Lesson 13: The shadow knows

Goals

- Calculate the unknown heights of objects by using proportional reasoning and explain (orally) the solution method.
- Justify (orally) why the relationship between the height of objects and the length of their shadows cast by the sun is approximately proportional.

Learning Targets

- I can model a real-world context with similar triangles to find the height of an unknown object.

Lesson Narrative

In this lesson, students examine the length of shadows of different objects. There appears to be a proportional relationship between the height of the object and the length of the shadow. Students use this relationship to predict the height of a lamppost given the length of its shadow. In order to justify the proportional relationship, students use the hypothesis that the rays of sunlight making the shadows are parallel, together with their knowledge of similar triangles. Finally, students go outside and make their own measurements of different objects and the lengths of their shadows and use this technique to estimate the height of a tall object.

This lesson involves modelling, not only because students interpret real-world data (both the given heights and shadow lengths and the measurements that they take themselves) but also because they need to make simplifying assumptions in order to justify why the relationship is proportional.

Building On

- Recognise and represent proportional relationships between quantities.

Addressing

- Use informal arguments to establish facts about the angle sum and exterior angle of triangles, about the angles created when parallel lines are cut by a transversal, and the angle-angle criterion for similarity of triangles. For example, arrange three copies of the same triangle so that the sum of the three angles appears to form a line, and give an argument in terms of transversals why this is so.

Building Towards

- Understand that a two-dimensional shape is similar to another if the second can be obtained from the first by a sequence of rotations, reflections, translations, and enlargements; given two similar two-dimensional shapes, describe a sequence that exhibits the similarity between them.

- Use informal arguments to establish facts about the angle sum and exterior angle of triangles, about the angles created when parallel lines are cut by a transversal, and the angle-angle criterion for similarity of triangles. For example, arrange three copies of the same triangle so that the sum of the three angles appears to form a line, and give an argument in terms of transversals why this is so.

Instructional Routines

- Stronger and Clearer Each Time
- Collect and Display
- Notice and Wonder

Required Materials

Measuring tapes

Required Preparation

Before doing the last activity, conduct the experiment ahead of time to ensure that shadow lengths will be cooperative at the time your class takes place. Also, make preparations to take your class outside. They will need measuring devices (tape measures, yard sticks, rulers) as well as a way to record their measurements.

Student Learning Goals

Let's use shadows to find the heights of an object.

13.1 Notice and Wonder: Long Shadows and Short Shadows

Warm Up: 5 minutes

The purpose of this warm-up is to show what happens when shadows are cast from a lamp versus the Sun. Later in this lesson, it is important that students understand that rays of sunlight that hit Earth are essentially parallel. While students may notice and wonder many things about these images, the length of the shadows (which are different for the pens near the lamp and appear to be same for the pens in the sunshine) is the most important discussion point.

The rest of this lesson will look at the shadows of the pens (and other objects) in detail, using the idea that the rays of sunlight hitting the pens are parallel (or nearly so).

Instructional Routines

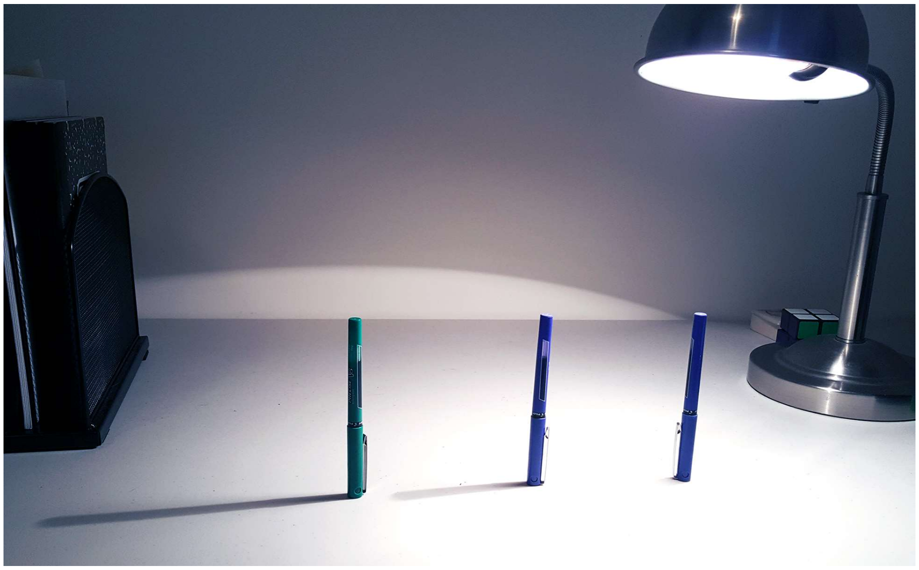
- Collect and Display
- Notice and Wonder

Launch

Arrange students in groups of 2. 2 minutes of quiet think time then share with a partner.

Student Task Statement

What do you notice? What do you wonder?



Student Response

Things students may notice:

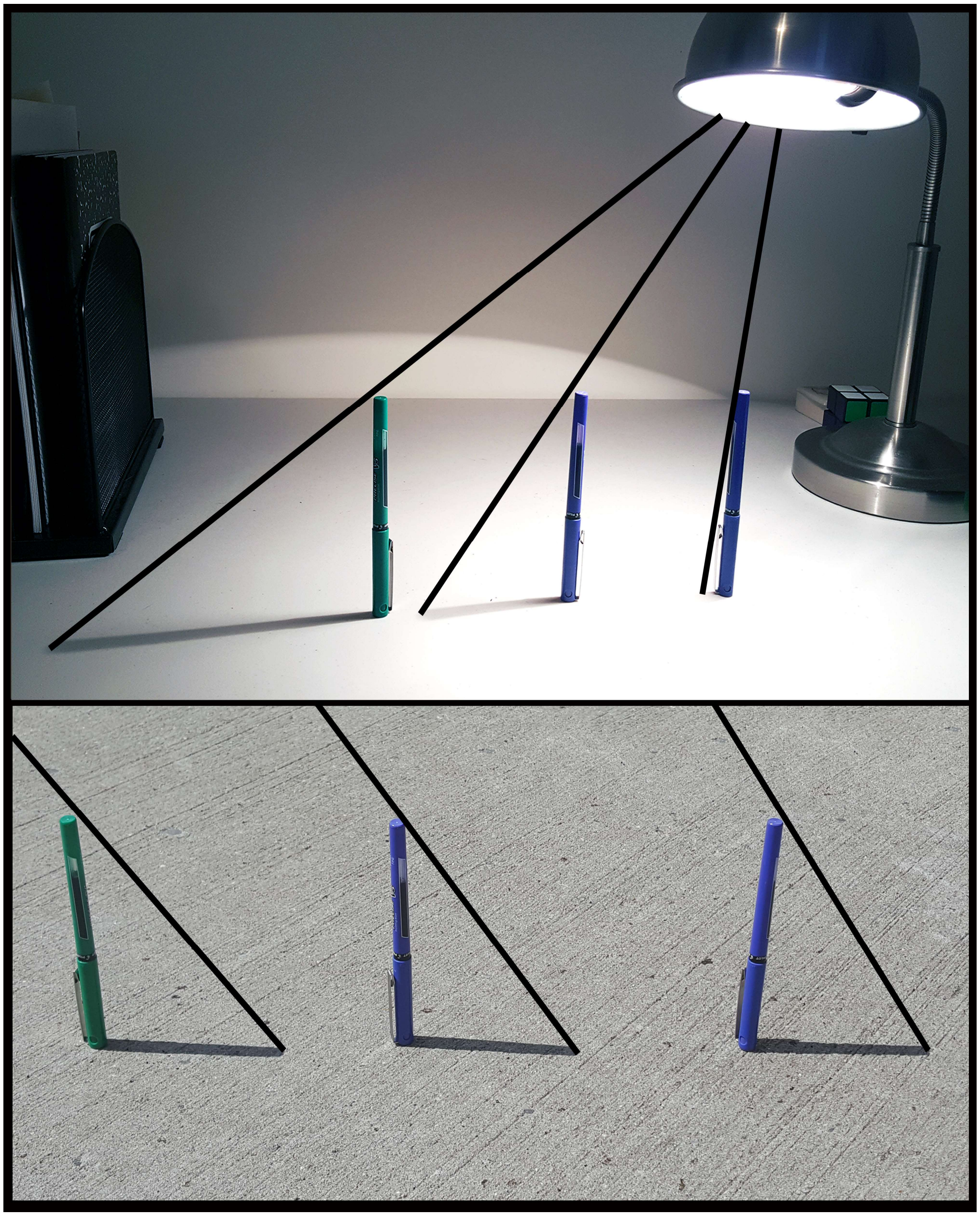
- It's the same pens in both photos.
- The pens have shadows in both photos.
- One photo was taken inside; one outside.
- One light source is a lamp; the other the Sun.
- In the photo where the light source is a lamp, the light is coming from the right. In the other photo, the light is coming from the left.
- In the photo where the light source is a lamp, the shadows are all different lengths. In the photo where the light source is the Sun, the shadows are all the same length.

Things students may wonder:

- Are there times when the pens outdoors don't leave any shadow?
- Why do the pens near the lamp have different length shadows?

Activity Synthesis

Ask students to share the things they noticed and wondered. Record and display their responses for all to see. After each response, ask the class if they agree or disagree and to explain alternative ways of thinking, referring back to the images each time.



If possible, show these images where the path of the light that reaches the top of each shadow is drawn in.

Use *Collect and Display* to gather words and phrases students use when describing why the pens do not have the same length of shadows in the first picture but do in the second. Make explicit connections to illustrate that the Sun's light hits the pens at the same angle.

13.2 Objects and Shadows

15 minutes

In this activity, students look at a photo of three people, a lamppost, and their shadows taken on a sunny day. They notice that there is an approximately proportional relationship between the height of an object and the length of its shadow. Then, they use what they know about proportional relationships and the length of a shadow to find the height of an object that is difficult to measure directly.

In this activity, students just notice that the relationship appears to be proportional based on inspecting several corresponding lengths. In the next activity, they will create a justification for why the relationship is proportional.

The given measurements are real measurements rounded to the nearest inch. Therefore, the given values are not in a perfectly proportional relationship. The quotient of each shadow length and its corresponding object's height is around, but not exactly, $\frac{2}{3}$.

Instructional Routines

- Collect and Display

Launch

Display the photo in the task, and ask students how they would go about measuring the height of each person and the lamppost. It would be straightforward to measure the height of the people using a yard stick or tape measure, but it would be difficult to measure the height of the lamppost. Tell students that even when something is too tall to measure directly, we can still figure out its height by using the length of its shadow (which, since it's on the ground, is easy to measure).

Draw students' attention to the measurements given in the table, and invite students to look for relationships in the table and use any relationships they notice to make a conjecture about the height of the lamppost.

Keep students in the same groups. 2 minutes of quiet work time and then students share thinking and continue working with a partner, followed by a whole-class discussion.

Representation: Internalise Comprehension. Represent the same information through different modalities by using a diagram. If students are unsure where to begin, suggest that they draw a diagram (a horizontal line for the shadow and a vertical line for the height for each person) and label them using the measurements provided in the table.

Supports accessibility for: Conceptual processing; Visual-spatial processing Speaking: Collect and Display. In order for students' own output to become a reference in developing mathematical language for this lesson, listen for and record the language students use to describe how they would measure the height of each person and the lamp post. As students work in pairs to determine the relationship between height and shadow length, listen for how students are comparing the quantities. Record phrases that students use, such as: "When I divide ...." and "Compared to the height, the shadow....". Over the course of this lesson, ask student to suggest revisions and updates to the display as they develop both new mathematical ideas and new ways of communicating ideas.
Design Principle(s): Support sense-making, Maximise meta-awareness

Anticipated Misconceptions

The task uses real measurements that were taken to the nearest inch. Because of the rounding, the values given are not in a perfectly proportional relationship, so students may hesitate to identify the relationship as proportional. If students struggle with this aspect of the activity, suggest that they start by coming up with a range of reasonable values for the height of the post. Also, share with them that the measurements were rounded to the nearest inch, so it's possible that the relationship is imperfect.

Some students may need help understanding the meaning of "conjecture." A simple definition to use is "a reasonable guess."

Student Task Statement

Here are some measurements that were taken when the photo was taken. It was impossible to directly measure the height of the lamppost, so that cell is blank.

	height (inches)	shadow length (inches)
younger boy	43	29
man	72	48
older boy	51	34
lamppost		114

1. What relationships do you notice between an object's height and the length of its shadow?

2. Make a conjecture about the height of the lamppost and explain your thinking.

Student Response

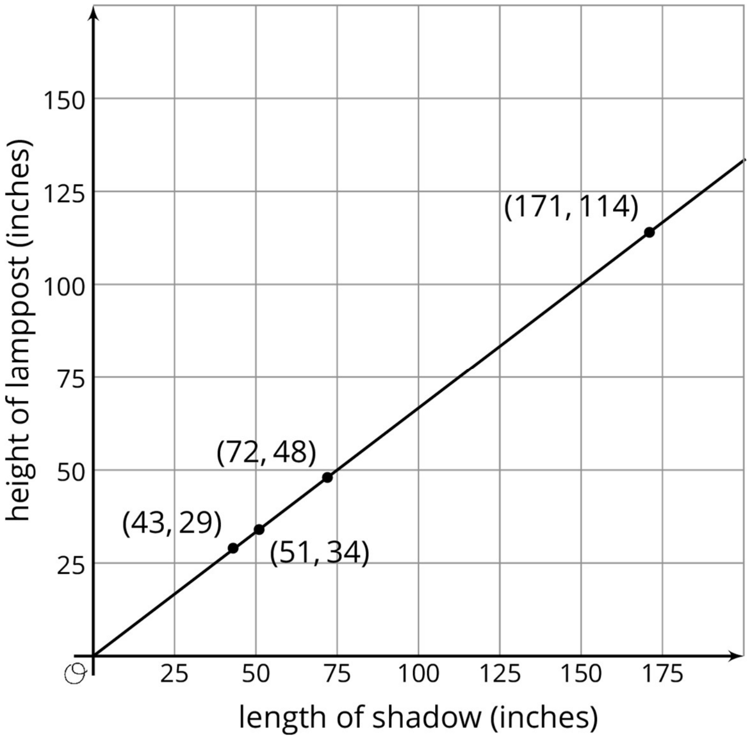
1. There appears to be a proportional relationship between the height of an object and the length of its shadow. All of the shadow lengths are approximately $\frac{2}{3}$ the height of the object.

2. Approximately 171 inches (or 14 feet 3 inches). Possible strategies: $114 \times \frac{3}{2}$, $114 \div \frac{2}{3}$, 150% of 114.

Activity Synthesis

Students should notice that the relationship appears to be approximately proportional. Highlight that the height to shadow length relationship is *not* exactly proportional because, for example, $\frac{43}{29}$ and $\frac{72}{48}$ are not equal (though they are very close). This could be because of rounding error in the measurements or other factors that make the real world differ slightly from a mathematical model. It may be interesting for students to speculate on some reasons (for example, perhaps the ground is not perfectly level, or perhaps one of the people or the lamppost is at a slightly different angle to the ground), though the next activity provides ample opportunity to discuss them, too.

If any students decide to create a graph of the associated heights and shadow lengths, it looks like this (or may have the axes reversed).



13.3 Justifying the Relationship

15 minutes

The purpose of this activity is for students to write a mathematical justification for the proportional relationship between heights and shadow lengths in the photo. A version of the photo is provided with some line segments drawn to strongly suggest an argument based on similar triangles. Given the work they have done up to this point, it is likely that students will recognise that similar triangles will be part of their justification. Many students need help understanding what components are important to include in their arguments.

Because the Sun is so very far away relative to its size, the rays that reach Earth are extremely close to parallel. This is essential in the argument, and you may decide to share this information with the students or ask them to think about the shadows of the pens in the warm-up. The prompt asks to show that the relationship is *approximately proportional,* leaving room for the fact that the Sun's rays may not be exactly parallel, or one of the people is not standing precisely perpendicular to the ground, or the ground may not be perfectly level.

Instructional Routines

- Stronger and Clearer Each Time

Launch

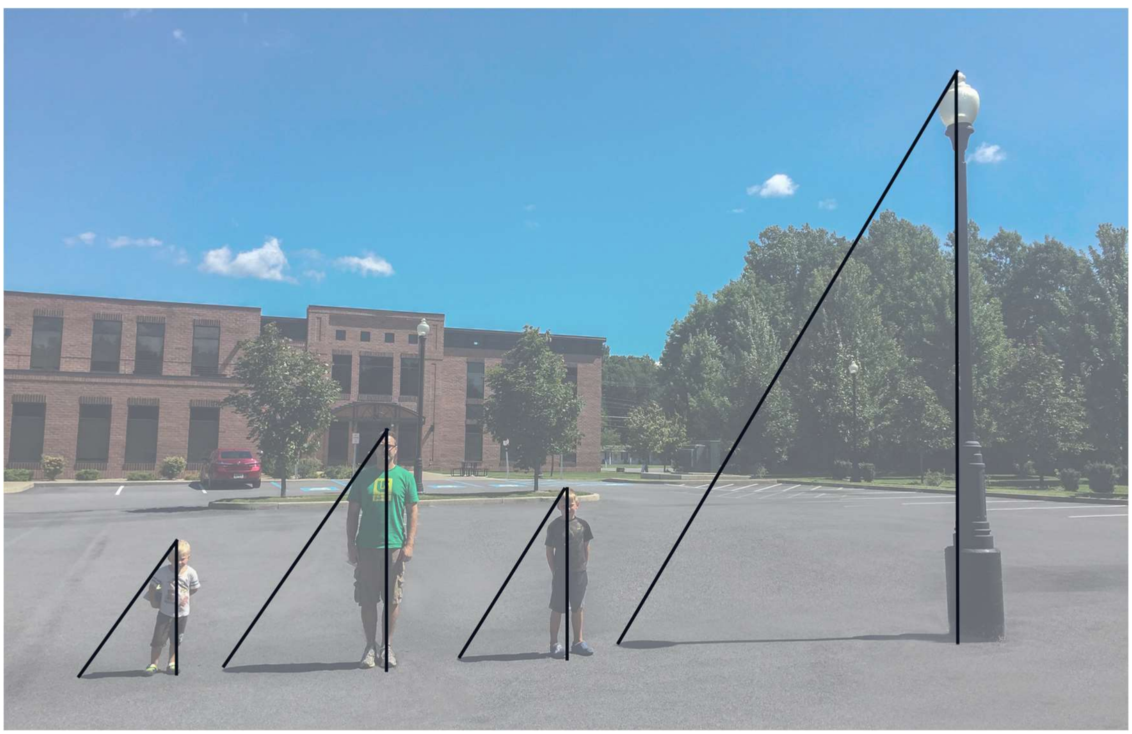
Keep students in the same groups. From the previous activity, the shadow measurements make us *suspect* there may be a proportional relationship between the object heights and the shadow lengths. Tell students that their goal here is to justify *why* the relationship is (approximately) proportional.

Anticipated Misconceptions

Students often struggle with deciding what is important to include in their explanation. A useful technique is to think about what you want to show, and then asking a series of questions. In this activity, we want to show that there is a proportional relationship between the side lengths in some triangles. Questions could include, "What types of triangles have sides that are in proportion? How do you know when triangles are similar triangles? Which pairs of angles do you know are congruent? Why are they congruent?" The answers to these questions are the building blocks of an argument.

Student Task Statement

Explain *why* the relationship between the height of these objects and the length of their shadows is approximately proportional.



Student Response

Choose to focus on one person and the lamppost. Since the Sun's rays making the shadows are approximately parallel, the angles where they strike the ground are congruent corresponding angles. Both the lamppost and the person are perpendicular to the ground, so they are both making right angles with the ground, so those angles are also congruent. The right-angled triangles formed by the lamppost and a person are similar by AA. Therefore, their corresponding sides are proportional.

Activity Synthesis

Debrief as a class. Invite selected students to share their explanations. Ask other students to restate, support, refine, or disagree with their arguments.

Emphasise that it is often necessary to make simplifying assumptions when modelling a real-world situation. For this geometric argument to work, for example, we have to assume that the light rays coming from the Sun are parallel, that the people and the lamppost are perpendicular to the ground, and that the ground is level.

Engagement: Develop Effort and Persistence. Break the class into small discussion groups and then invite a representative from each group to report back to the whole class. *Supports accessibility for: Attention; Social-emotional skills Speaking, Listening: Stronger and Clearer Each Time.* After students respond to the prompt “Explain why the relationship between the height of these objects and the length of their shadows is approximately proportional,” ask each student to meet with 2–3 other partners in a row for feedback. Provide students with prompts for feedback that will help them strengthen their ideas and clarify their language (e.g., “How do you know that the triangles are similar?”, and “Which pairs of angles are congruent?”, etc.). Students can borrow ideas and language from each partner to refine and clarify their original explanation. This will help students revise and refine both their reasoning and their verbal and written output.
Design Principles(s): Optimise output (for explanation); Maximise meta-awareness

13.4 The Height of a Tall Object

20 minutes

This activity could be done on a sunny day. You should try it out ahead of time to ensure that the shadows created in your part of the world at the time your class takes place are cooperative! Either find a tall object outside that all students will find the height of, or let students choose a tall object (for example: a flagpole, a building, or a tree). It should be tall enough that its height can’t be easily measured directly, on level ground, and perpendicular to the ground. Students head outside with tape measures (or other measuring devices) and use what they’ve learned in this lesson to figure out the height of the tall object.

This is a modelling task as the goal of the task is to solve a real-world problem (find the height of some object), and students need to devise and justify a method to do this. They have developed the tools in this lesson but need to apply them appropriately.

Launch

Tell students that they are going to apply what they have learned about shadow lengths of different objects to estimate the height of an object outside. Either tell them which object to use or explain the parameters for choosing an appropriate object.

Students require tape measures, yardsticks, or rulers, and a way to record their measurements. 5–10 minutes to take measurements and do calculations followed by a whole-class discussion.

Engagement: Develop Effort and Persistence. Provide prompts or checklists to increase length of on-task orientation in the face of distractions. For example, provide a checklist that chunks the various steps of the activity into a set of manageable tasks. This checklist may include necessary materials, a list of objects to choose from, steps to find the height, and an exemplar.
Supports accessibility for: Attention; Social-emotional skills

Student Task Statement

1. Head outside. Make sure that it is a sunny day and you take a measuring device (like a tape measure or metre stick) as well as a pencil and some paper.
2. Choose an object whose height is too large to measure directly. Your teacher may assign you an object.
3. Use what you have learned to figure out the height of the object! Explain or show your reasoning.

Student Response

Answers vary.

Activity Synthesis

The main technique that students will apply to solve this problem is likely proportional reasoning. Make sure to highlight the explanation for *why* the relationship is proportional uses angles made by parallel lines cut by a transversal (studied in the previous unit) and properties of similar triangles (a focus of this unit).

Students may think of the height of the object and the length of its shadow as a pair (x, y) lying on a line whose gradient is known. They know the shadow length and are looking for the corresponding height. While this line of reasoning could be considered Year 8 work, students now have the new language of gradient (in addition to unit rate and constant of proportionality) to use in these calculations.

Lesson Synthesis

The activities in this lesson are a good example of using mathematics to model a real-world situation.

- The data is not always perfect (in this case, the object height to shadow length relationship was only approximately proportional).
- To explain why the object height to shadow length relationship might be proportional, simplifying assumptions were needed.
- Mathematical models can be used to make accurate guesses or predictions about quantities that are difficult or impossible to measure directly. In this case, a relationship between the height of an object and the length of its shadow was observed from easier-to-get measurements and justified by reasoning about similar triangles. Once we knew this relationship existed and why it existed, we could reasonably expect the same relationship to hold for a very tall object nearby at the same time of day, and use the length of the tall object's shadow to find its height.
- An interesting historical connection: over 2 000 years ago, the ancient Greek mathematician Eratosthenes also studied shadows closely (in a slightly different way) and used this to estimate the circumference of Earth with an error of less than 2%!