3.6}$$

By Taylor's theorem if $f \in C^n[a,b]$ with $n = 4$ or 5,

$$\begin{aligned} &f(a + 1.5h) - f(a + 0.5h) \\ &= \frac{f_2 - f_0}{2} - \frac{h^2}{16}\left(f_2'' - \frac{h^2}{12}f_2^{\text{iv}} - \left(f_0'' - \frac{h^2}{12}f_0^{\text{iv}}\right)\right) + \hat{R}_0^n, \end{aligned} \tag{3.7}$$

where $|\hat{R}_0^n| \leqq Kh^n\|f^{(n)}\|_\infty$. A subtraction of (3.6) from (3.7), use of the left end condition F2 and the interpolation property of s followed by a division by $h^2/16$ gives $e_0 - e_2 = R_0^n$, where $|R_0^n| \leqq Kh^{n-2}\|f^{(n)}\|_\infty$. The same bound holds for $|e_{m-2} - e_m|$ so use of Lemma 2 gives a system like (3.5) with a modified first and last row. Again a matrix reduction allows Lemma 4 to be applied giving the result. The result for D1 follows easily from (1.13)′ without requiring matrix reduction.

It is interesting to compare the three homogeneous end conditions H1, H2 and H3. If the spline identities (1.13)′ and (1.12)′ are applied to H1 it may be seen that H1 is a linear combination of H2 and H3 in the proportion 9 parts of H2 to −2 parts of H3. Since H1 is as computationally involved as H3 but is contaminated by the lower order error of H2, it would seem that the use of either H2 or H3 should be preferred.

One example of such an application would be the use of collocation techniques over $\mathrm{Sp}(\pi_m, n+3)$ to approximate the solution of the nonlinear two-point boundary value problem

$$D^n u = f(x, u, \cdots, u^{(n-1)}), \qquad a < x < b,$$

with boundary conditions

$$\sum_{j=0}^{n-1} [a_{ij}u^{(j)}(a) + b_{ij}u^{(j)}(b)] = 0, \qquad 1 \leqq i \leqq n,$$

where $\mathrm{Sp}_0(\pi_m, n+3)$ is the space of $C^{n+2}[a, b]$ polynomial splines of degree $n+3$ in each subinterval of π_m which satisfy the boundary conditions. In this setting the basic collocation equations are simply

$$s_i^{(n)} = f(x_i, s_i, \cdots, s_i^{(n-1)}), \qquad 0 \leqq i \leqq m, \tag{3.8}$$

for $s \in \mathrm{Sp}_0(\pi_m, n+3)$. To have an algebraically well-defined problem, two additional conditions are required. Since the error analysis of collocation methods using (3.8) involves the study of uniformly bounded projectors taking any $g \in C[a, b]$ into some $\mathrm{Sp}(\pi_m, 3)$-interpolate of g (see Lucas and Reddien [11]) any of the end conditions of Table 1 which are feasible in this setting could be considered. This would evidently include at most H1 – F3. In [11] the projector corresponding to H1, P_k, was studied, and it was shown that if $u \in C^{n+4}[a, b]$, then

$$\|D^j(u - s)\|_{L^\infty[a,b]} \leqq K_j h^4 \|u^{(n+4)}\|_{L^\infty[a,b]}, \qquad 0 \leqq j \leqq n. \tag{3.9}$$

Apparently H2 would give similar accuracy but with a simpler band matrix, and H3 would give smaller constants in (3.9) and hence superior accuracy with a matrix of the same band type as in the H1 method. Some numerical experimentation with H3 in place of H1 has confirmed this expectation for a specific second order problem with an order of magnitude improvement for the error in $u'' - s''$. First derivative error was improved somewhat and the error in function values was not changed appreciably. This seems consistent with the published results in Tables 2.1 and 3.1 of [11] and the remark following them concerning the P_k projector.

THEOREM 6. *Let s be the* $\mathrm{Sp}(\pi_m, 3)$*-interpolate of f with end conditions* H4, F3, D2, DD2, DD3 *or* DD4. *If $f \in C^n[a, b]$ for $n = 4$, 5 or 6, then*

$$\max_{0 \leqq i \leqq m} \left| f_i'' - \frac{h^2}{12} f_i^{\text{iv}} - s_i'' \right| \leqq K h^{n-2} \|f^{(n)}\|_{L^\infty[a,b]}.$$

Proof. The technique of proof is that of Theorem 5. In general it consists of establishing a Taylor's theorem result related to the given end conditions, after possibly transforming them by the use of spline identities. Then Lemma 2 is used in combination with these results to give a matrix system differing from (3.5) only in the first and last rows. After possibly performing a matrix reduction, the results will follow by Lemma 4. The details are omitted.

COROLLARY. *Let s be the* Sp$(\pi_m, 3)$-*interpolate of f with any of the end conditions of Theorem* 6. *If* $f \in C^6[a, b]$, *then*

$$\max_{0 \leqq i \leqq m} \left| f_i'' - \frac{h^2}{12} f_i^{\text{iv}} + \frac{h^4}{360} f_i^{\text{vi}} - s_i'' \right| \leqq K h^4 \| f^{(6)} \|_{L^\infty[a,b]}.$$

It is possible to develop end conditions of increasingly high order by generalizing or combining together lower order end conditions in appropriate ways. We end this section with a particularly simple 8th order method.

THEOREM 7. *Let s be the* Sp$(\pi_m, 3)$-*interpolate of f with end conditions* DD5. *If* $f \in C^n[a, b]$ *for* $n = 4, 5$ *or* 6, *then*

$$\max_{0 \leqq i \leqq m} \left| f_i'' - \frac{h^2}{12} f_i^{\text{iv}} - s_i'' \right| \leqq K h^{n-2} \| f^{(n)} \|_{L^\infty[a,b]}. \tag{3.10}$$

If $f \in C^n[a, b]$ *for* $n = 6, 7$ *or* 8, *then*

$$\max_{0 \leqq i \leqq m} \left| f_i'' - \frac{h^2}{12} f_i^{\text{iv}} + \frac{h^4}{360} f_i^{\text{vi}} - s_i'' \right| \leqq K h^{n-2} \| f^{(n)} \|_{L^\infty[a,b]}. \tag{3.11}$$

Proof. Assume $f \in C^n[a, b]$, where $n = 6, 7$ or 8. Denote the terms on the left of (3.11) by $\bar{e}_i$ as in Lemma 3. By Taylor's theorem,

$$\begin{aligned} &7\left(f_0'' - \frac{h^2}{12} f_0^{\text{iv}} + \frac{h^4}{360} f_0^{\text{vi}}\right) + 46\left(f_1'' - \frac{h^2}{12} f_1^{\text{iv}} + \frac{h^4}{360} f_1^{\text{vi}}\right) \\ &\quad + 7\left(f_2'' - \frac{h^2}{12} f_2^{\text{iv}} + \frac{h^4}{360} f_2^{\text{vi}}\right) = 2f_0'' + 56 f_1'' + 2 f_2'' + R_0^n, \end{aligned} \tag{3.12}$$

where $|R_0^n| \leqq K h^{n-2} \| f^{(n)} \|_\infty$. Subtracting DD5 from (3.12) gives $7\bar{e}_0 + 46\bar{e}_1 + 7\bar{e}_2 = R_0^n$. Likewise $7\bar{e}_{m-2} + 46\bar{e}_{m-1} + 7\bar{e}_m = R_m^n$, where $|R_m^n| \leqq K h^{n-2} \| f^{(n)} \|_\infty$. The bound (3.11) follows by use of Lemma 3 and, after a matrix reduction, Lemma 4. The error bound (3.10) may be derived in a similar way.

4. Error bounds for locally smooth functions over locally uniform partitions. The following local convergence theorem gives conditions under which the hypotheses of Theorems 1 and 2 are satisfied for a certain Sp$(\pi_m, 3)$-interpolate of an arbitrary bounded function which is smooth inside a subinterval where the partitions are locally uniform. While a specific low order end condition is used for the spline interpolation, it should be clear that the results are not sensitive to this choice.

THEOREM 8. *Let* $a \leqq a' < \alpha < \beta < b' \leqq b$. *Suppose f is a bounded function over* $[a, b]$ *with* $f \in C^n[a', b']$ *for some* n, $4 \leqq n \leqq 8$. *Let* $\sigma \geqq 1$ *and* π *be any partition in* $P_\sigma[a, b]$ *such that* π *restricted to* $[a', b']$ *is uniform with partition size h. Let s be the* Sp$(\pi, 3)$-*interpolate of f with end conditions* $s''(a) = s''(b) = 0$. *Then there exists a constant* C_n *dependent on f but independent of* π *such that*

$$\max_{i_0 \leqq i \leqq i_1} \left| f_i'' - \frac{h^2}{12} f_i^{\text{iv}} - s_i'' \right| \leqq C_n h^{n-2} \quad \text{if } 4 \leqq n \leqq 6, \tag{4.1}$$

and

$$\max_{i_0 \leqq i \leqq i_1} \left| f_i'' - \frac{h^2}{12} f_i^{\text{iv}} + \frac{h^4}{360} f_i^{\text{vi}} - s_i'' \right| \leqq C_n h^{n-2} \quad \textit{if } 6 \leqq n \leqq 8, \tag{4.2}$$

where x_{i_0} is the first point of π in $[\alpha, \beta]$ and x_{i_1} is the last. Moreover, for $n = 4$, the bound (4.1) is valid for arbitrary $\pi \in P_\sigma[a, b]$ with $h = \max h_i$.

Proof. Let g be any function in $C^4[a, b] \cap C^n[a', b']$ such that $f(x) = g(x)$ for all $x \in [a', b']$. Let $|||a_i||| = \max\{|a_i| : i_0 \leqq i \leqq i_1\}$ and s_g be the Sp $(\pi, 3)$-interpolate of g satisfying $s_g''(a) = s_g''(b) = 0$. Then

$$\begin{aligned} \left|\left|\left| f_i'' - \frac{h^2}{12} f_i^{\text{iv}} - s_i'' \right|\right|\right| &\leqq \left|\left|\left| f_i'' - \frac{h^2}{12} f_i^{\text{iv}} - \left(g_i'' - \frac{h^2}{12} g_i^{\text{iv}} \right) \right|\right|\right| \\ &\quad + \left|\left|\left| g_i'' - \frac{h^2}{12} g_i^{\text{iv}} - (s_g)_i'' \right|\right|\right| + |||(s_g)_i'' - s_i''||| . \end{aligned} \tag{4.3}$$

The first term on the right side of (4.3) is zero. We now show that the second is of the right order.

Suppose $4 \leqq n \leqq 6$. Letting $a_i = h_i/(h_i + h_{i+1})$ and $b_i = 1 - a_i$, by Taylor's theorem,

$$\begin{aligned} &a_i\left(g_{i-1}'' - \frac{h^2}{12} g_{i-1}^{\text{iv}} \right) + 2\left(g_i'' - \frac{h^2}{12} g_i^{\text{iv}} \right) + b_i\left(g_{i+1}'' - \frac{h^2}{12} g_{i+1}^{\text{iv}} \right) \\ &\quad = \frac{6}{h_i + h_{i+1}} \left[\frac{g_{i+1} - g_i}{h_{i+1}} - \frac{g_i - g_{i-1}}{h_i} \right] + R_i^n, \qquad 1 \leqq i \leqq m - 1, \end{aligned} \tag{4.4}$$

where $|R_i^n| \leqq K_1 h^2 \|g^{\text{iv}}\|_{L^\infty[a,b]}$. Moreover if $x_i \in (a' + h, b' - h)$, then

$$|R_i^n| \leqq K_2 h^{n-2} \|f^{(n)}\|_{L^\infty[a',b']}.$$

Finally, letting

$$R_0^n = a_1\left(g_0'' - \frac{h^2}{12} g_0^{\text{iv}} \right), \quad R_m^n = b_{m-1}\left(g_m'' - \frac{h^2}{12} g_m^{\text{iv}} \right)$$

and

$$e_i = g_i'' - \frac{h^2}{12} g_i^{\text{iv}} - (s_g)_i'', \qquad 0 \leqq i \leqq m,$$

it follows from (1.12) that

$$\begin{bmatrix} 2 & b_1 & & & & \\ a_2 & 2 & b_2 & & & \\ & a_3 & 2 & b_3 & & \\ & & & \cdots & & \\ & & & a_{m-2} & 2 & b_{m-2} \\ & & & & a_{m-1} & 2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ \vdots \\ e_{m-2} \\ e_{m-1} \end{bmatrix} = \begin{bmatrix} R_1^n - R_0^n \\ R_2^n \\ R_3^n \\ \vdots \\ R_{m-2}^n \\ R_{m-1}^n - R_m^n \end{bmatrix} \tag{4.5}$$

By an inequality of Kershaw [8], writing (4.5) as $AE = B$, the elements of $A^{-1} = \{a_{ij}^{-1}\}$ are bounded by

$$|a_{ij}^{-1}| \leqq \tfrac{4}{3}(0.5)^{|i-j|+1}, \qquad 1 \leqq i, j \leqq m - 1. \tag{4.6}$$

Therefore, letting $\Delta = \min\{\alpha - a', b' - \beta\}$,

$$|||e_i||| \leqq 2K_2 h^{n-2} \|f^{(n)}\|_{L^\infty[a',b']} + 0.5^{[\Delta/h]}(16/3)\|B\|_\infty,$$

where $[\cdot]$ is the greatest integer function. Since $(0.5)^{[\Delta/h]} h^{-k} \to 0$ as $h \to 0$ for any k, it follows that $|||e_i||| \leqq Kh^{n-2}$.

We now consider the last term in (4.3). Let $r(x) = s_g(x) - s(x)$. Then $r''_0 = r''_m = 0$. By the spline identity (1.12),

$$\begin{aligned} &a_i r''_{i-1} + 2r''_i + b_i r''_{i+1} \\ &= \frac{6}{h_i + h_{i+1}}\left[\frac{r_{i+1} - r_i}{h_{i+1}} - \frac{r_i - r_{i-1}}{h_i}\right], \qquad 1 \leqq i \leqq m - 1. \end{aligned} \tag{4.7}$$

Denoting the right side of (4.7) by c_i, $\max|c_i| \leqq Kh^{-2}$ since $h_i \leqq \sigma h$ and f and g are both bounded. Also $|||c_i||| = 0$. Another application of (4.6) thus gives for $i_0 \leqq i \leqq i_1$,

$$|r''_i| \leqq \frac{2}{3}\sum_{j=1}^{m-1} (0.5)^{|i-j|}|c_j| \leqq \frac{16}{3}(0.5)^{[\Delta/h]} Kh^{-2}.$$

Thus $|||r''_i||| \leqq Kh^{n-2}$ giving (4.1). The proof for (4.2) is similar.

COROLLARY. *Let $a \leqq a' < \alpha < \beta < b' \leqq b$ and f be a bounded function over $[a, b]$ with $f \in C^4[a', b']$. For any $\pi \in P_\sigma[a, b]$ with $\bar{h}$ sufficiently small if s is the* Sp $(\pi, 3)$-*interpolate of f satisfying $s''(a) = s''(b) = 0$, then*

$$\|D^j(f - s)\|_{L^\infty[\alpha,\beta]} \leqq K\bar{h}^{4-j}, \qquad 0 \leqq j \leqq 2.$$

K is dependent on f but independent of π.

Proof. By Theorem 8 if $a \leqq a' < a'' < \alpha < \beta < b'' < b' \leqq b$, for any $x_i \in \pi$ with $x_i \in [a'', b'']$, $|f''_i - s''_i| \leqq K\bar{h}^2$. But for $\bar{h}$ sufficiently small, the first and last partition points of $[a'', b'']$, x_{i_0} and x_{i_1}, will be such that $a'' \leqq x_{i_0} < \alpha < \beta < x_{i_1} \leqq b''$. The result follows by Lemma 1.

Kammerer and Reddien [7] have established a similar result, but they required that either $\pi \in P_1[a, b]$ or $f \in C^1[a, b]$.

5. A numerical example. We will illustrate these results by giving a variety of errors for interpolating cubic splines with end conditions DD1, D1, F3 or DD5. Note these end conditions are of order 4, 5, 6 and 8. The function $f(x) = e^x \cos 5x$ will be interpolated over $[0, 1]$ with the uniform partition spacing $h = 0.05$. The approximate rates of decrease of the error, $O(h^\alpha)$, where α is computed from the observed decrease in the error from $h = 1/16$ to $h = 1/20$, is given in parentheses. (See Table 2.)

The set $\pi_{20} = \{\text{partition points}\}$, $M = \{\text{midpoints}\}$ and $L = \{\text{midpoints} \pm h/\sqrt{12}\}$. Table 3 illustrates the great improvements possible in estimating f'', f''' and f^{iv} at the partition points by use of finite difference operators and higher order end conditions such as F3 or DD5.

TABLE 2

L^∞ and point errors and rates with four end conditions

$h = 0.05$	DD1	D1	F3	DD5
$\|f - s\|_{L^\infty[0,1]}$	0.000039(4.10)	0.000025(3.99)	0.000025(4.02)	0.000025(4.03)
$\|f' - s'\|_{L^\infty[0,1]}$	0.00292 (3.10)	0.00150 (2.99)	0.00149 (3.02)	0.00149 (3.02)
$\|f'' - s''\|_{L^\infty[0,1]}$	0.317 (2.03)	0.313 (2.03)	0.314 (2.03)	0.314 (2.03)
$\max_{x \in \pi_{20} \cup M} \|f'(x) - s'(x)\|$	0.00292 (3.10)	0.000248(3.66)	0.000205(4.01)	0.000250(4.01)
$\max_{x \in L} \|f''(x) - s''(x)\|$	0.149 (2.10)	0.0131 (2.56)	0.0118 (3.01)	0.0138 (2.95)
$\max_{x \in M} \|f'''(x) - s'''(x)\|$	5.33 (1.11)	0.606 (2.01)	0.606 (2.01)	0.646 (1.82)

TABLE 3

Derivative errors and rates at knots using finite difference operators

$h = 0.05$	DD1	D1	F3	DD5
$\max_{1 \leq i \leq 19} \left\| f_i'' - \frac{s_{i+1}'' + 10 s_i'' + s_{i-1}''}{12} \right\|$	0.0255(2.08)	0.00161(2.79)	0.000869(3.96)	0.000736(3.94)
$\max_{2 \leq i \leq 18} \left\| f_i''' - \frac{-s_{i+2} + 14(s_{i+1}'' - s_{i-1}'') + s_{i-2}''}{24h} \right\|$	0.689(0.970)	0.0652 (2.32)	0.0171 (4.04)	0.0171 (4.02)
$\max_{1 \leq i \leq 19} \left\| f_i^{iv} - \frac{s_{i+1}'' - 2s_i'' + s_{i-1}''}{h^2} \right\|$	119. (0.008)	11.2 (1.02)	2.39 (2.08)	0.0107 (3.78)

In all cases the rates of decrease are, to the closest integer, those given by Theorems 1 and 2 in conjunction with Theorems 4, 5, 6 and 7.

6. Conclusions. From §§ 2 and 3 we have seen that for partitions $\pi \in P_1[a, b]$ and families $\mathfrak{F}$ of periodic functions the periodic end conditions (1.8) give Sp $(\pi, 3)$-interpolates of smooth functions f which yield approximations of f', f'', f''' and f^{iv} of order up to $O(h^4)$ at selected points. Moreover if the end conditions are chosen carefully these same results generalize to nonperiodic functions. In §§ 2 and 4 it was shown that these approximations are locally valid for general partitions $\pi \in P_\sigma[a, b]$ and bounded functions at certain points inside regions where the partitions are all locally uniform and the function is in a high order local smoothness class.

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