Lesson 10: Piecewise linear functions

Goals

- Calculate the different rates of change of a piecewise linear function using a graph, and interpret (orally and in writing) the rates of change in context.
- Create a model of a non-linear function using a piecewise linear function, and describe (orally) the benefits of having more or less line segments in the model.

Learning Targets

- I can create graphs of non-linear functions with pieces of linear functions.

Lesson Narrative

This lesson picks up on the idea planted in the previous lesson about creating linear models for data. Specifically, in some situations where a quantity changes at different constant rates over different time intervals, we can model the situation with a piecewise linear function. Students look at temperature data, which changes at different, almost-constant rates during different parts of the day. They also use piecewise linear graphs to find information about the real-life situation they represent. The focus of this lesson is not necessarily to find equations for the piecewise linear functions (though students may choose to do so in some instances), but rather to study the graphs qualitatively and to calculate and compare the different rates of change.

Addressing

- Use functions to model relationships between quantities.
- Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from two (x, y) values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values.
- Describe qualitatively the functional relationship between two quantities by analysing a graph (e.g., where the function is increasing or decreasing, linear or nonlinear). Sketch a graph that exhibits the qualitative features of a function that has been described verbally.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Stronger and Clearer Each Time
- Co-Craft Questions
- Discussion Supports

- Notice and Wonder

Student Learning Goals

Let's explore functions built out of linear pieces.

10.1 Notice and Wonder: Lines on Dots

Warm Up: 5 minutes

This warm-up connects to the previous lesson and is meant to elicit ideas that will be helpful for students in the next activity in this lesson. Students should notice that the points in the graph are not connected and wonder how well the lines model the sections of data they span, which is explored further in the next activity. For that reason, the discussion should focus on collecting all the things students notice and wonder but not explaining the things they wonder.

Instructional Routines

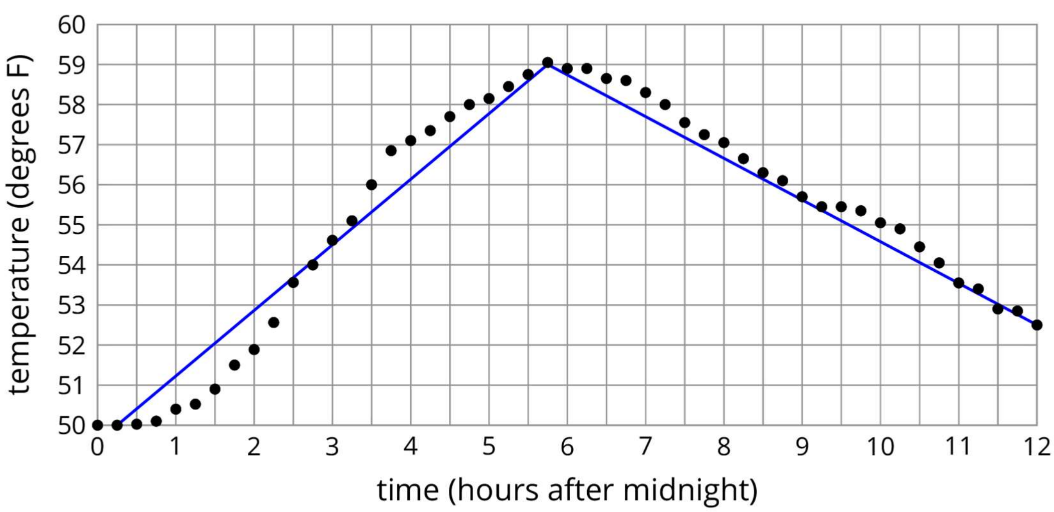
- Notice and Wonder

Launch

Tell students they will look at a graph and their job is to think of at least one thing they notice and at least one thing they wonder about the graph. Display the graph for all to see and give 1 minute of quiet think time.
Ask students to give a signal when they have noticed or wondered about something.

Student Task Statement

What do you notice? What do you wonder?



Student Response

Things students may notice:

- Not many of the points are on the blue line.
- The temperature gets warmer and then cooler as time goes on.
- At about 5:45 it is the warmest at about 59 degrees.
- The second blue line connects the highest point and the point furthest to the right.

Things students may wonder:

- What location does this data represent?
- Why is it warmer at 6:00 am then it is at noon?
- Why aren't the points connected?
- Why is the second line lower than almost all the points?

Activity Synthesis

Ask students to share their ideas. Record and display the responses for all to see. If no one mentions that the dots are not connected or what they think the blue lines mean, bring these ideas to their attention and tell them they will be working more with these ideas in the next activity.

10.2 Modelling Recycling

10 minutes

In this activity, students work with a graph that clearly cannot be modelled by a single linear function, but pieces of the graph could be reasonably modelled using different linear functions, leading to the introduction of piecewise linear functions. Students find the gradients of their piecewise linear model and interpret them in the context.

Monitor for students who:

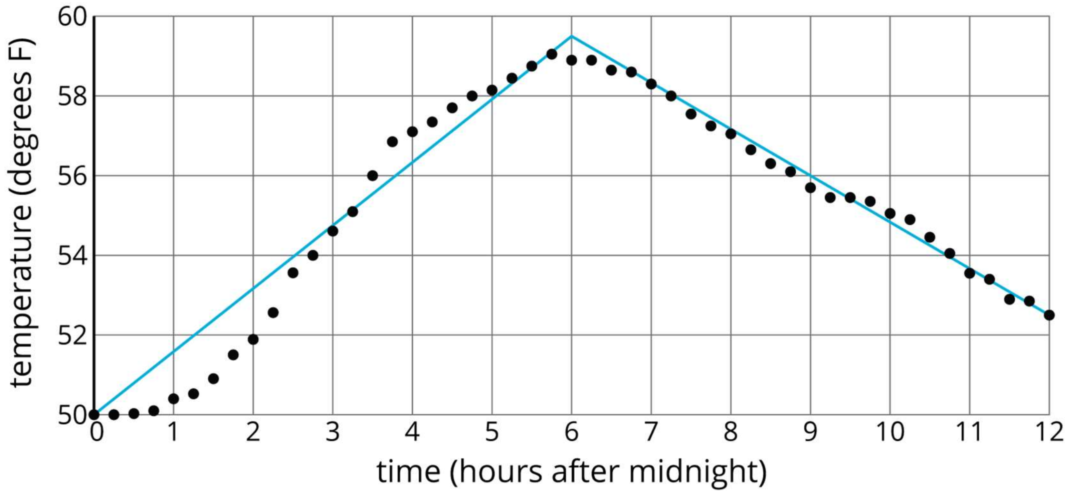
- Choose different numbers of line segments to represent the function (e.g., 3, 4, and 5 line segments).
- Choose different endpoints for the line segments (e.g., two students have chosen 4 line segments but different placement of those line segments).

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Discussion Supports

Launch

Arrange students in groups of 2. Display images for all to see. Tell students that sometimes we model functions with multiple line segments in different places. These models are called piecewise linear functions. For example, here are two different piecewise linear models of the same temperature data (note that the first image is not the same as the image in the warm-up):



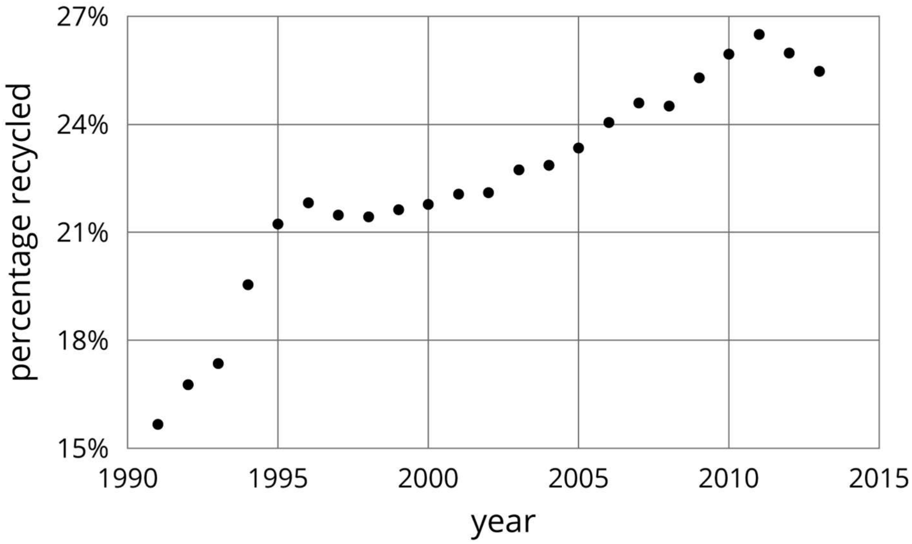
Give students 3–5 minutes of quiet work time and then time to share their responses with their partners. Follow with a whole-class discussion.

Representation: Develop Language and Symbols. Add to the display created in an earlier lesson of important terms and vocabulary. Add the following term and continue to maintain the display for reference throughout the remainder of the unit: piecewise linear functions.

Supports accessibility for: Memory; Language Speaking: Discussion Supports. Amplify mathematical language students use to communicate about piecewise linear functions. As students share what they notice between the two graphs, revoice their statements using the term "piecewise linear functions" while gesturing. Invite students to use the term "piecewise linear function" when describing what they noticed. Some students may benefit from chorally repeating the phrases that include the word "piecewise linear function" in context.

Design Principle(s): Support sense-making; Optimise output (for explanation)

Student Task Statement

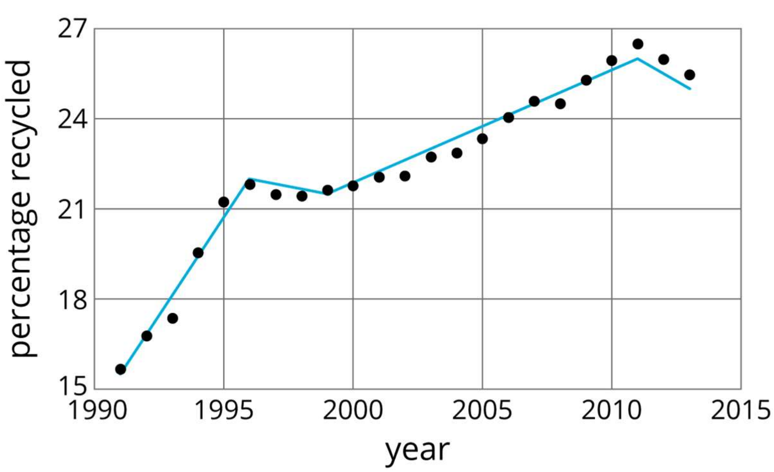


1. Approximate the percentage recycled each year with a piecewise linear function by drawing between three and five line segments to approximate the graph.

2. Find the gradient for each piece. What do these gradients tell you?

Student Response

1. Answers vary. Here is a piecewise linear model of the graph using four line segments.



2. Answers vary. The endpoints of the four line segments given in the answer to the previous part are: (1991,16), (1996,22), (1999,21.5), (2011,26), and (2013,25), so we can find that the four line segments have respective gradients $\frac{6}{5}$, $\frac{-1}{6}$, $\frac{3}{8}$, and $\frac{-1}{2}$. These gradients describe the approximate rate at which the percent recycled increased or decreased over those times. For example, from 2011 to 2013, the percent recycled decreased by approximately 0.5 percent per year.

Activity Synthesis

Select previously identified students to share their responses and display each student's graph for all to see. Sequence the student responses in order from fewest to greatest number of line segments.

To highlight the use and interpretation of piecewise linear models, ask:

- "What do the gradients of the different lines mean?" (The gradients of the lines tell us the rate of change of the different linear pieces for the specific intervals of time.)
- "Can we use this information to predict information for recycling in the future?" (If the data continues to decrease as it does from 2011 to 2013, yes. If the data starts to increase again, our model may not make very good predictions.)
- "What are the benefits of having fewer line segments in the piecewise linear function? What are the benefits of having more line segments?" (Fewer line segments are easier to write equations for and help show long-term trends. More line segments give a more accurate model of the data.)

10.3 Dog Bath

15 minutes

The purpose of this activity is to give students more exposure to working with a situation that can be modelled with a piecewise linear function. Here, the situation has already been modelled and students must calculate the rate of change for the different pieces of the model and interpret it in the context. A main discussion point should be around what the different rates of change mean in the situation and connecting features of the graph to the events in the context.

Instructional Routines

- Co-Craft Questions

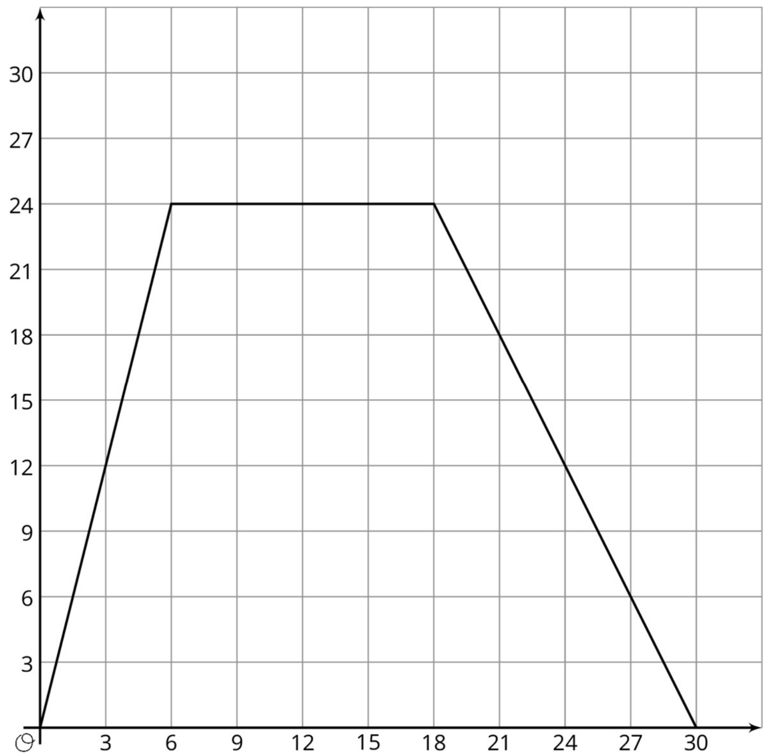
Launch

Students in groups of 2. Give 3–5 minutes of quiet work time and then ask them to share their responses with a partner and reach an agreement. Follow with a whole-class discussion.

Conversing, Writing: Co-Craft Questions. Display the situation and the graph without revealing the questions that follow. Invite students to write mathematical questions that could be answered by the graph. Invite students to share their questions with a partner before selecting 2–3 to share with the whole class. Highlight questions that ask students to make sense of the rate of change in context. Next, reveal the questions of the activity. This routine allows students to produce the language of mathematical questions and talk about the relationship between quantities and rates of change that are represented graphically. *Design Principle(s): Maximise meta-awareness; Support sense-making*

Student Task Statement

Elena filled up the tub and gave her dog a bath. Then she let the water out of the tub.



1. The graph shows the amount of water in the tub, in gallons, as a function of time, in minutes. Add labels to the graph to show this.
2. When did she turn off the water tap?
3. How much water was in the tub when she bathed her dog?
4. How long did it take for the tub to drain completely?
5. At what rate did the tap fill the tub?
6. At what rate did the water drain from the tub?

Student Response

1. Horizontal axis: “time (minutes)” (or equivalent). Vertical axis: “volume of water (gallons)” (or equivalent).
2. After 6 minutes. The volume of water is increasing until $t = 6$, so this must be when Elena turned off the tap.
3. 24 gallons. She bathed her dog after she turned the water off, so we look to the graph to see the volume of water at $t = 6$.

4. 12 minutes. The volume of water starts decreasing at $t = 18$ minutes, and takes until $t = 30$ to completely drain.

5. 4 gallons per minute. It took 6 minutes to fill 24 gallons, giving a rate of 4 gallons per minute $\left(\frac{24}{6}\right)$.

6. 2 gallons per minute. It took 12 minutes to drain 24 gallons, giving a rate of 2 gallons per minute $\left(\frac{24}{12}\right)$.

Activity Synthesis

The purpose of this discussion is for students to make sense of what the rate of change and other features of the model mean in the context of this situation. Consider asking the following questions:

- "Did the tub fill faster or drain faster? How can you tell?" (The tub filled faster. The gradient of the line representing the interval the water was filling the tub is steeper than the gradient of the line representing the interval the water was draining from the tub.)
- "If you were going to write a linear equation for the first piece of the graph, what would you use for m? For b?" (I would use $m = 4$ and $b = 0$, because the tub filled at a rate of 4 gallons per minute and the initial amount of water was 0.)
- "Which part of the graph represents the 2 gallons per minute you calculated?" (The last part of the piecewise function has a gradient of -2, which is when the tub was draining at 2 gallons per minute.)

10.4 Distance and Speed

Optional: 5 minutes

This activity is similar to the previous activity in that students are interpreting a graph and making sense of what situation the graph is representing. The difference here is that the specificity with numbers has been removed, so students need to think a bit more abstractly about what the changes in the graph represent and how they connect to the situation.

Monitor for students describing the graph with different levels of detail, particularly for any students who state that the car got up to speed faster than the car slowed down to 0.

Instructional Routines

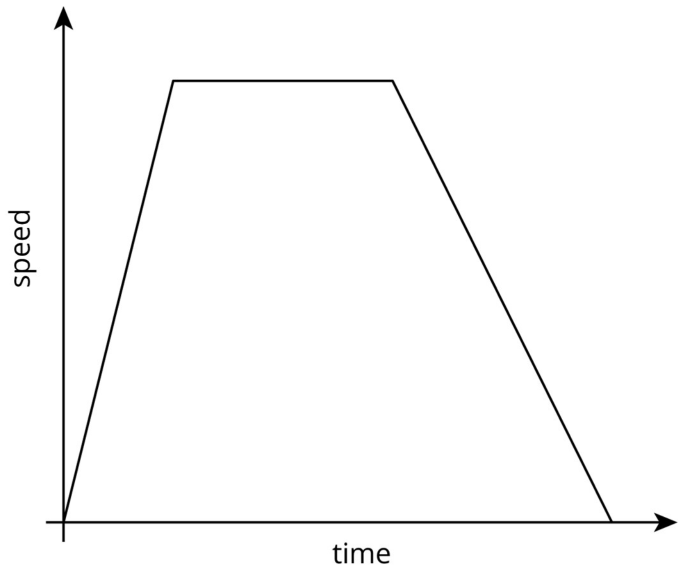
- Anticipate, Monitor, Select, Sequence, Connect
- Stronger and Clearer Each Time

Launch

Give students 1–2 minutes of quiet work time, followed by a whole-class discussion.

Engagement: Develop Effort and Persistence. Break the class into small discussion groups and then invite a representative from each group to report back to the whole class. All group members should be prepared to share if invited.
Supports accessibility for: Attention; Social-emotional skills

Student Task Statement



The graph shows the speed of a car as a function of time. Describe what a person watching the car would see.

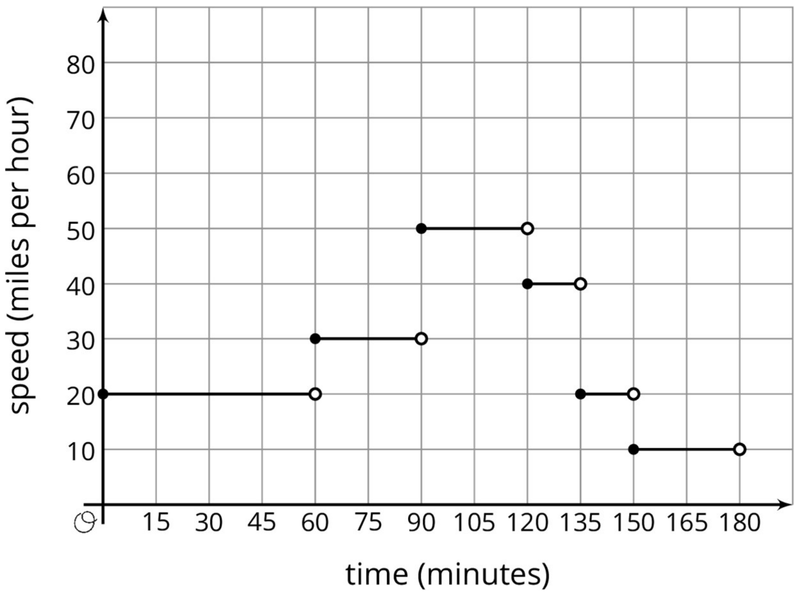
Student Response

The car begins at rest and then quickly picks up speed. After some time, it reaches its maximum speed, stays at that speed for a while, and then gradually slows back down until it comes to a stop.

Are You Ready for More?

The graph models the speed of a car as a function of time during a 3-hour trip. How far did the car go over the course of the trip?

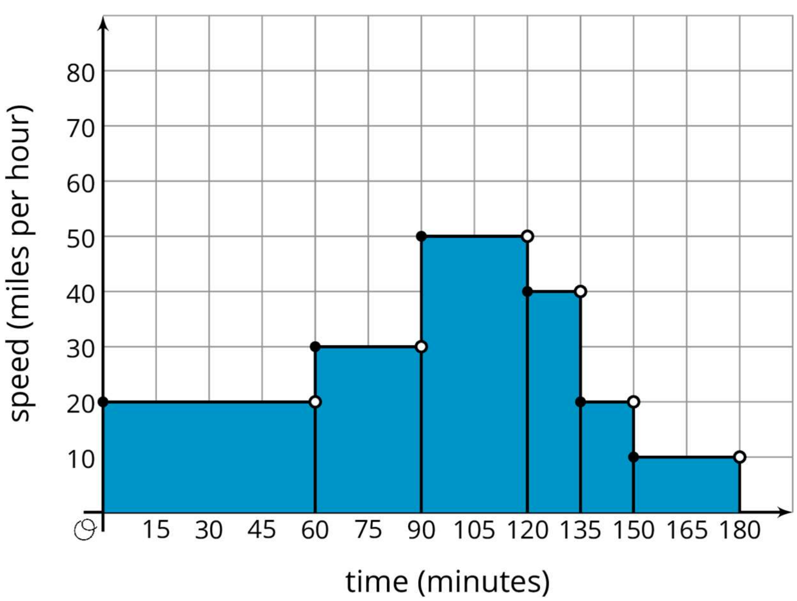
There is a nice way to visualise this quantity in terms of the graph. Can you find it?



Student Response

We can divide the trip into line segments over which the speed is constant, and use "distance $= speed \times time$." Over the first hour, the car travelled 20 miles per hour, for a total of 20 miles. Then for half an hour, the car travelled 30 miles per hour, for a total of 15 miles. Similarly, the remaining four line segments correspond to distances travelled of 25 miles, 10 miles, 5 miles, and 5 miles. Summing these gives a total of 80 miles.

The quantity "$speed \times time$" is represented graphically by a length measured along the x-axis times a height measured along the y-axis, giving the area of a rectangle. The distance travelled over each line segment is the area of the rectangle under that line segment, and the total distance is the total area of the shaded region.



Activity Synthesis

The purpose of this activity is for students to connect what is happening in a graph to a situation. Display the graph for all to see. Select previously identified students to share the situation they came up with. Sequence students from least descriptive to most descriptive. Have students point out the parts on the graph as they share their story about the situation.

Consider asking the following questions:

- "Did the car speed up faster or slow down faster? How do you know?" (The car sped up faster because the first part of the model is steeper than the third part of the model.)
- "How did you know that the car stayed that speed for a period of time?" (The graph stays at the same height for a while, so the speed was not changing during that time.)

Reading, Writing, Conversing: Stronger and Clearer Each Time. Use this routine to provide students with a structured opportunity to revise and refine the explanation of their thinking about the shape and patterns in the graph. After students have written a response to the task statement, ask students to meet with 2–3 partners in a row for feedback. Each time, partners should explain their thinking without reading from their written work. Provide students with prompts for feedback that will help students strengthen their ideas and clarify their language (e.g., "How do you know that from the graph?", "What in the graph makes you think that?", "Can you give an example?", etc.). Students can borrow ideas and language from each partner to strengthen their final version.
Design Principle(s): Optimise output (for explanation)

Lesson Synthesis

Ask students, "How would you describe a piecewise linear function to someone who has never seen one?" and give 1 minute of quiet think time and then time to share their response with a partner. Invite partners to share their responses with the class while recording them for all to see.
(A piecewise linear function is a function whose graph is pieced together out of line segments. For different ranges of input, the output is changing at different approximately constant rates so a different line is used for each range.) If students don't include the different constant rates over different intervals of the independent variable, make sure that is made clear.

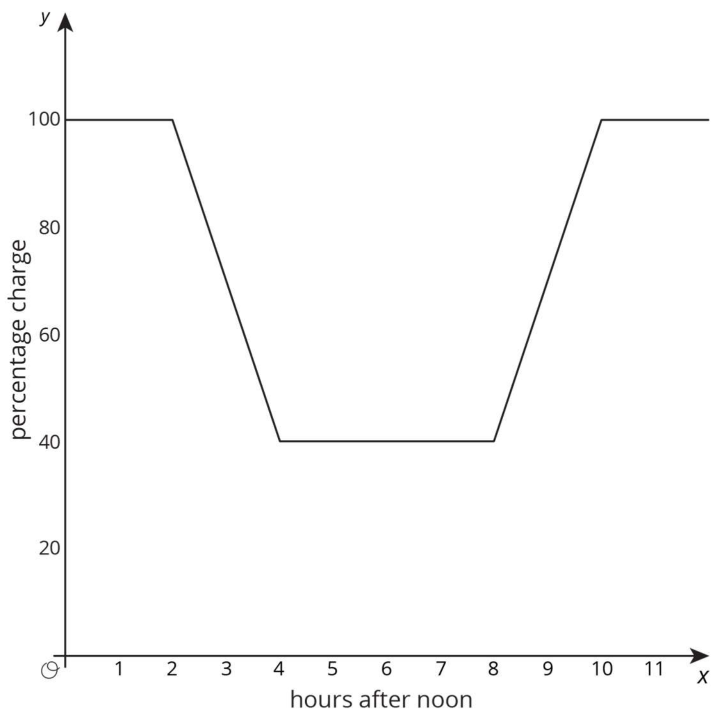
If time allows, ask students, "Can you think of another situation that changes at different constant rates over time?" and give partners 1 minute of think time before selecting groups to share their situations.

10.5 Lin's Phone Charge

Cool Down: 5 minutes

Student Task Statement

Lin uses an app to graph the charge on her phone.



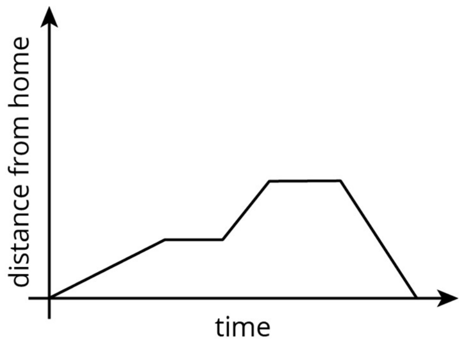
1. When did she start using her phone?
2. When did she start charging her phone?
3. While she was using her phone, at what rate was Lin's phone battery dying?

Student Response

1. Lin started using her phone 2 hours after noon or at 2:00 p.m., since that is where the negative gradient begins.
2. Lin started charging her phone 8 hours after noon or at 8:00 p.m., since that is where the positive gradient begins.
3. The battery was dying at 30% per hour since it decreased 60% over 2 hours.

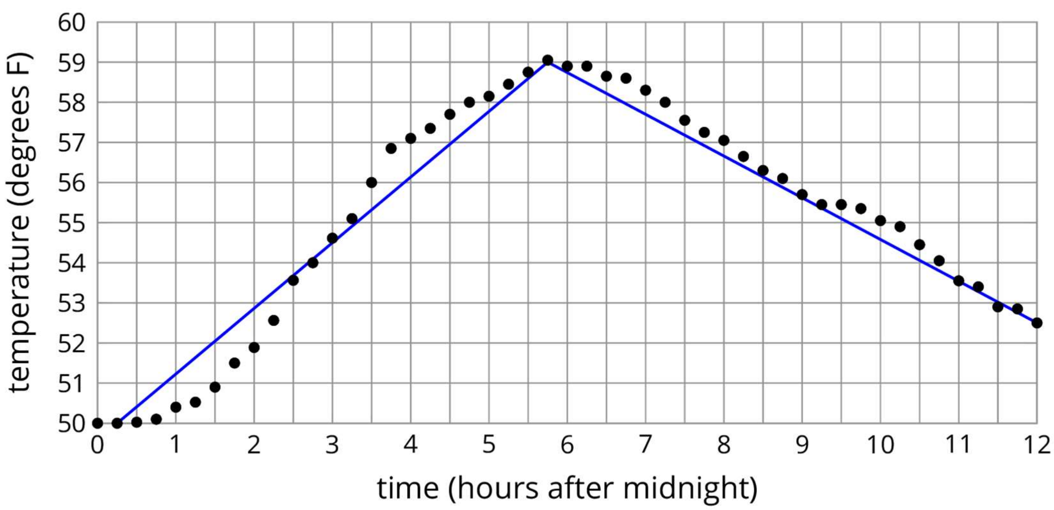
Student Lesson Summary

This graph shows Andre cycling to his friend's house where he hangs out for a while. Then they cycle together to the store to buy some groceries before racing back to Andre's house for a movie night. Each line segment in the graph represents a different part of Andre's travels.



This is an example of a piecewise linear function, which is a function whose graph is pieced together out of line segments. It can be used to model situations in which a quantity changes at a constant rate for a while, then switches to a different constant rate.

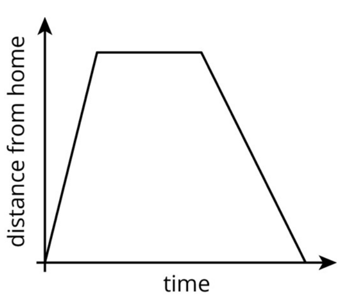
We can use piecewise functions to represent stories, or we can use them to model actual data. In the second example, temperature recordings at several times throughout a day are modelled with a piecewise function made up of two line segments. Which line segment do you think does the best job of modelling the data?



Lesson 10 Practice Problems

1. **Problem 1 Statement**

   The graph shows the distance of a car from home as a function of time.



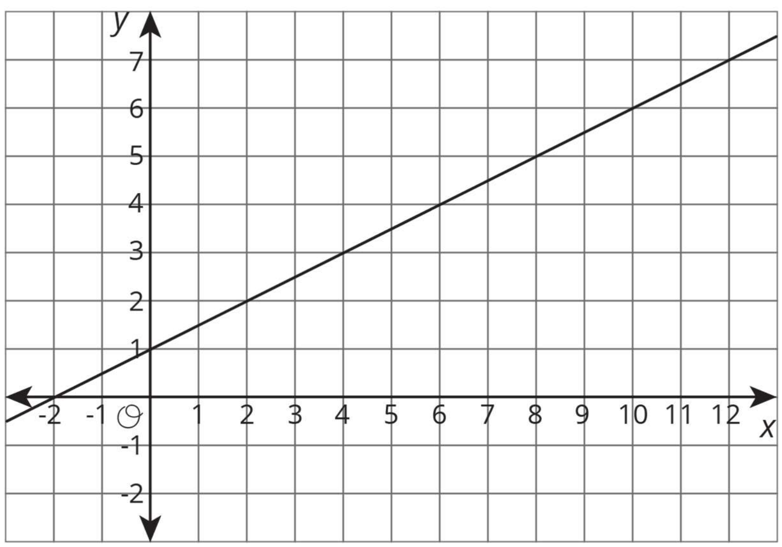
Describe what a person watching the car may be seeing.

Solution

Answers vary. Sample response: The car is driven away from home, then waits. The car is then driven back home at a slower speed than it was when driven away from home.

2. **Problem 2 Statement**

The equation and the graph represent two functions. Use the equation $y = 4$ and the graph to answer the questions.



a. When x is 4, is the output of the equation or the graph greater?

b. What value for x produces the same output in both the graph and the equation?

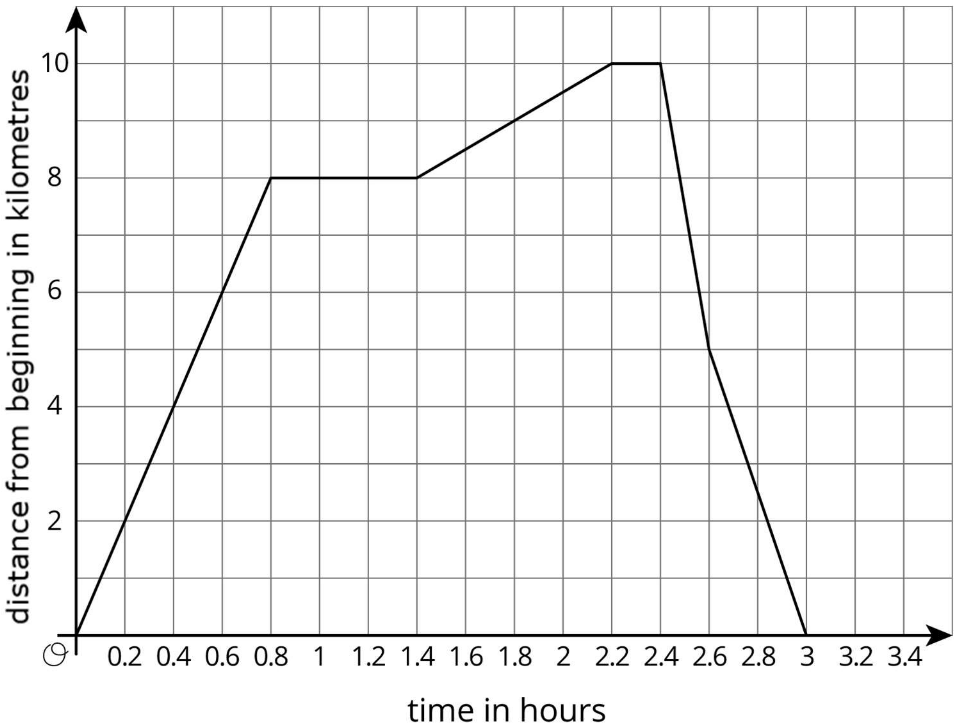
Solution

a. Equation

b. 6

3. **Problem 3 Statement**

This graph shows a trip on a cycle trail. The trail has markers every 0.5 km showing the distance from the beginning of the trail.



a. When was the cyclist going the fastest?

b. When was the cyclist going the slowest?

c. During what times was the cyclist going away from the beginning of the trail?

d. During what times was the cyclist going back towards the beginning of the trail?

e. During what times did the cyclist stop?

Solution

a. Between 2.4 and 2.6 hours

b. Between 1.4 and 2.2 hours, except the times the cyclist stopped

c. Between 0 and 0.8 hours and between 1.4 and 2.2 hours because the cyclist was stopped between 0.8 and 1.4 hours

d. Between 2.4 and 3 hours

e. Between 0.8 and 1.4 hours and between 2.2 and 2.4 hours

4. **Problem 4 Statement**

The expression -$25t + 1250$ represents the volume of liquid of a container after t seconds. The expression $50t + 250$ represents the volume of liquid of another container after t seconds. What does the equation $-25t + 1250 = 50t + 250$ mean in this situation?

Solution

Responses vary. Sample response: The equation says that the volume in one container is equal to the volume in the other container. This equation can be solved for t to find the time at which both containers have the same volume.