Lesson 8: Keeping track of all possible outcomes

Goals

- Compare and contrast (in writing) different methods for representing the sample space of a compound event, and evaluate (orally) their usefulness.
- Determine the total number of possible outcomes for a compound event, and justify the reasoning (orally, in writing, and using other representations).
- Interpret or create a list, table, or tree diagram that represents the sample space of a compound event.

Learning Targets

- I can write out the sample space for a multi-step experiment, using a list, table, or tree diagram.

Lesson Narrative

In this lesson, students practise listing the sample space for a compound event. They make use of the structure of tree diagrams, tables, and organised lists as methods of organising this information. Students notice that the total number of outcomes in the sample space for an experiment that can be thought of as being performed as a sequence of steps can be found by multiplying the number of possible outcomes for each step in the experiment.

In the next lesson, students will use sample spaces to calculate the probability of compound events.

Addressing

- Represent sample spaces for compound events using methods such as organised lists, tables and tree diagrams. For an event described in everyday language (e.g., “rolling double sixes”), identify the outcomes in the sample space which compose the event.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Collect and Display
- Discussion Supports
- Think Pair Share

Student Learning Goals

Let’s explore sample spaces for experiments with multiple parts.

8.1 How Many Different Meals?

Warm Up: 5 minutes

The purpose of this warm-up is to elicit methods students are already using to organise their understanding of different outcomes. In this lesson, students are asked to use different structures to think about the outcomes of experiments that involve multiple steps, so this activity should give you an idea of how students are approaching the problem on their own.

Instructional Routines

- Think Pair Share

Launch

Arrange students in groups of 2. Tell students they should organise their work so it can be understood by others. Give students 1 minute of quiet think time, 3 minutes for partner discussion, and follow up with a whole-class discussion.

Student Task Statement

How many different meals are possible if each meal includes one main course, one side dish, and one drink?

main courses	side dishes	drinks
grilled chicken	salad	milk
turkey sandwich	applesauce	juice
pasta salad	—	water

Student Response

There are 18 different meals.

Activity Synthesis

Select several groups to share their methods for organising their thoughts about the different meals that are possible.

Consider these questions for the discussion:

- “How did you know you counted all of the different possible meals?”
- “How did you know you didn’t repeat any meals?”

8.2 Lists, Tables, and Trees

15 minutes

In this activity, students learn 3 different methods for writing the sample spaces of multi-step experiments and explore their use in a few different situations. Since the calculated probability of an event depends on the number of outcomes in the sample space, it is important to be able to find this value in an efficient way. In the discussion, students will explore how different methods may be useful in different situations.

As students work on the second set of questions, monitor for students who:

1. Always use a list format to write out the sample space.
2. Always use a tree format to write out the sample space.
3. Change which representations they use for different questions.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Collect and Display

Launch

Allow 5 minutes for students to answer the first set of questions, then pause the class to discuss the methods.

Poll the class for their favourite methods in the given situation and display the results for all to see. Ask at least 1 student for each representation for their reason they believe that method is their favourite.

Following the discussion, give students another 5 minutes of quiet work time to finish the questions. Follow with a whole-class discussion.

Representing, Writing: Collect and Display. As students discuss the question: "Which method do you prefer for this situation?", write down the words and phrases students use to explain their reasoning. Listen for the language students use to describe when each method is efficient and useful for different situations. As students review the language collected in the visual display, encourage students to revise and improve how ideas are communicated. For example, a phrase such as: "I prefer the table because it is neater" can be improved with the phrase "I prefer the table because it shows all the outcomes of the sample space in an organised manner." This routine will provide feedback to students in a way that supports sense-making while simultaneously increasing meta-awareness of language.
Design Principle(s): Support sense-making (for representation); Maximise meta-awareness

Anticipated Misconceptions

Some students may have trouble interpreting the tree diagram. Help students see that a single outcome is represented by following the "branches" from the point furthest to the left until they reach the end of a branch on the right side. It may help for students to write the full outcome on the diagram as well. In Priya's picture, next to the uppermost 1, a student could write H1.

Student Task Statement

Consider the experiment: Flip a coin, and then roll a dice.

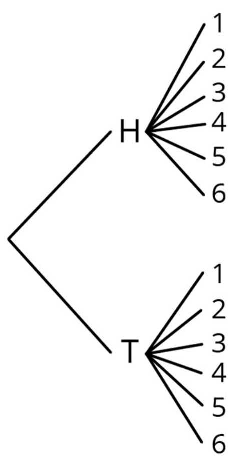
Elena, Kiran, and Priya each use a different method for finding the sample space of this experiment.

- Elena carefully writes a list of all the options: Heads 1, Heads 2, Heads 3, Heads 4, Heads 5, Heads 6, Tails 1, Tails 2, Tails 3, Tails 4, Tails 5, Tails 6.

- Kiran makes a table:

	1	2	3	4	5	6
H	H1	H2	H3	H4	H5	H6
T	T1	T2	T3	T4	T5	T6

- Priya draws a tree with branches in which each pathway represents a different outcome:



1. Compare the three methods. What is the same about each method? What is different? Be prepared to explain why each method produces all the different outcomes without repeating any.

2. Which method do you prefer for this situation?

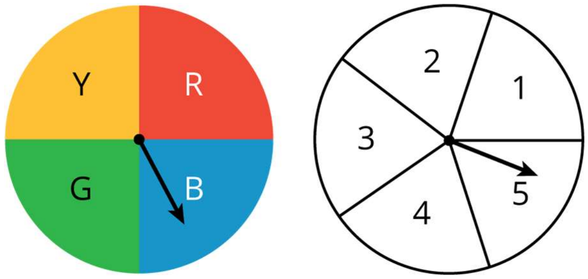
   Pause here so your teacher can review your work.

3. Find the sample space for each of these experiments using any method. Make sure you list every possible outcome without repeating any.

   a. Flip a 10p coin, then flip a 5p coin, and then flip a penny. Record whether each lands heads or tails up.

   b. Han’s cupboard has: a blue shirt, a grey shirt, a white shirt, blue trousers, khaki trousers, and black trousers. He must select one shirt and one pair of trousers to wear for the day.

   c. Spin a colour, and then spin a number.



d. Spin the hour hand on an analogue clock, and then choose a.m. or p.m.

Student Response

1. Answers vary. Sample response: They all show the 12 possible outcomes, but Elena's method might get more difficult with more options.

2. Answers vary. Sample response: Since there are not very many outcomes in this experiment, I might use Elena's method since it is quick and I am careful.

    a. HHH, HHT, HTH, THH, HTT, THT, TTH, TTT

    b. Blue shirt and blue trousers, blue shirt and khaki trousers, blue shirt and black trousers, green shirt and blue trousers, green shirt and khaki trousers, green shirt and black trousers, white shirt and blue trousers, white shirt and khaki trousers, white shirt and black trousers.

    c. R1, R2, R3, R4, R5, B1, B2, B3, B4, B5, G1, G2, G3, G4, G5, Y1, Y2, Y3, Y4, Y5

    d. 1 a.m., 2 a.m., 3 a.m., 4 a.m., 5 a.m., 6 a.m., 7 a.m., 8 a.m., 9 a.m., 10 a.m., 11 a.m., 12 a.m., 1 p.m., 2 p.m., 3 p.m., 4 p.m., 5 p.m., 6 p.m., 7 p.m., 8 p.m., 9 p.m., 10 p.m., 11 p.m., 12 p.m.

Activity Synthesis

The purpose of this discussion is to think about the different methods of writing the sample space and when each might be useful. The discussion is also meant to make the connection between the methods and the number of outcomes in the sample space so that students can quickly find the number of outcomes without writing out all the possibilities.

Select previously identified students to share their group's strategies for answering the questions in the sequence identified in the Activity Narrative. For students who used the same representation throughout, ask, "Why did you choose to use this same strategy for all the questions? What are the benefits of this strategy? Did you encounter any problems using the strategy?" For students who changed strategies for different questions, ask, "How did you decide which strategy to use for each question?"

Consider these questions for the discussion:

- "What structure did you use for each situation to make sure all the different outcomes were included without duplicating any?"

- “Would each of Elena’s, Kiran’s, and Priya’s methods work for flipping the different coins?” (A table would not work since there are three parts.)
- “Count the number of outcomes in each sample space. Is there a way to find the *number* of outcomes without writing all the possibilities? Explain or show your reasoning.” (Note the connection of Kiran’s table structure of “2 rows of 6 outcomes” or Priya’s tree diagram of “2 groups of 6” to 2×6 in the experiment where they flipped a coin and rolled a dice.)

Action and Expression: Internalise Executive Functions. Provide students with a graphic organiser to support their participation during the synthesis. Invite students to describe what to look for to compare the similarities and differences between the three sample space methods, and include examples for each.
Supports accessibility for: Conceptual processing; Organisation

8.3 How Many Sandwiches?

10 minutes

In this activity, students practise using their understanding of ways to calculate the number of outcomes in the sample space without writing out the entire sample space. Many situations with multiple steps have very large sample spaces for which it is not helpful to write out the entire sample space, but it is still useful to know the number of outcomes in the sample space. In this activity, students find the number of different sandwiches that can be made from available options.

Instructional Routines

- Discussion Supports

Launch

Explain to students that the sandwich makers are instructed to put a certain amount of each item on the sandwich for each selection. Therefore, if a person really loves tomatoes, he should ask for tomatoes twice as his two veggie choices. Give students 5 minutes of quiet work time followed by a whole-group discussion.

Action and Expression: Internalise Executive Functions. Chunk this task into more manageable parts to support students who benefit from support with organisational skills in problem solving. For example, pause to check for understanding of determining the number of outcomes after 3–5 minutes of work time.
Supports accessibility for: Organisation; Attention

Anticipated Misconceptions

Some students may attempt to write out the entire sample space. Encourage them to write a few outcomes to get the idea of what is possible, but let them know that the answer for question 1 is well over 1 000, so finding a pattern or way to calculate the answer might be more efficient.

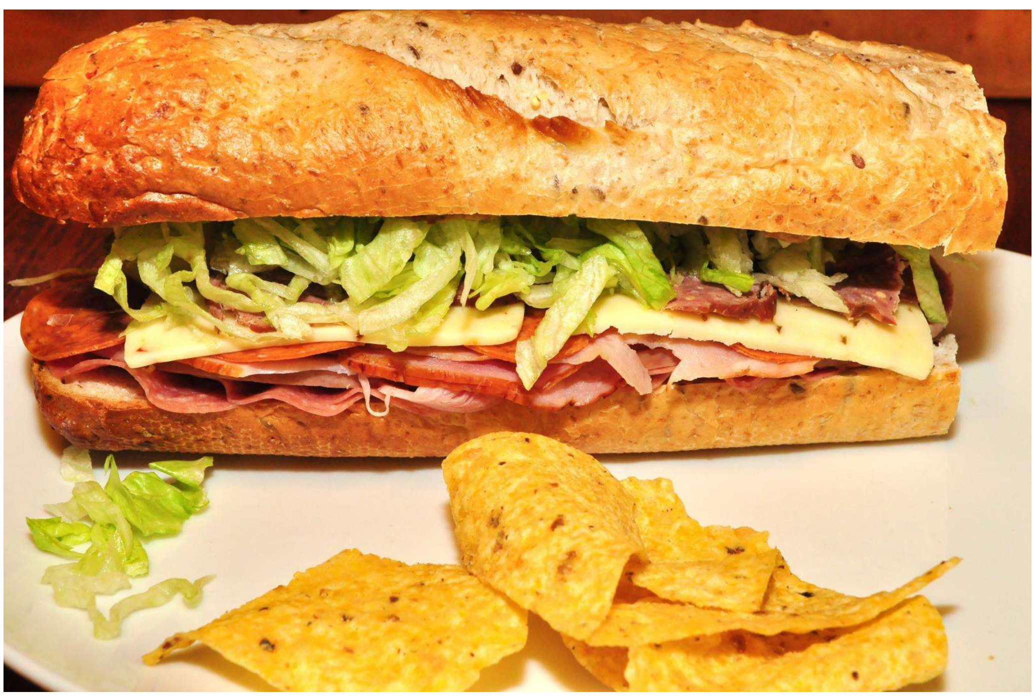
Some students may not count the “none” option for cheese as a distinct choice. Show these students that a sandwich like “Italian bread, tuna, provolone, lettuce, and tomatoes” is different than a sandwich like “Italian bread, tuna, no cheese, lettuce, and tomatoes” and should be counted separately.

Some students may notice that the order of the vegetable selection may not matter. For example, selecting lettuce then tomato would create a similar sandwich to one with tomato then lettuce selected. Some sandwich shops may offer more of the first option, so we could ask students to consider the sandwiches as different based on this idea. Otherwise, there are only 720 sandwiches possible since there are only 15 different options for 2 vegetables and $3 \times 4 \times 4 \times 15 = 720$.

Student Task Statement

1. A submarine sandwich shop makes sandwiches with one kind of bread, one protein, one choice of cheese, and *two* vegetables. How many different sandwiches are possible? Explain your reasoning. You do not need to write out the sample space.

   - Breads: Italian, white, wheat
   - Proteins: Tuna, ham, turkey, beans
   - Cheese: Provolone, Swiss, American, none
   - Vegetables: Lettuce, tomatoes, peppers, onions, pickles



2. Andre knows he wants a sandwich that has ham, lettuce, and tomatoes on it. He doesn’t care about the type of bread or cheese. How many of the different sandwiches would make Andre happy?

3. If a sandwich is made by randomly choosing each of the options, what is the probability it will be a sandwich that Andre would be happy with?

Student Response

1. 1 200 options since $3 \times 4 \times 4 \times 5 \times 5 = 1\,200$.

2. 12 sandwiches, since there are 3 options for bread and 4 options for cheese that can be made and $3 \times 4 = 12$.

3. $\frac{12}{1\,200}$ (or equivalent) since there are 12 sandwiches that would make Andre happy out of a total of 1 200 sandwiches possible.

Are You Ready for More?

Describe a situation that involves three parts and has a total of 24 outcomes in the sample space.

Student Response

Answers vary. Sample response: Flip a coin, then select between rock, paper, scissors, then select a letter from the word BATH. This has 24 outcomes, since $2 \times 3 \times 4 = 24$.

Activity Synthesis

The purpose of the discussion is to help students understand the calculations behind the solutions of these problems.

Some questions for discussion:

- "Describe how the tree of sandwich options would look without drawing it out." (The first column would have the 3 options for bread. Coming out from each of those options would be 4 branches for each of the proteins. From each of these there would be 4 more branches for each of the cheese options. Each of those would have 5 branches for the veggies. Finally, there would be 5 branches for the veggies again since the sandwich has 2 of them.)

- "How is the tree connected to the calculation of the size of the sample space?" (Since the first two choices [bread and protein] are 3 groups of 4 branches, there are 12 options for those two choices. When adding the cheese option, there are 12 groups of 4 or 48 options. For the first veggie, there are 48 groups of 5 things or 240 options. Finally, there are 240 groups of 5 veggies for the last option, giving a total of 1 200 outcomes.)

- "If the two veggie choices had to be different, would there be a higher or lower total number of possible sandwiches? Explain your reasoning." (Fewer, since an item like "Italian bread, ham, Swiss, onions, and onions" was an option before, but is not an option with the new restriction.)

Speaking: Discussion Supports. Use this routine to support whole-class discussion. For each observation or response that is shared, ask students to restate and/or revoice what they heard using precise mathematical language. Consider providing students time to restate what they hear to a partner, before selecting one or two students to share with the class. Ask the original speaker if their peer was accurately able to restate their thinking. Call students' attention to any words or phrases that helped to clarify the original statement. This will provide more students with an opportunity to produce language as they interpret the reasoning of others.
Design Principle(s): Support sense-making

Lesson Synthesis

Consider asking these discussion questions:

- "What are some methods for writing out the sample space of a chance experiment that consists of multiple steps?" (Trees, tables, and lists.)
- "How does the tree method relate to finding the number of outcomes in a sample space?" (Each path from the start to the end of the "branches" represents one outcome in the sample space, so counting all the paths will give you the number of items in the sample space.)
- "Why is it important to know the number of outcomes in a sample space when finding probability?" (Probability can be found by $\frac{k}{n}$ where k represents the number of outcomes in the event and n represents the number of outcomes in the sample space.)

8.4 Random Points

Cool Down: 5 minutes

The cool-down asks students to count the number of outcomes in the sample space for an experiment with multiple parts. Students should be allowed to use any of the methods explored in this lesson to arrive at the answer. The size of the sample space and the number of items in a subset of the sample space will be used to find probabilities for events in subsequent lessons.

Student Task Statement

Andre is reviewing proportional relationships. He wants to practise using a graph that goes through a point so that each coordinate is between 1 and 10.

1. For the point, how many outcomes are in the sample space?
2. For how many outcomes are the x-coordinate and the y-coordinate the same number?

Student Response

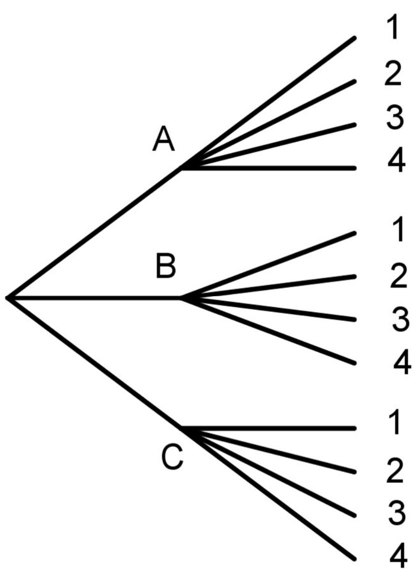
1. There are 100 outcomes in the sample space since $10 \times 10 = 100$.

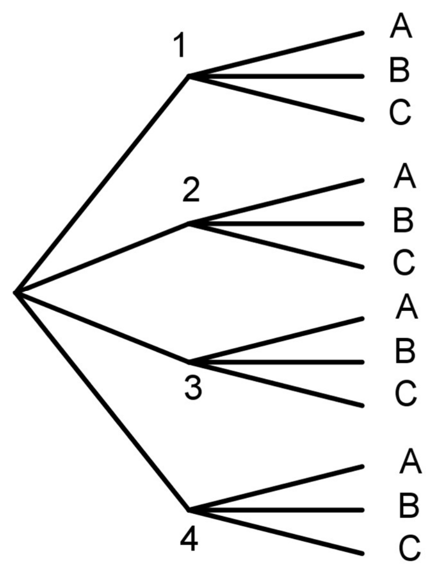
2. 10. (1,1), (2,2), (3,3), (4,4), (5,5), (6,6), (7,7), (8,8), (9,9), (10,10).

Student Lesson Summary

Sometimes we need a systematic way to count the number of outcomes that are possible in a given situation. For example, suppose there are 3 people (A, B, and C) who want to run for the president of a club and 4 different people (1, 2, 3, and 4) who want to run for vice president of the club. We can use a *tree*, a *table*, or an *ordered list* to count how many different combinations are possible for a president to be paired with a vice president.

With a tree, we can start with a branch for each of the people who want to be president. Then for each possible president, we add a branch for each possible vice president, for a total of $3 \times 4 = 12$ possible pairs. We can also start by counting vice presidents first and then adding a branch for each possible president, for a total of $3 \times 4 = 12$ possible pairs.





A table can show the same result:

	1	2	3	4
A	A, 1	A, 2	A, 3	A, 4
B	B, 1	B, 2	B, 3	B, 4
C	C, 1	C, 2	C, 3	C, 4

So does this ordered list:

A1, A2, A3, A4, B1, B2, B3, B4, C1, C2, C3, C4

Lesson 8 Practice Problems

Problem 1 Statement

Noah is planning his birthday party. Here is a tree showing all of the possible themes, locations, and days of the week that Noah is considering.

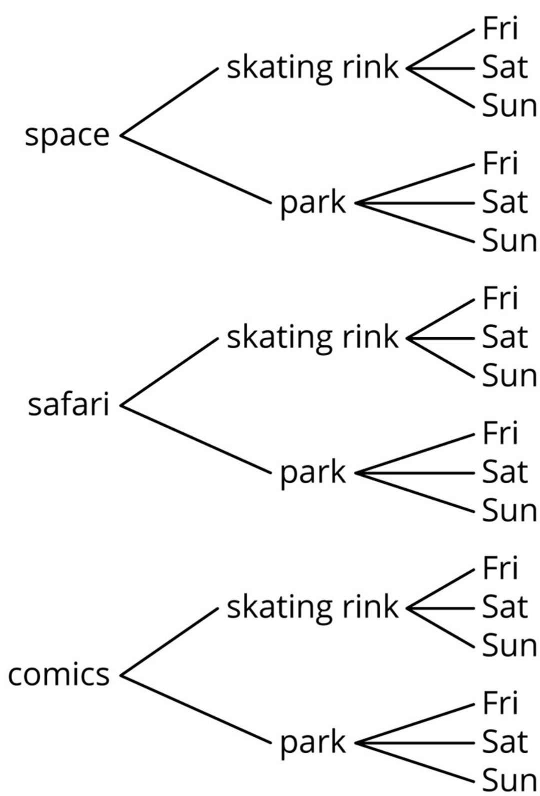
a. How many themes is Noah considering?

b. How many locations is Noah considering?

c. How many days of the week is Noah considering?

d. One possibility that Noah is considering is a party with a space theme at the skating rink on Sunday. Write two other possible parties Noah is considering.

e. How many different possible outcomes are in the sample space?



Solution

a. 3 themes

b. 2 locations

c. 3 days

d. Answers vary. Sample response: Noah is considering a comics themed party at the skating rink on Saturday or a safari themed party at the park on Friday.

e. 18 outcomes (The number of outcomes is given by $3 \times 2 \times 3$ or by counting the branches in the tree diagram.)

Problem 2 Statement

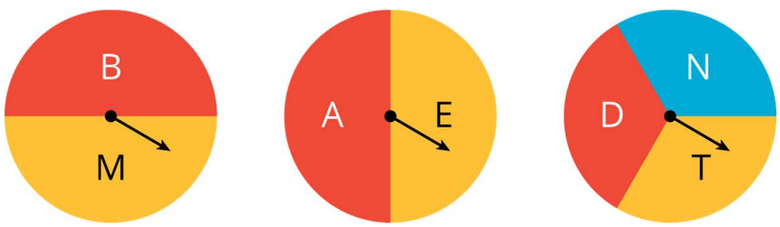
For each event, write the sample space and tell how many outcomes there are.

a. Lin selects one type of lettuce and one dressing to make a salad.

Lettuce types: iceberg, romaine
Dressings: ranch, Italian, French

b. Diego chooses rock, paper, or scissors, and Jada chooses rock, paper, or scissors.

c. Spin these 3 spinners.



Solution

a. 6 outcomes: iceberg and Italian, iceberg and ranch, iceberg and French, romaine and Italian, romaine and ranch, romaine and French

b. 9 outcomes: rr, rp, rs, pr, pp, ps, sr, sp, ss

c. 12 outcomes: bat, bet, mat, met, ban, ben, man, men, bad, bed, mad, med

Problem 3 Statement

A simulation is done to represent kicking 5 penalties in a single game with a 72% probability of making each one. A 1 represents making the kick and a 0 represents missing the kick.

trial	result
1	10101
2	11010
3	00011
4	11111
5	10011

Based on these results, estimate the probability that 3 or more kicks are made.

Solution

$\frac{4}{5}$

Problem 4 Statement

There is a bag of 50 marbles.

- Andre takes out a marble, records its colour, and puts it back in. In 4 trials, he gets a green marble 1 time.
- Jada takes out a marble, records its colour, and puts it back in. In 12 trials, she gets a green marble 5 times.
- Noah takes out a marble, records its colour, and puts it back in. In 9 trials, he gets a green marble 3 times.

Estimate the probability of getting a green marble from this bag. Explain your reasoning.

Solution

Answers vary. Sample response: A good estimate of the probability of getting a green marble comes from combining Andre, Jada, and Noah’s trials. They took a marble out of the bag a total of 25 times and got a green marble 9 of those times. So, the probability of getting a green marble appears to be close to $\frac{9}{25} = 0.36$. Since there are 50 marbles in the bag, it is a reasonable estimate that 18 of the 50 marbles are green, though this is not guaranteed.