Lesson 11: What is the same?

Goals

- Compare and contrast (orally and in writing) side lengths, angle sizes, and areas using translations, rotations or reflections to explain why a shape is or is not congruent to another.
- Comprehend that congruent shapes have equal corresponding side lengths, angle sizes, and areas.
- Describe (orally and in writing) two shapes that can be moved to one another using a sequence of translations, rotations and reflections as “congruent.”

Learning Targets

- I can decide visually whether or not two shapes are congruent.

Lesson Narrative

In this lesson, students explore what it means for shapes to be “the same” and learn that the term **congruent** is a mathematical way to talk about shapes being the same that has a precise meaning. Specifically, they learn that two shapes are congruent if there is a sequence of translations, rotations, and reflections that moves one to the other. They learn that shapes that are congruent can have different orientations, but corresponding lengths and angles are equal. Agreeing upon and formulating the definition of congruence requires careful use of precise language and builds upon all of the student experiences thus far in this unit, moving shapes and trying to make them match up.

For shapes that are not congruent, what property can be identified in one that is not shared by the other? This could be an angle, a side length, or the size of the shape. For shapes that are congruent, is there any way to tell other than experimenting with tracing paper? In some cases, like the rectangles, students discover that looking at the length and width is enough to decide if they are congruent.

In earlier years, deciding if two shapes are the “same” usually involves making sure that they are the same general shape (for example, triangles or circles) and that the size is the same. As shapes become more complex and as we develop new ways to measure them (angles for example), something more precise is needed. The definition of congruence here states that two shapes are congruent if there is a sequence of translations, rotations, and reflections that matches one shape up exactly with the other. This definition has many advantages:

- It does not require measuring all side lengths or angles.
- It applies equally well to all shapes, not just polygons.
- It is precise and unambiguous: certain moves are allowed and two shapes are congruent when one can be moved to align exactly with the other.

The material treated here will be taken up again later on from a more abstract point of view. It is essential for students to gain experience executing translations, rotations and reflections with a variety of tools (tracing paper, coordinates, technology) to develop the intuition that they will need when they study these moves (or transformations) in greater depth later.

Addressing

- Verify experimentally the properties of rotations, reflections, and translations:
- Understand that a two-dimensional shape is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent shapes, describe a sequence that exhibits the congruence between them.

Building Towards

- Understand that a two-dimensional shape is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent shapes, describe a sequence that exhibits the congruence between them.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Stronger and Clearer Each Time
- Collect and Display
- Think Pair Share

Required Materials

Geometry toolkits
tracing paper, graph paper, coloured pencils, scissors, and an index card to use as a straightedge or to mark right angles, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Required Preparation

Coloured pencils are supposed to be a usual part of the geometry toolkit, but they are called out here because one activity asks students to shade rectangles using different colours.

Student Learning Goals

Let’s decide whether shapes are the same.

11.1 Find the Right Hands

Warm Up: 5 minutes

In this activity, students get their first formal introduction to the idea of mirror orientation. The easiest way to decide which are the right hands is to hold one's hands up and rotate them until they match a particular shape (or don't). This prepares them for a discussion about whether shapes with different mirror orientation are the same or not.

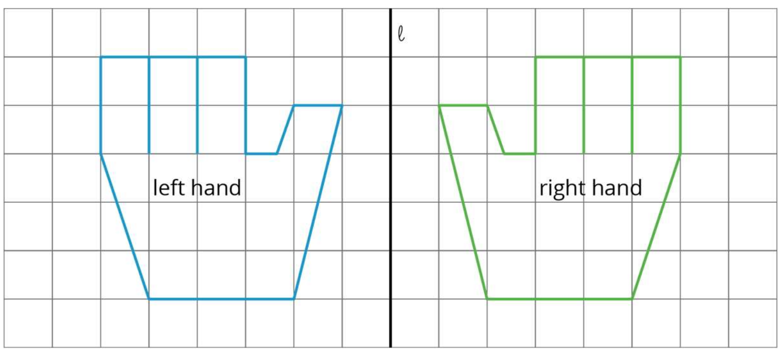
Instructional Routines

- Think Pair Share

Launch

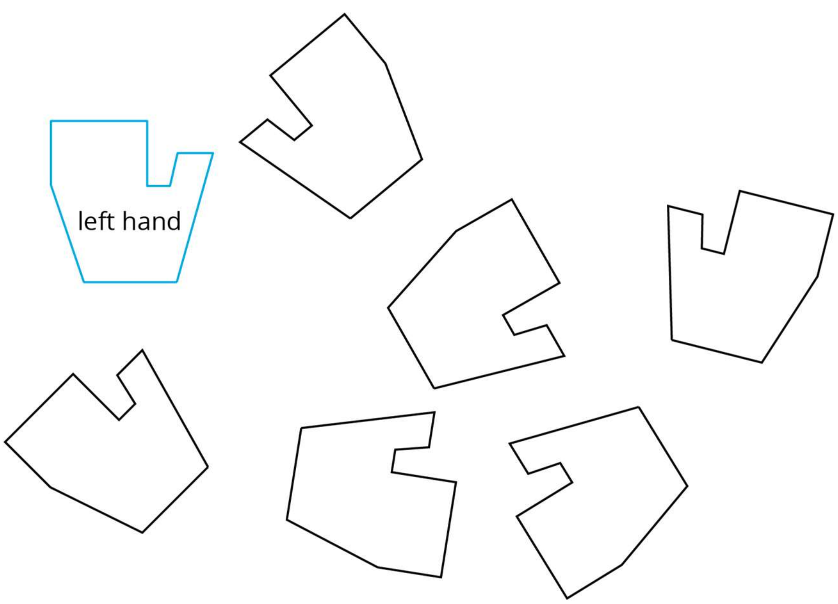
Arrange students in groups of 2, and provide access to geometry toolkits. Give 2 minutes of quiet work time, followed by time for sharing with a partner and a whole-class discussion.

Show students this image or hold up both hands and point out that our hands are mirror images of each other. These are hands shown from the back. If needed, clarify for students that all of the hands in the task are shown from the back.

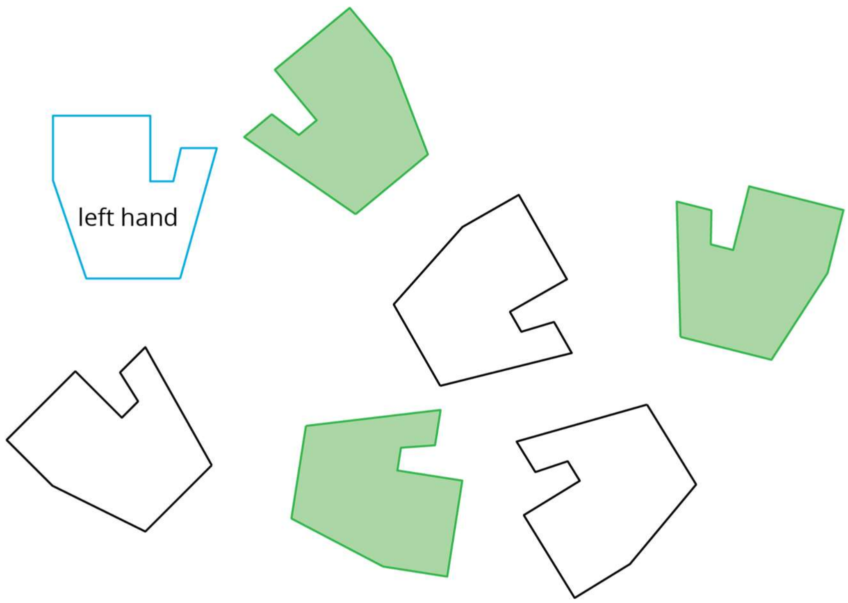


Student Task Statement

A person's hands are mirror images of each other. In the diagram, a left hand is labelled. Shade all of the right hands.



Student Response



Activity Synthesis

Ask students to think about the ways in which the left and right hands are the same, and the ways in which they are different.

Some ways that they are the same include:

- The side lengths and angles on the left and right hands match up with one another.
- If a left hand is flipped, it can match it up perfectly with a right hand (and vice versa).

Some ways that they are different include:

- They cannot be lined up with one another *without* flipping one of the hands over.
- It is not possible to make a physical left and right hand line up with one another, except as "mirror images."

11.2 Are They the Same?

15 minutes

In previous work, students learned to identify translations, rotations, and reflections. They started to study what happens to different shapes when these transformations are applied. They used sequences of translations, rotations, and reflections to build new shapes and to study complex configurations in order to compare, for example, vertical angles made by a pair of intersecting lines. Starting in this lesson, translations, rotations and reflections are used to formalise what it means for two shapes to be the same, a notion which students have studied and applied since primary school.

In this activity, students express what it means for two shapes to be the same by considering carefully chosen examples. Students work to decide whether or not the different pairs of shapes are the same. Then the class discusses their findings and comes to a consensus for what it means for two shapes to be the same: the word "same" is replaced by "congruent" moving forward.

There may be discussion where a reflection is required to match one shape with the other. Students may disagree about whether or not these should be considered the same and discussion should be encouraged.

Monitor for students who use these methods to decide whether or not the shapes are the same and invite them to share during the discussion:

- Observation (this is often sufficient to decide that they are not the same): Encourage students to articulate what feature(s) of the shapes help them to decide that they are not the same.
- Measuring side lengths using a ruler or angles using a protractor: Then use differences among these measurements to argue that two shapes are not the same.
- Cutting out one shape and trying to move it on top of the other: A variant of this would be to separate the two images and then try to put one on top of the other or use tracing paper to trace one of the shapes. This is a version of applying transformations studied extensively prior to this lesson.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Collect and Display

Launch

Give 5 minutes of quiet work time followed by a whole-class discussion. Provide access to geometry toolkits.

Action and Expression: Develop Expression and Communication. Invite students to talk about their ideas with a partner before writing them down. Display sentence frames to support students when they explain their ideas. For example, "This pair of shapes is/is not the same because..." or "If I translate/rotate/reflect, then…."
Supports accessibility for: Language; Organisation Conversing, Representing: Collect and Display. As students work on comparing shapes, circulate and listen to students talk. Record common or important phrases (e.g., side length, rotated, reflected, etc.), together with helpful sketches or diagrams on a display. Pay particular attention to how students are using transformational language while determining whether the shapes are the same. Write students' words and sketches on a visual display to refer back to during whole-class discussions throughout this lesson and the rest of the unit. This will help students use

mathematical language during their group and whole-class discussions.
Design Principle(s): Support sense-making

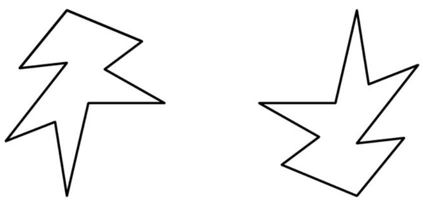
Anticipated Misconceptions

Students may think all of the shapes are the same because they are the same general shape at first glance. Ask these students to look for any differences they can find among the pairs of shapes.

Student Task Statement

For each pair of shapes, decide whether or not they are the same.

A



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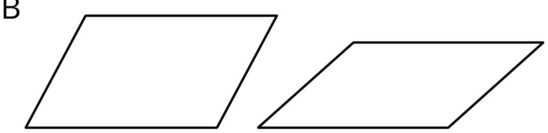

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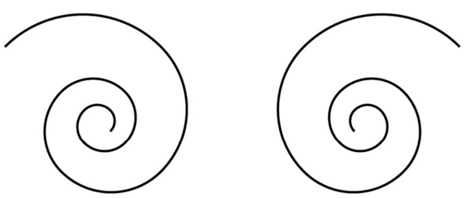
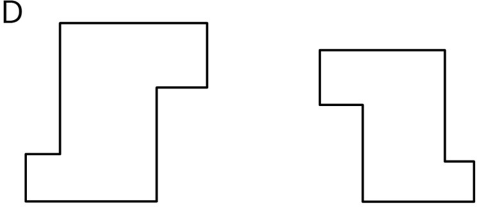
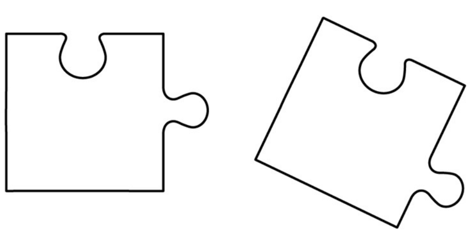


D



E



Student Response

1. The two shapes are the same. Rotating the shape on the left (by 180 degrees) around the top point and moving it down and to the right it matches up perfectly with the shape on the right.
2. They are not the same. Possible strategy: The side lengths of the shapes are the same but the angles are not. The shape on the right is more squished down (and has less area) so they are not the same.
3. They are the same. Possible strategy: The shapes are both curly arrows and look like they are the same size. Reflecting in a vertical line halfway between the two shapes, they appear to match up perfectly with one another. *Or:* They are not the same. Possible strategy: The curly arrow on the left moves in a clockwise direction while the curly arrow on the right moves in an anti-clockwise direction.
4. They are not the same. Possible strategy: The general shapes are the same and the angles match up but the side lengths are different. The shape on the left is bigger than the shape on the right.
5. They are not the same. Possible strategy: The part that sticks out of the right side is higher on the first piece and lower on the second piece. Building a puzzle, both shapes would not fit in the same spot.

Activity Synthesis

For each pair of shapes, poll the class. Count how many students decided each pair was the same or not the same. Then for each pair of shapes, select at least one student to defend their reasoning. (If there is unanimous agreement over any of the pairs of shapes, these can be dealt with quickly, but allow the class to hear at least one argument for each pair of shapes.)

Sequence these explanations in the order suggested in the Activity Narrative: general observations, taking measurements, and applying translations, rotations and reflections with the aid of tracing paper.

The most general and precise of these criteria is the third which is the foundation for the mathematical definition of congruence: The other two are consequences. The moves allowed by translations, rotations and reflections do not change the shape, size, side lengths, or angles.

There may be disagreement about whether or not to include reflections when deciding if two shapes are the same. Here are some reasons to include reflections:

- A shape and its reflected image can be matched up perfectly (using a reflection).
- Corresponding angles and side lengths of a shape and its reflected image are the same.

And here are some reasons against including reflections:

- A left foot and a right foot (for example) do not work exactly the same way. If we literally had two left feet it would be difficult to function normally!
- Translations and rotations can be enacted, for example, by putting one sheet of tracing paper on top of another and physically translating or rotating it. For a reflection the typical way to do this is to lift one of the sheets and flip it over.

If this disagreement doesn't come up, ask students to think about why someone might conclude that the pair of shapes in C were not the same. Explain to students that people in the world can mean many things when they say two things are "the same." In mathematics there is often a need to be more precise, and one *kind of* "the same" is **congruent**. (Two shapes *are* congruent if one is a reflection of the other, but one could, if one wanted, define a different term, a different kind of "the same," where flipping was not allowed!)

Explain that shape A is **congruent** to shape B if there is a sequence of translations, rotations, and reflections which make shape A match up exactly with shape B.

Combining this with the earlier discussion a few general observations about congruent shapes include

- Corresponding sides of congruent shapes are congruent.
- Corresponding angles of congruent shapes are congruent.
- The areas of congruent shapes are equal.

What can be "different" about two congruent shapes? The location (they don't have to be on top of each other) and the orientation (requiring a reflection to move one to the other) can be different.

11.3 Area, Perimeter, and Congruence

10 minutes

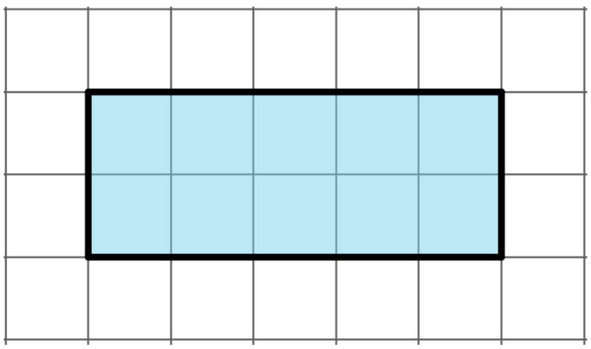
Sometimes people characterise congruence as "same size, same shape." The problem with this is that it isn't clear what we mean by "same shape." All of the shapes in this activity have the same shape because they are all rectangles, but they are not all congruent. Students examine a set of rectangles and classify them according to their area and perimeter. Then they identify which ones are congruent. Because congruent shapes have the same side lengths, congruent rectangles have the same perimeter. But rectangles with the same perimeter are *not* always congruent. Congruent shapes, including rectangles, also have the same area. But rectangles with the same area are *not* always congruent.

Instructional Routines

- Stronger and Clearer Each Time
- Think Pair Share

Launch

Tell students that they will investigate further how finding the area and perimeter of a shape can help show that two shapes are not congruent. It may have been a while since students have thought about the terms area and perimeter. If necessary, to remind students what these words mean and how they can be calculated, display a rectangle like this one for all to see. Ask students to explain what perimeter means and how they can find the perimeter and area of this rectangle.



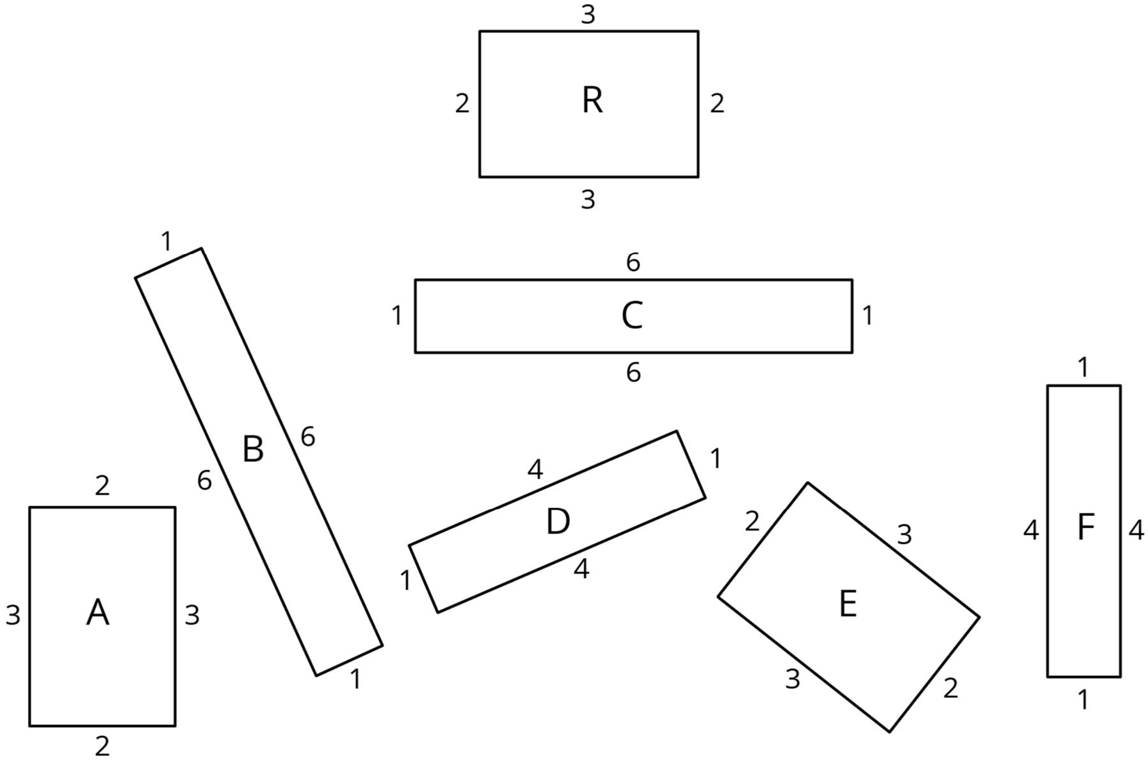
Arrange students in groups of 2. Provide access to geometry toolkits (coloured pencils are specifically called for). Give 2 minutes for quiet work time followed by sharing with a partner and a whole-class discussion.

Representation: Internalise Comprehension. Chunk this task into more manageable parts to differentiate the degree of difficulty or complexity. For example, provide students with a subset of the rectangles to start with and introduce the remaining rectangles once students have completed their initial set of comparisons.
Supports accessibility for: Conceptual processing; Organisation

Anticipated Misconceptions

Watch for students who think about the final question in terms of “same shape and size.” Remind them of the definition of congruence introduced in the last activity.

Student Task Statement



1. Which of these rectangles have the same area as rectangle R but different perimeter?

2. Which rectangles have the same perimeter as rectangle R but different area?

3. Which have the same area *and* the same perimeter as rectangle R?

4. Use materials from the geometry tool kit to decide which rectangles are **congruent**. Shade congruent rectangles with the same colour.

Student Response

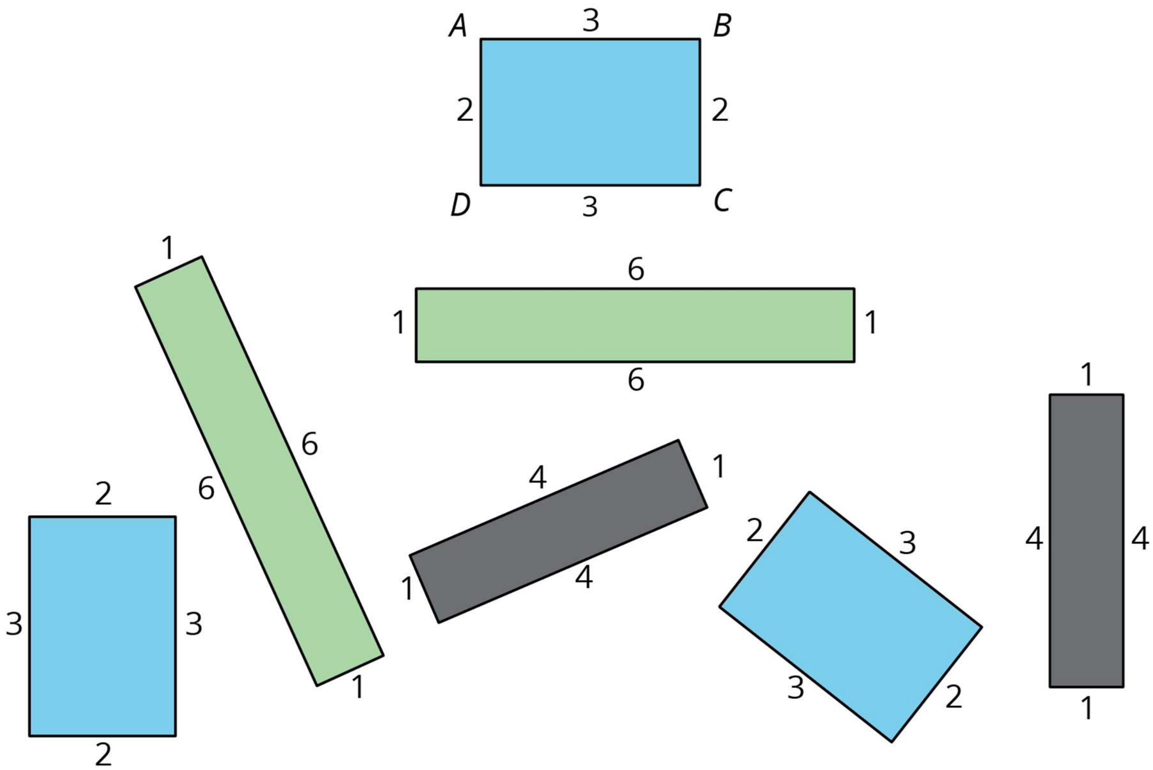
The perimeter of rectangle R is 10 units since $3 + 2 + 3 + 2 = 10$ while its area is 6 square units since $2 \times 3 = 6$. All of the rectangles in the picture share at least one of these properties (either the perimeter or the area) but only the 2 unit by 3 unit rectangles share both:

1. Rectangles B and C have the same area (6 square units) but different perimeter (14 units)

2. Rectangles D and F have the same perimeter (10 units) but different area (4 square units)

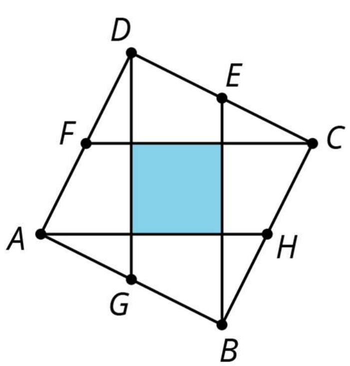
3. Rectangles A and E have the same area and perimeter: only their position and orientation on the page is different.

4. The 2 by 3 rectangles are congruent to rectangle R. In each case, rectangle R can be translated and rotated so that it matches up perfectly with the 2-by-3 rectangle. The same argument shows that rectangles B and C are congruent as are rectangles D and F.



Are You Ready for More?

In square $ABCD$, points E, F, G, and H are midpoints of their respective sides. What fraction of square $ABCD$ is shaded? Explain your reasoning.



Student Response

$\frac{1}{5}$ of square $ABCD$ is shaded. Reasoning varies. Sample reasoning: Transform the unshaded pieces into four congruent squares that are each congruent to the shaded square.

It is interesting to generalise this problem such that points E, F, G, and H partition the sides of $ABCD$ in a ratio other than 1: 1.

Activity Synthesis

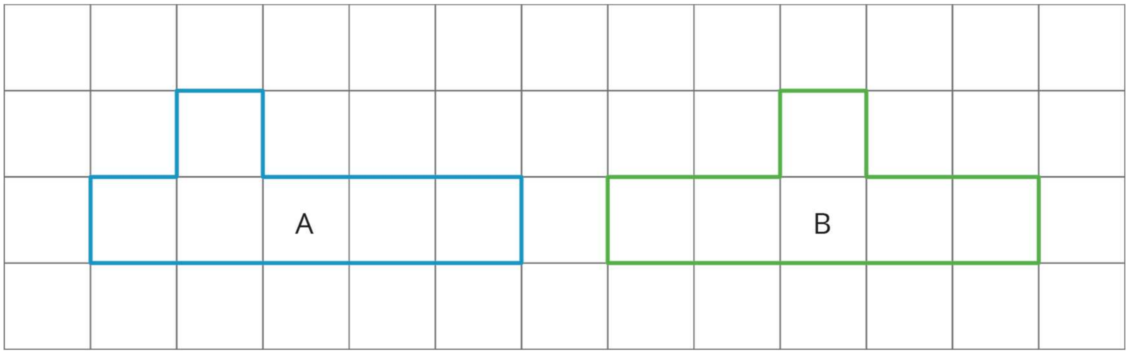
Invite students who used the language of transformations to answer the final question to describe how they determined that a pair of rectangles are congruent.

Perimeter and area are two different ways to measure the size of a shape. Ask the students:

- "Do congruent rectangles have the same perimeter? Explain your reasoning." (Yes. translations, rotations and reflections do not change distances, and so congruent rectangles have the same perimeter.)
- "Do congruent rectangles have the same area? Explain your reasoning." (Yes. translations, rotations and reflections do not change area *or* translations, rotations and reflections do not change distances and so do not change the length times the width in a rectangle.)
- "Are rectangles with the same perimeter always congruent?" (No. Rectangles D and F have the same perimeter but they are not congruent.)
- "Are rectangles with the same area always congruent?" (No. Rectangles B and C have the same area but are not congruent.)

One important take away from this lesson is that measuring perimeter and area is a good method to show that two shapes are *not* congruent if these measurements differ. When the measurements are the same, more work is needed to decide whether or not two shapes are congruent.

A risk of using rectangles is that students may reach the erroneous conclusion that if two shapes have both the same area and the same perimeter, then they are congruent. If this comes up, challenge students to think of two shapes that have the same area and the same perimeter, but are not congruent. Here is an example:



Writing, Speaking: Stronger and Clearer Each Time. Use this routine to give students a structured opportunity to revise their written strategies for deciding which rectangles are congruent. Give students time to meet with 2–3 partners to share and get feedback on their

responses. Display prompts for feedback that will help individuals strengthen their ideas and clarify their language. For example, "How was a sequence of transformations used to…?", "What properties do the shapes share?", and "What was different and what was the same about each pair?" Students can borrow ideas and language from each partner to strengthen their final product.
Design Principle(s): Optimise output (for explanation)

Lesson Synthesis

Ask students to state their best definition of **congruent**. (Two shapes are congruent when there is a sequence of translations, rotations, and reflections that take one shape to the other.)

Some important concepts to discuss:

- "How can you check if two shapes are congruent?" (For rectangles, the side lengths are enough to tell. For more complex shapes, experimenting with transformations is needed.)
- "Are a shape and its mirror image congruent?" (Yes, because a reflection takes a shape to its mirror image.)
- "What are some ways to know that two shapes are not congruent?" (Two shapes are not congruent if they have different areas, side lengths, or angles.)
- "What are some properties that are shared by congruent shapes?" (They have the same number of sides, same length sides, same angles, same area.)

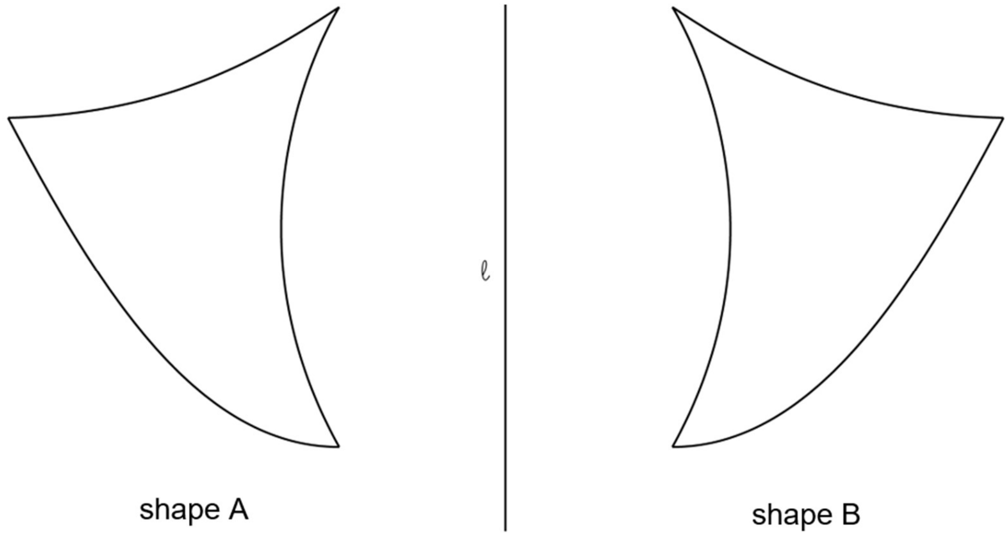
11.4 Mirror Images

Cool Down: 5 minutes

Throughout this unit, students have been using translations, rotations, and reflections to move shapes in the plane. In this lesson, students have learned that shape A is congruent to shape B when there is a sequence of translations, rotations, and reflections that take shape A to shape B. Here they apply this to two non-polygonal shapes, one of which is a reflection of the other.

Student Task Statement

Shape B is the image of shape A when reflected in line ℓ. Are shape A and shape B congruent? Explain your reasoning.

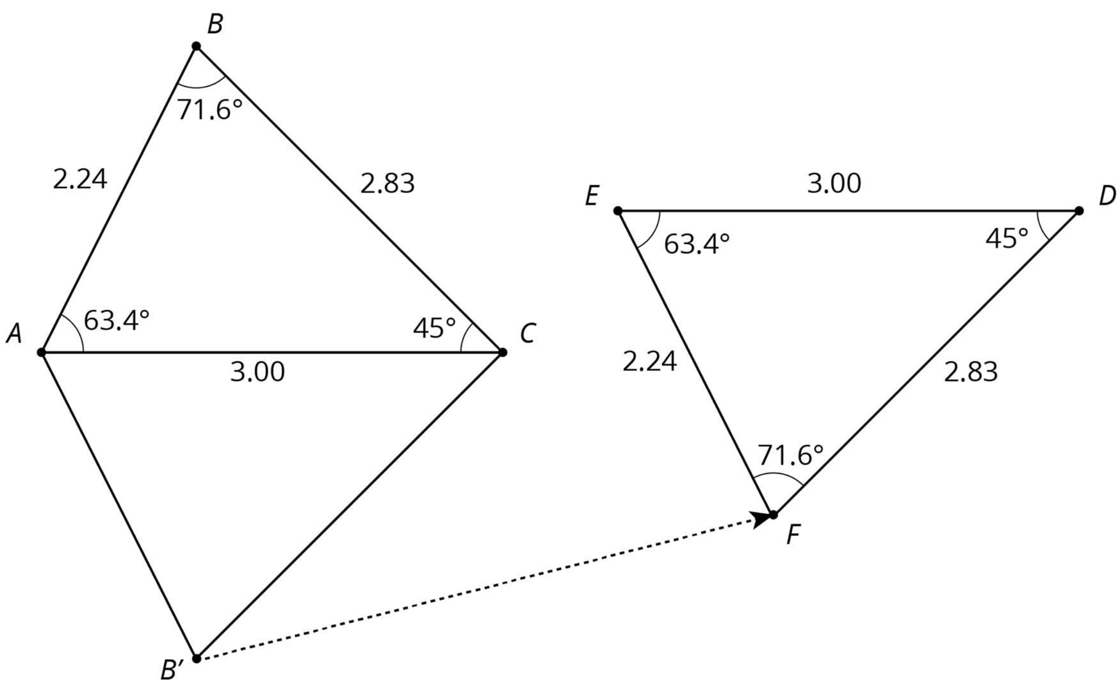


Student Response

Yes, they are congruent. There is a translation, rotation or reflection that takes one shape to the other, so they are congruent.

Student Lesson Summary

Congruent is a new term for an idea we have already been using. We say that two shapes are congruent if one can be lined up exactly with the other by a sequence of translations, rotations and reflections. For example, triangle EFD is congruent to triangle ABC because they can be matched up by reflecting triangle ABC in AC followed by the translation shown by the arrow. Notice that all corresponding angles and side lengths are equal.



Here are some other facts about congruent shapes:

- We don’t need to check all the measurements to prove two shapes are congruent; we just have to find a sequence of translations, rotations and reflections that match up the shapes.
- A shape that looks like a mirror image of another shape can be congruent to it. This means there must be a reflection in the sequence of transformations that matches up the shapes.
- Since two congruent polygons have the same area and the same perimeter, one way to show that two polygons are *not* congruent is to show that they have a different perimeter or area.

Glossary

- congruent

Lesson 11 Practice Problems

1. **Problem 1 Statement**

   If two rectangles have the same perimeter, do they have to be congruent? Explain how you know.

   Solution

   No. Two non-congruent rectangles can have the same perimeter. For example, a rectangle with side lengths 3 inches and 4 inches is not congruent to a rectangle with side lengths 2 inches and 5 inches. Even though the angles of all rectangles have the same size, when two shapes are congruent all side lengths and angles are the same.

2. **Problem 2 Statement**

   Draw two rectangles that have the same area, but are *not* congruent.

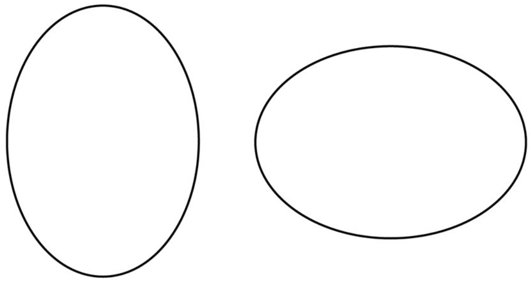
   Solution

   Answers vary. Sample response: a 2-by-6 rectangle and a 3-by-4 rectangle.

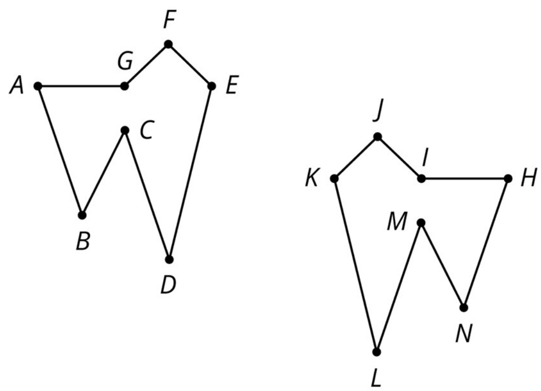
3. **Problem 3 Statement**

   For each pair of shapes, decide whether or not the two shapes are congruent. Explain your reasoning.

   a.



b.



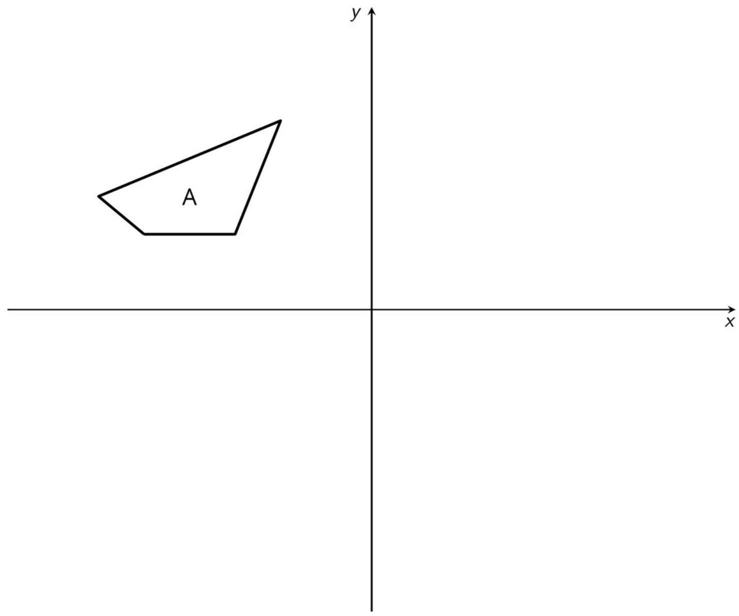
Solution

a. These appear to be congruent. If the shape on the right is traced, it can be moved over and it appears to match up perfectly with the shape on the left. This can be done with a rotation (90 degrees clockwise) and then a translation.

b. These appear to be congruent. If $ABCDEFG$ is reflected in a vertical line and then translated, it appears to land on top of $HNMLKJI$.

4. **Problem 4 Statement**

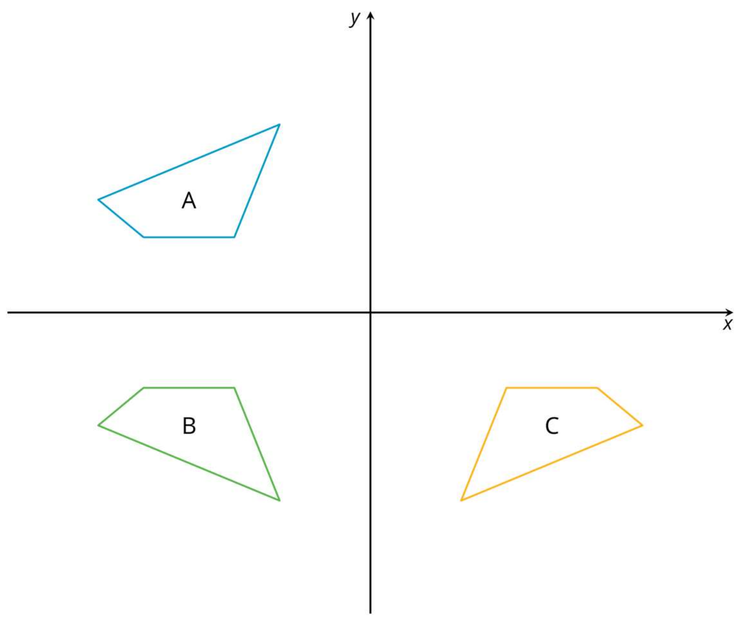
a. Reflect quadrilateral A in the x-axis. Label the image quadrilateral B. Reflect quadrilateral B in the y-axis. Label the image C.



b. Are quadrilaterals A and C congruent? Explain how you know.

Solution

a.



b. Yes, because there is a translation, rotation or reflection taking A to C, the two shapes are congruent.

5. **Problem 5 Statement**

The point (-2,-3) is rotated 90 degrees anti-clockwise using centre (0,0). What are the coordinates of the image?

a. (-3,-2)

b. (-3,2)

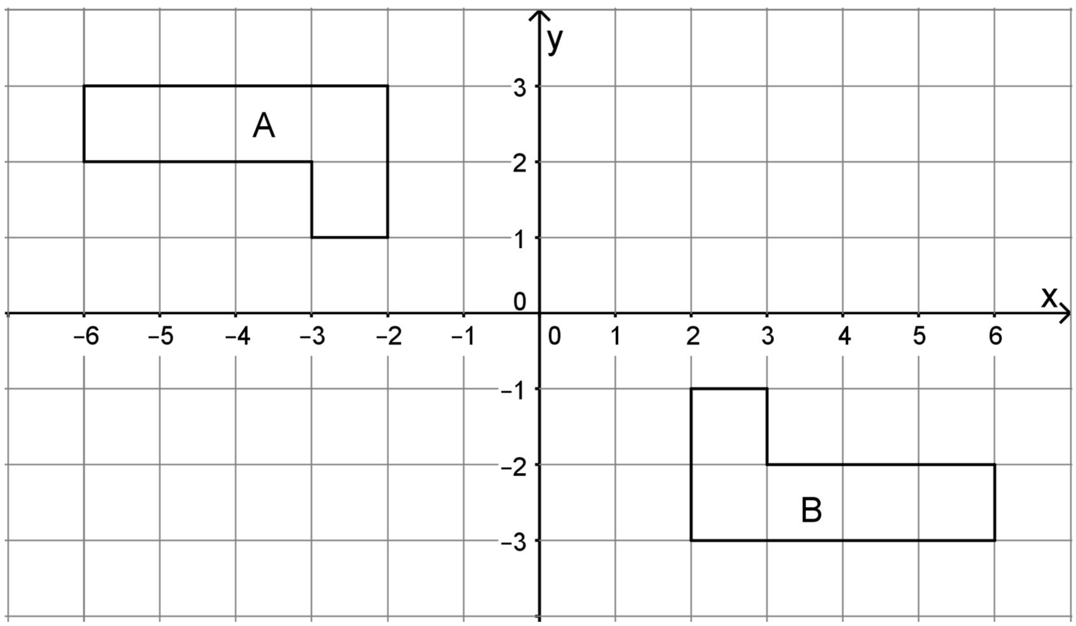
c. (3,-2)

d. (3,2)

Solution C

6. **Problem 6 Statement**

Describe a transformation that takes polygon A to polygon B.



Solution

Answers vary. Sample response: Rotate polygon A 180 degrees around (0,0).