MEI Mathematics Education Innovation

Lesson 6: Estimating probabilities using simulation

Goals

- Comprehend the that term “simulation” (in written and spoken language) refers to a chance experiment used to represent a real-world situation.
- Describe (orally and in writing) a simple chance experiment that could be used to simulate a real-world event.
- Perform a simulation, and use the results to estimate the probability of a simple event in a real-world situation (using words and other representations).

Learning Targets

- I can simulate a real-world situation using a simple experiment that reflects the probability of the actual event.

Lesson Narrative

This lesson introduces the idea of simulation. Different groups of students use different chance experiments that are designed to enable you to approximate the probability of a real world event.

Students follow a process similar to what they used in previous lessons for calculating relative frequencies (the activities in which students were rolling a 1 or 2 on a dice or drawing a green block out of a bag). The distinction in this lesson is that the outcomes students are tracking are from an experiment designed to represent the outcome of some other experiment that would be harder to study directly. Students see that a simulation depends on the experiment used in the simulation being a reasonable stand-in for the actual experiment of interest.

This lesson works with estimating the probability of simple events in preparation for students being able to estimate the probability of compound events in upcoming lessons.

Building On

- Convert a rational number to a decimal using long division; know that the decimal form of a rational number terminates in 0s or eventually repeats.

Addressing

- Investigate chance processes and develop, use, and evaluate probability models.
- Understand that the probability of a chance event is a number between 0 and 1 that expresses the likelihood of the event occurring. Larger numbers indicate greater likelihood. A probability near 0 indicates an unlikely event, a probability around $\frac{1}{2}$ indicates an event that is neither unlikely nor likely, and a probability near 1 indicates a likely event.

- Approximate the probability of a chance event by collecting data on the chance process that produces it and observing its long-run relative frequency, and predict the approximate relative frequency given the probability. For example, when rolling a dice 600 times, predict that a 3 or 6 would be rolled roughly 200 times, but probably not exactly 200 times.
- Develop a probability model (which may not be uniform) by observing frequencies in data generated from a chance process. For example, find the approximate probability that a spinning penny will land heads up or that a tossed paper cup will land open-end down. Do the outcomes for the spinning penny appear to be equally likely based on the observed frequencies?
- Design and use a simulation to generate frequencies for compound events. For example, use random digits as a simulation tool to approximate the answer to the question: If 40% of donors have type A blood, what is the probability that it will take at least 4 donors to find one with type A blood?

Instructional Routines

- Discussion Supports
- Think Pair Share
- Which One Doesn't Belong?

Required Materials

Dice
cubes with sides numbered from 1 to 6

Paper bags
Paper clips
Pre-printed slips, cut from copies of the blackline master

Diego's Walk - Set A wait *more* than 1 minute	Diego's Walk - Set A wait *more* than 1 minute	Diego's Walk - Set A wait *more* than 1 minute	Diego's Walk - Set A wait *more* than 1 minute	Diego's Walk - Set A wait *more* than 1 minute
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Required Preparation

Print and cut up slips and spinners from the Diego's Walk blackline master. Provide each group of 3 supplies for 1 type of simulation: choosing a situation slip from a bag, spinning a spinner, or rolling 2 dice. The supplies for each simulation include:

- a paper bag containing a set of slips cut from the blackline master
- a spinner cut from the blackline master, a pencil and a paper clip
- 2 standard dice

Student Learning Goals

Let's simulate real-world situations.

6.1 Which One Doesn't Belong: Spinners

Warm Up: 5 minutes

This warm-up prompts students to compare four images. It encourages students to explain their reasoning, hold mathematical conversations, and gives you the opportunity to hear how they use terminology and talk about characteristics of the images in comparison to one another. To allow all students to access the activity, each image has one obvious reason it does not belong. Encourage students to use appropriate terminology (e.g. The bottom left spinner is the only one with an outcome that has a probability greater than 0.5).

During the discussion, listen for important ideas and terminology that will be helpful in upcoming work of the unit.

Instructional Routines

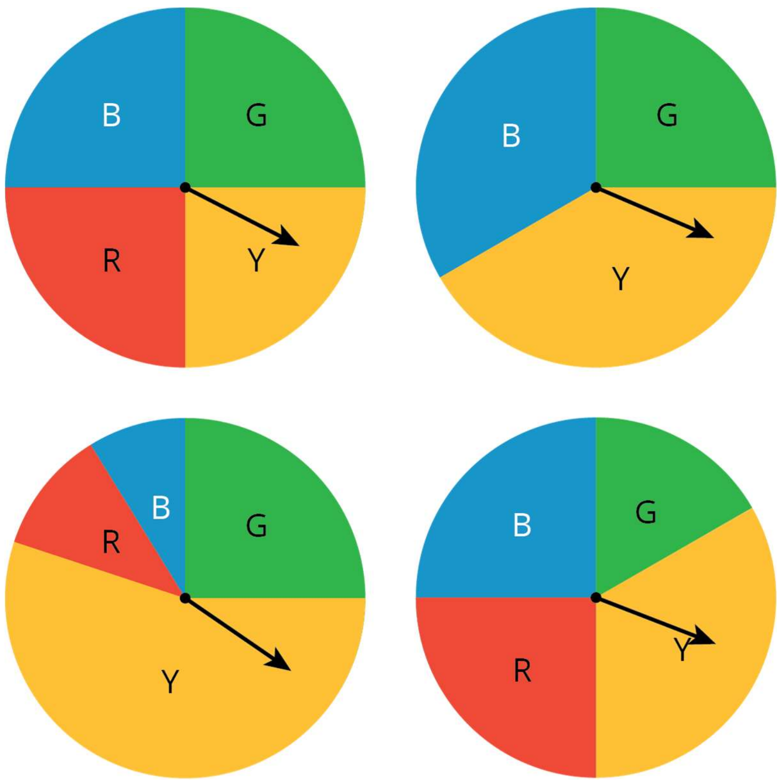
- Which One Doesn't Belong?

Launch

Arrange students in groups of 2–4. Display the image for all to see. Ask students to indicate when they have noticed which image does not belong and can explain why. Give students 2 minutes of quiet think time and then time to share their thinking with their group. After everyone has conferred in groups, ask the group to offer at least one reason each image doesn't belong. Follow with a whole-class discussion.

Student Task Statement

Which spinner doesn't belong?



Student Response

Answers vary. Sample responses:

- The top left spinner does not belong since all of the outcomes are equally likely.
- The top right spinner does not belong since it only has 3 possible outcomes.
- The bottom left spinner does not belong since it has an outcome that is more likely than the other three combined.
- The bottom right does not belong since the green outcome is not one fourth of the circle.

Activity Synthesis

Ask each group to share one reason why a particular image does not belong. Record and display the responses for all to see. After each response, ask the class if they agree or disagree. Since there is no single correct answer to the question of which one does not belong, attend to students' explanations and ensure the reasons given are correct. During the discussion, ask students to explain the meaning of any terminology they use, such as probability. Also, press students on unsubstantiated claims.

6.2 Diego's Walk

20 minutes

In this activity, students estimate the probability of a real-world event by simulating the experience with a chance experiment. Students see that that multiple simulation methods can result in similar estimates for the probability of the actual event.

Launch

Arrange students in groups of 3. Prepare each group with supplies for 1 type of simulation: choosing a slip from a bag, spinning a spinner, or rolling 2 dice. The supplies for each of these simulations include:

- a bag containing a set of slips from the blackline master
- a spinner cut from the blackline master, a pencil and a paper clip
- 2 standard dice

Set up the following simulation by telling the students: Diego must cross a busy intersection at a crossing on his way to school. Some days he is able to cross immediately or wait only a short while. Other days, he must wait for more than 1 minute for the signal to indicate he may cross the street. We will simulate his luck at this intersection using different methods and estimate his probability of waiting more than 1 minute.

Teacher note: The bag of papers and spinner are designed to have a probability of 0.7 to wait more than 1 minute. The dice have a probability of approximately 0.72 to wait more than 1 minute. To the extent that the students are estimating the probabilities, these are close enough to give similar results.

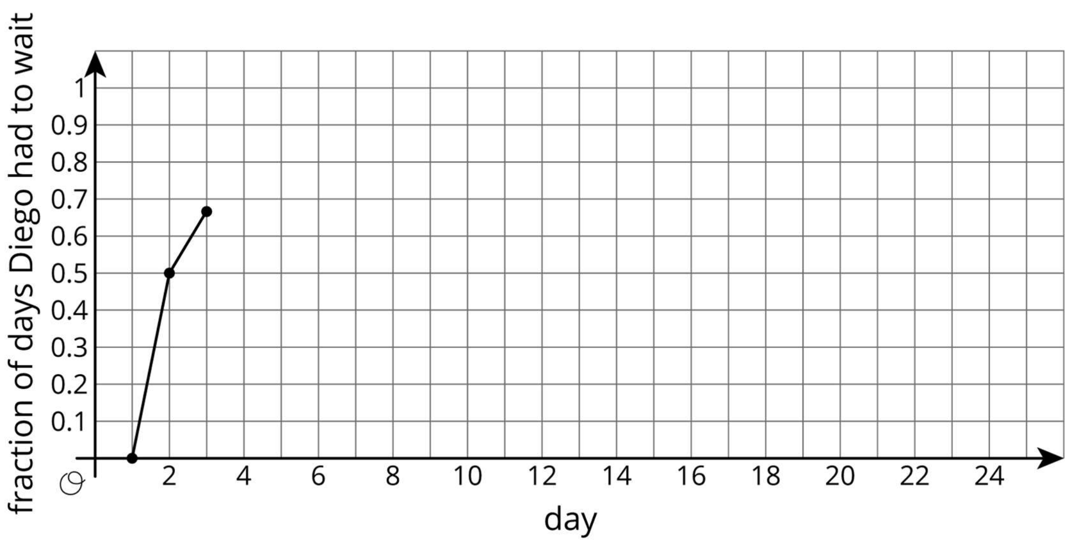
Give students 15 minutes for group work followed by a whole-class discussion.

Representation: Internalise Comprehension. Check in with students after the first 3-5 minutes of work time. Check to make sure students have attended to all parts of the simulation to record one day on the graph.
Supports accessibility for: Conceptual processing; Organisation

Student Task Statement

Your teacher will give your group the supplies for one of the three different simulations. Follow these instructions to simulate 15 days of Diego’s walk. The first 3 days have been done for you.

- Simulate one day:
    - If your group gets a bag of papers, reach into the bag, and select one paper without looking inside.
    - If your group gets a spinner, spin the spinner, and see where it stops.
    - If your group gets two dice, roll both dice, and add the numbers that land face up. A sum of 2–8 means Diego has to wait.
- Record in the table whether or not Diego had to wait more than 1 minute.
- Calculate the total number of days and the cumulative fraction of days that Diego has had to wait so far.
- On the graph, plot the number of days and the fraction that Diego has had to wait. Connect each point by a line.
- If your group has the bag of papers, put the paper back into the bag, and shake the bag to mix up the papers.
- Pass the supplies to the next person in the group.



day	Does Diego have to wait more than 1 minute?	total number of days Diego had to wait	fraction of days Diego had to wait
1	no	0	$\frac{0}{1} = 0.00$

2	yes	1	$\frac{1}{2} = 0.50$
3	yes	2	$\frac{2}{3} \approx 0.67$
4			
5			
6			
7			
8			
9			
10			
11			
12			
13			
14			
15			

1. Based on the data you have collected, do you think the fraction of days Diego has to wait after the 16th day will be closer to 0.9 or 0.7? Explain or show your reasoning.

2. Continue the simulation for 10 more days. Record your results in this table and on the graph from earlier.

day	Does Diego have to wait more than 1 minute?	total number of days Diego had to wait	fraction of days Diego had to wait
16			
17			
18			
19			
20			
21			
22			
23			
24			
25			

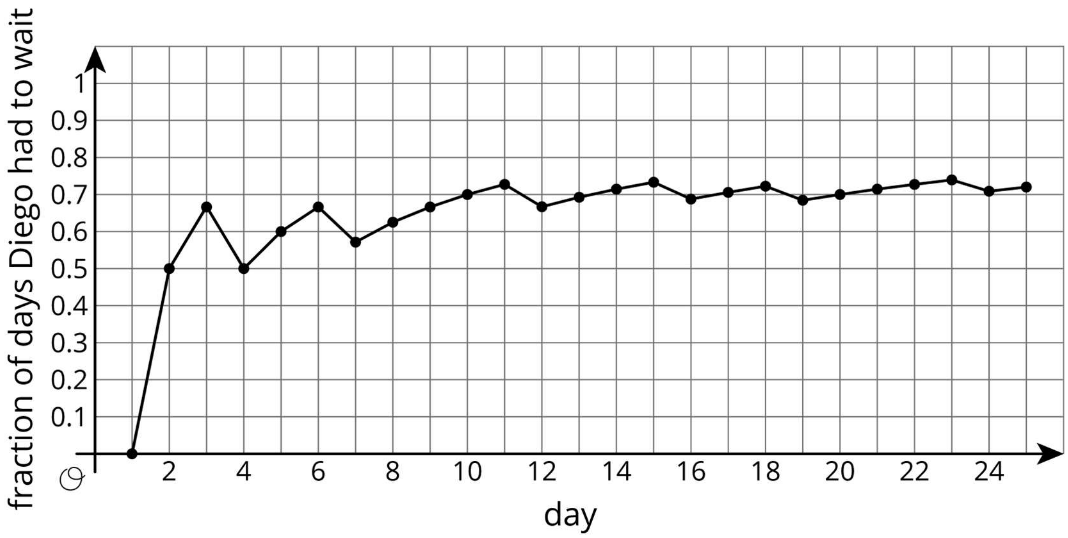
3. What do you notice about the graph?

4. Based on the graph, estimate the probability that Diego will have to wait more than 1 minute to cross the crossing.

Student Response

1. Answers vary. Sample response: Probably closer to 0.7 since our fraction after 16 trials was $\frac{11}{16} \approx 0.69$ which is closer to 0.7 than 0.9.

2. Answers vary. Sample response:



3. Answers vary. Sample response: At the beginning, the graph jumped up and down a lot, but seems to be levelling out near the end.

4. Answers vary. Sample response: I estimate the probability to be 0.7, since the graph seems to be levelling out there.

Are You Ready for More?

Let's look at why the values tend to not change much after doing the simulation many times.

1. After doing the simulation 4 times, a group finds that Diego had to wait 3 times. What is an estimate for the probability Diego has to wait based on these results?

    a. If this group does the simulation 1 more time, what are the two possible outcomes for the fifth simulation?

    b. For each possibility, estimate the probability Diego has to wait.

    c. What are the differences between the possible estimates after 5 simulations and the estimate after 4 simulations?

2. After doing the simulation 20 times, this group finds that Diego had to wait 15 times. What is an estimate for the probability Diego has to wait based on these results?

    a. If this group does the simulation 1 more time, what are the two possible outcomes for the twenty-first simulation?

b. For each possibility, estimate the probability Diego has to wait.

c. What are the differences between the possible estimates after 21 simulations and the estimate after 20 simulations?

3. Use these results to explain why a single result after many simulations does not affect the estimate as much as a single result after only a few simulations.

Student Response

1. $\frac{3}{4}$

   a. Either he has to wait or does not on the fifth day.

   b. Estimates are either $\frac{4}{5}$ or $\frac{3}{5}$.

   c. $\frac{1}{20} = 0.05$ or $\frac{3}{20} = 0.15$ since since $\frac{4}{5} - \frac{3}{4} = \frac{1}{20}$ and $\frac{3}{4} - \frac{3}{5} = \frac{3}{20}$.

2. $\frac{15}{20}$, which is equal to $\frac{3}{4}$.

   a. Either he has to wait or does not on the twenty-first day.

   b. Estimates are either $\frac{15}{21}$ or $\frac{16}{21}$.

   c. $\frac{1}{28} \approx 0.036$ or $\frac{1}{84} \approx 0.012$ since $\frac{15}{20} - \frac{15}{21} = \frac{1}{28}$ and $\frac{16}{21} - \frac{15}{20} = \frac{1}{84}$.

3. Explanations vary. Sample explanation: As the number of simulations grows, the denominator of each fraction grows, while the difference in the numerators remains 1. So, differences in the estimates between one simulation and the next will not change the estimate as much after many simulations.

Activity Synthesis

The purpose of this discussion is for students to understand why simulations are useful in place of actual experiments.

Select at least one group for each of the simulation methods to display the materials they used to run their simulation and explain the steps involved in using their materials.

Ask students, "Why do you think these simulations are more useful than actually doing the experiment many times?" (It would take a lot of time and work for Diego to walk to school more than usual, but it is easy to do the simulation many times quickly.)

Select students to share what they noticed about the graph of the fraction of days Diego had to wait as the simulated days went on.

6.3 Designing Experiments

10 minutes

In this activity, students have the opportunity to design their own simulations that could be used to estimate probabilities of real-life events. Students attend to precision by assigning each possible outcome for the real-life experiment to a corresponding outcome in their simulation in such a way that the pair of outcomes have the same probability. In the discussion following the activity, students are asked to articulate how these simulations could be used to estimate probabilities of certain events.

Instructional Routines

- Discussion Supports
- Think Pair Share

Launch

Keep students in groups of 3. Give students 5 minutes quiet work time to design their own experiments, followed by small-group discussion to compare answers for the situations and whole-class discussion.

As students work, monitor for students who are using the same chance events for multiple scenarios (for example, always using a spinner) and encourage them to think about other ways to simulate the event.

Engagement: Provide Access by Recruiting Interest. Leverage choice around perceived challenge. Invite students to select 2–3 of the situations to complete.
Supports accessibility for: Organisation; Attention; Social-emotional skills

Anticipated Misconceptions

Students may think that the number of outcomes in the sample space must be the same in the simulation as in the real-life situation. Ask students how we could use the results from the roll of a standard dice to represent a situation with only two equally likely outcomes. (By making use of some extra options to count as "roll again.")

Student Task Statement

For each situation, describe a chance experiment that would fairly represent it.

1. Six people are going out to lunch together. One of them will be selected at random to choose which restaurant to go to. Who gets to choose?
2. After a robot stands up, it is equally likely to step forward with its left foot or its right foot. Which foot will it use for its first step?
3. In a computer game, there are three tunnels. Each time the level loads, the computer randomly selects one of the tunnels to lead to the castle. Which tunnel is it?
4. Your school is taking 4 buses of students on a field trip. Will you be assigned to the same bus that your maths teacher is riding on?

Student Response

Answers vary. Sample responses:

1. Assign each person a number from 1 to 6, then roll a dice. The person whose number is face up on the dice is the one to choose.
2. Flip a coin. If it is heads, it should step with the left foot. If it is tails, it should step with the right.
3. Label each tunnel “left,” “right,” and “middle.” Make a spinner that has three equal sections with one of these labels in each section. Spin the spinner and the castle is behind the one the spinner lands on.
4. Assign a colour to each bus. Put one block of each colour in a bag, then reach in and pull out one block. If the block is red, then you are on the same bus.

Activity Synthesis

The purpose of this discussion is for students to think more deeply about the connections between the real-life experiment and the simulation.

Select partners to share the simulations they designed for each of the situations.

Some questions for discussion:

- "How could a standard dice be used to simulate the situation with the buses?" (Each bus is assigned a number 1 through 4. If the dice ends on 5 or 6, roll again.)
- "If one of the buses was numbered with your maths teacher's favourite number and you wanted to increase the probability of that bus being selected, how could you change the simulation to do this?" (Add more of the related outcome. For example, using the standard dice as in the previous discussion question, the bus with the favourite number could be assigned numbers 4 and 5 while the other buses are still 1 through 3.)
- Two of the tunnels in the video game lead to a swamp that ends the game. How could you use your simulation to estimate the probability of choosing one of those two tunnels? (Since all of the tunnels are equally likely to lead to the swamp, it can be assumed that "left" and "right" lead to the swamp. Spin the spinner many times and use the fraction of times it ends on "left" or "right" to estimate the probability of ending the game. It should happen $\frac{2}{3}$ or about 67% of the time.)
- "You and a friend are among the people going to lunch. How could you use the simulation you designed to estimate the probability that you or your friend will be the one to choose the restaurant?" (My friend and I will be represented by 1 and 2 on a dice. Roll the dice a lot of times and find the fraction of times 1 or 2 appear, then estimate the probability that we will be the ones selected.)

Speaking: Discussion Supports. Use this routine to support whole-class discussion. After students share the simulations they designed, display the following sentence frames to help students respond: "I agree because ….” or "I disagree because ….” Encourage students to use mathematical language to support their response. This will support rich and inclusive discussion about how to simulate a real-world situation using a simple experiment that reflects the probability of the actual event.
Design Principle(s): Support sense-making, Cultivate conversation

Lesson Synthesis

Consider asking these discussion questions:

- "What is a simulation?"
- "Why might you want to run a simulation rather than the actual event?" (Simulations are easier and usually faster to do multiple times, so using them to get an estimate of the probability of an event is sometimes preferred.)
- "If you conduct a few trial simulations of a situation and record the fraction of outcomes for which a particular event occurs, how might you know that you have done enough simulations to have a good estimate of the probability of that event happening?" (When the fractions seem to not be changing very much based on how accurate you want your estimate to be.)

6.4 Video Game Weather

Cool Down: 5 minutes

In this activity, students use their understanding of simulations to design a chance experiment that can be easily repeated while mimicking another situation with the same probability of a certain event.

Student Task Statement

In a video game, the chance of rain each day is always 30%. At the beginning of each day in the video game, the computer generates a random integer between 1 and 50. Explain how you could use this number to simulate the weather in the video game.

Student Response

Answers vary. Sample response: If the number is between 1 and 15, the video game should create a rainy day. If the number is between 16 and 50, the video game should not create a rainy day.

Student Lesson Summary

Sometimes it is easier to estimate a probability by doing a **simulation**. A simulation is an experiment that approximates a situation in the real world. Simulations are useful when it is hard or time-consuming to gather enough information to estimate the probability of some event.

For example, imagine Andre has to transfer from one bus to another on the way to his music lesson. Most of the time he makes the transfer just fine, but sometimes the first bus is late and he misses the second bus. We could set up a simulation with slips of paper in a bag. Each paper is marked with a time when the first bus arrives at the transfer point. We select slips at random from the bag. After many trials, we calculate the fraction of the times that he missed the bus to estimate the probability that he will miss the bus on a given day.

Glossary

- simulation

Lesson 6 Practice Problems

Problem 1 Statement

The weather forecast says there is a 75% chance it will rain later today.

a. Draw a spinner you could use to simulate this probability.

b. Describe another way you could simulate this probability.

Solution

Answers vary. Sample response:

a. A circle with $\frac{3}{4}$ coloured blue and labelled "rain" and the other $\frac{1}{4}$ left white and labelled "no rain."

b. Put 4 marbles in a bag, three blue and one white. The blue marbles represent "rain."

Problem 2 Statement

An experiment will produce one of ten different outcomes with equal probability for each. Why would using a standard dice to simulate the experiment be a bad choice?

Solution

A standard dice only has 6 outcomes, so it cannot produce all 10 possibilities from the experiment.

Problem 3 Statement

An ice cream shop offers 40 different flavours. To simulate the most commonly chosen flavour, you could write the name of each flavour on a piece of paper and put it in a bag. Draw from the bag 100 times, and see which flavour is chosen the most. This simulation is not a good way to figure out the most-commonly chosen flavour. Explain why.

Solution

Answers vary. Sample response: Drawing from the bag is random, but people do not usually randomly choose ice cream flavours.

Problem 4 Statement

Each set of three numbers represents the lengths, in units, of the sides of a triangle. Which set can *not* be used to make a triangle?

a. 7, 6, 14

b. 4, 4, 4

c. 6, 6, 2

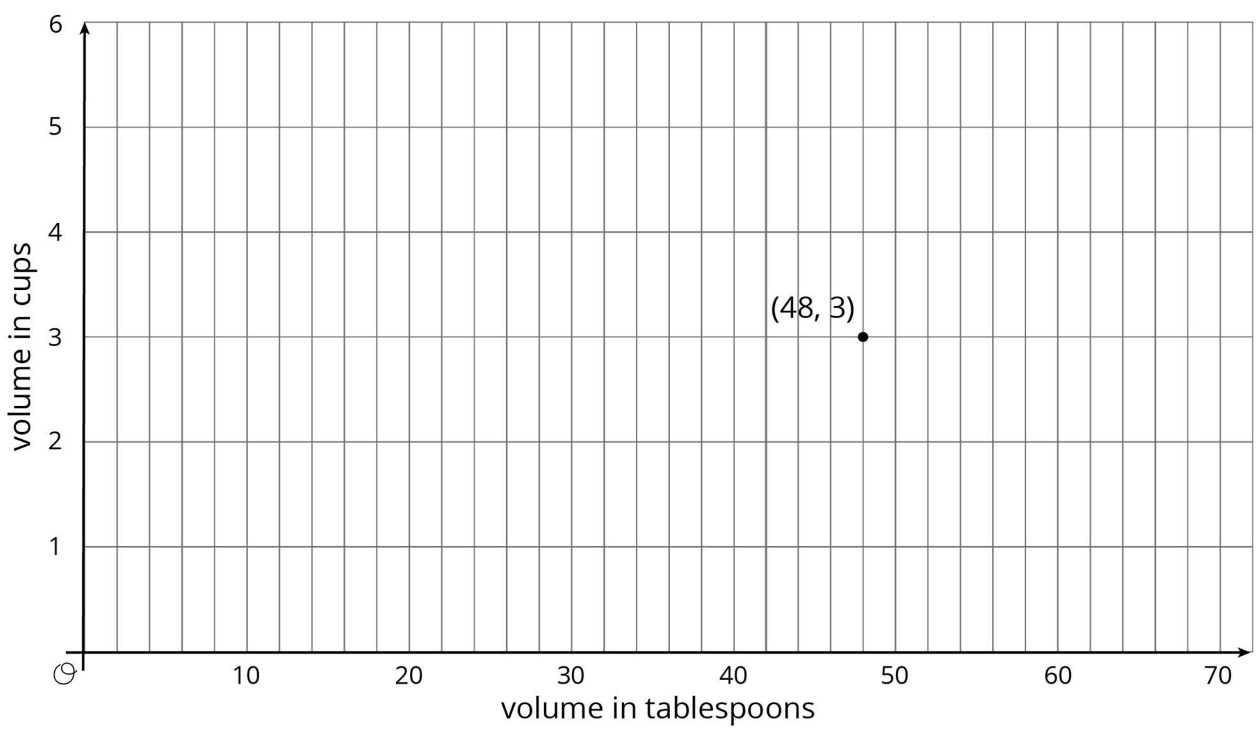
d. 7, 8,13

Solution

a. 7, 6, 14

Problem 5 Statement

There is a proportional relationship between a volume measured in cups and the same volume measured in tablespoons. 48 tablespoons is equivalent to 3 cups, as shown in the graph.



a. Plot and label some more points that represent the relationship.

b. Use a straightedge to draw a line that represents this proportional relationship.

c. For which value y is $(1, y)$ on the line you just drew?

d. What is the constant of proportionality for this relationship?

e. Write an equation representing this relationship. Use c for cups and t for tablespoons.

Solution

a. See below

b. See below

c. $\frac{1}{16}$

d. $\frac{1}{16}$ cups per tablespoon

e. $c = \frac{1}{16}t$ or equivalent