

## Lesson 5: More than two choices

### Goals

- Apply reasoning about ratios and percentages to analyse (orally and in writing) voting situations involving more than two choices.
- Choose and justify (orally) which voting system seems the fairest for dealing with more than two choices.
- Compare and contrast (orally and in writing) different voting systems for dealing with more than two choices, i.e., plurality, runoff, and instant runoff.

### Lesson Narrative

This lesson is optional. It is the second of three lessons that explores the mathematics of voting. The activities in this lesson build on each other and on the previous lesson. As with all lessons in this unit, all related topics have been addressed in prior units; this lesson provides an *optional* opportunity to go more deeply and make connections between domains.

The five activities in this lesson deal with elections in which there are more than two choices. For example, if there are three choices, then the top vote getter might be approved by only 34% of the voters. Students explore several different rules for determining the winner: plurality, runoff, and instant runoff, and discover that the rules can give different results from the same set of voter preferences. They think about which voting rule more fairly represents the opinions of the voters. The mathematics in these activities emphasises quantitative reasoning in a real-world situation.

Most of the activities use students' skills from earlier units to reason about ratios and proportional relationships in the context of real-world problems. While some of the activities do not involve much calculation, they all require serious thinking.

Most importantly, this lesson addresses topics that are important for citizens in a democracy to understand. Teachers may wish to collaborate with a citizenship/PSHE teacher to learn how the fictional KS3 situations in this lesson relate to real-world elections.

### Building On

- Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, bar models, double number line diagrams, or equations.

### Addressing

- Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, bar models, double number line diagrams, or equations.
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## Instructional Routines

- Discussion Supports

## Student Learning Goals

Let's explore different ways to determine a winner.

## 5.1 Cross Curriculum Day

### Optional: 5 minutes

This is the first of five activities about elections where there are more than two choices. This introductory activity gets students thinking about the fairness of a voting rule. If the choice with the most votes wins, it's possible that the winning choice was preferred by only a small percentage of the voters.

### Launch

Students work alone and share solutions with whole class.

### Student Task Statement

Students in a year 7 class were asked, "What activity would you most like to do for cross curriculum day?" The results are shown in the table.

activity	number of votes
softball game	16
scavenger hunt	10
dancing talent show	8
marshmallow throw	4
no preference	2

1. What percentage of the class voted for softball?
2. What percentage did not vote for softball as their first choice?

### Student Response

1. 40% voted for softball since  $\frac{16}{40} = \frac{2}{5} = 40\%$ .
2. 60% did not vote for softball, at least as their first choice.

### Activity Synthesis

Poll the class about the answers. (40% of the class voted for softball, so a majority of the class did not vote for softball as their first choice.) Ask if it is possible to determine whether softball was a highly rated choice by those who voted for another cross curriculum activity. (Not without holding another vote.)

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In this voting system the *plurality* wins, the choice with the most votes, even if it is less than 50%.

## 5.2 School Lunches (Part 1)

### Optional: 30 minutes

This activity presents a method for deciding the winner of an election with more than two choices: runoff voting. If no choice has a majority of votes, then one or more choices with the fewest votes are eliminated and another vote is held between the remaining choices. Repeat until one choice gets a majority of the votes.

Students learn the technique of analysing the results by holding their own vote. A fictitious story (choosing a company to supply school lunches) is provided for students to vote on a situation with four choices, each of which may have some positive and negative aspects. They follow two different systems of voting rules to see how results can differ depending on the rule system used. Students use quantitative reasoning to analyse and compare two different voting rules.

Note: This activity includes a lot of teacher-directed voting activity. Students periodically stop to record information and determine the winner, or the need to do another round of voting, or reflect on the results.

Here is the situation to vote on: Imagine the kitchen that usually prepares our school lunches is closed for repairs for a week. We get to choose which of four catering companies will feed everyone for a week. You can choose only one caterer. The school has found four catering companies that will supply a week of lunches for everyone. No changes in the menus are possible.

Make sure students understand the situation. Students vote by drawing symbols next to the four menu choices or on pre-cut voting slips of paper.

choice	symbol
A. Meat Lovers	
B. Vegetarian	
C. Something for Everyone	
D. Concession Stand	

### Voting System #1. Plurality: Conduct a vote using the plurality wins voting system:

If there is an even number of students in the class, vote yourself to prevent a tie at the end. Ask students to raise their hand if the lunch plan was their first choice. Record the votes in a table for all to see.

lunch plan	number of votes
A. Meat Lovers	

B. Vegetarian	
C. Something for Everyone	
D. Concession Stand	

Students work through questions 1 and 2 in the activity in groups. Then they discuss question 3 as a whole class.

"How could we measure how satisfied people are with the result? For example, people whose top choice was the winner will be very satisfied. People whose last choice was the winner will be very dissatisfied."

Students vote with a show of hands, and record the votes.

what choice did you rank the winner?	number of people	% of people
top choice (star)		
second choice (smiley)		
third choice (square face)		
last choice (X)		

### **Voting system #2: Runoff**

Students work through questions 4–6 alternating between conducting the next round of voting as a whole class and analysing the results in their groups. "Use your same choices that you recorded. We'll count the votes in a different, more complicated way. If one choice did not get a majority, we hold a runoff vote. Eliminate the choice that got the fewest votes. Then we vote again. If your first choice is out, vote for your second choice."

- Record the votes in a table like the first one, except that one of the choices will be gone.
- "Did the same choice get the most votes both times?" (Sometimes no. Results may vary in your class.)
- "Did one of the choices get a majority?" (If so, that choice wins. If not, eliminate the choice with the fewest votes and vote again. Repeat until one choice gets a majority of the votes.)

Again ask for satisfaction with the results of the voting. Record numbers in column 2.

What choice did you rank the winner?	Number of people	% of people
top choice (star)		
second choice (smiley)		
third choice (square face)		
last choice (X)		

Students calculate percentages in the last column and work on question 7 in groups.

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## Instructional Routines

- Discussion Supports

### Launch

Arrange students in groups of 2–4.

Introduce the situation: "When there are more than two choices, it's often hard to decide which choice should win. For example, in the cross curriculum day question, softball got the most votes, but only 40% of the votes were for softball and 60% were not for softball. But were these really votes against softball, or did some of those people like softball, but just liked another choice more?"

In this lesson, we'll try two voting systems. We'll vote on an imaginary situation: choosing a caterer to supply student lunches."

See the Activity Narrative for instructions for conducting the activity.

*Representation: Internalise Comprehension.* Activate or supply background knowledge about the fairness of a voting rule. Some students may benefit from watching a physical demonstration of the runoff voting process. Invite students to engage in the process by offering suggested directions as you demonstrate.

*Supports accessibility for: Visual-spatial processing; Organisation* *Conversing: Discussion Supports.* Prior to voting and calculating results, invite discussion about the four menus as part of the democratic process. Display images of any foods that are unfamiliar to students such as, hummus, liver, pork cutlets, pitta, beef stew, meat loaf. Provide students with the following questions to ask each other: "What are the pros and cons of this menu?" "What would you like or dislike?" Students may have adverse feelings toward certain foods due to personal preferences or beliefs. Allow students time in small group to share ideas in order to better connect with the idea of making personal decisions and the purpose of voting.

*Design Principle(s): Cultivate conversation; Support sense-making*

### Anticipated Misconceptions

Students may not know what some of the foods are. You can either explain, or tell them that it's up to them to vote for unknown foods or not.

- Liver is an internal organ, not muscle meat. Many people don't like it.
- Hummus is a bean dip made of chickpeas. Pitta is middle eastern flatbread.

The voting rules are somewhat complicated. Acting out the voting process should make things more clear.

### Student Task Statement

Suppose students at our school are voting for the lunch menu over the course of one week. The following is a list of options provided by the caterer.

Menu 1: Meat Lovers

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- Meat loaf
- Hot dogs
- Pork cutlets
- Beef stew
- Liver and onions

Menu 2: Vegetarian

- Vegetable soup and peanut butter sandwich
- Hummus, pitta, and veggie sticks
- Veggie burgers and fries
- Chef's salad
- Cheese pizza every day
- Double desserts every day

Menu 3: Something for Everyone

- Chicken nuggets
- Burgers and fries
- Pizza
- Tacos
- Leftover day (all the week's leftovers made into a casserole)
- Bonus side dish: pea jello (green gelatin with canned peas)

Menu 4: Concession Stand

- Choice of hamburger or hot dog, with fries, every day

To vote, draw one of the following symbols next to each menu option to show your first, second, third, and last choices. If you use the slips of paper from your teacher, use only the column that says "symbol."



1st choice



2nd choice



3rd choice



4th choice

1. Meat Lovers \_\_\_\_\_
2. Vegetarian \_\_\_\_\_
3. Something for Everyone \_\_\_\_\_
4. Concession Stand \_\_\_\_\_

Here are two voting systems that can be used to determine the winner.

- Voting system #1. *Plurality*: The option with the most first-choice votes (stars) wins.
- Voting system #2. *Runoff*: If no choice received a majority of the votes, leave out the choice that received the fewest first-choice votes (stars). Then have another vote.

If your first vote is still a choice, vote for that. If not, vote for your second choice that you wrote down.

If there is still no majority, leave out the choice that got the fewest votes, and then vote again. Vote for your first choice if it's still in, and if not, vote for your second choice. If your second choice is also out, vote for your third choice.

1. How many people in our class are voting? How many votes does it take to win a majority?
  2. How many votes did the top option receive? Was this a majority of the votes?
  3. People tend to be more satisfied with election results if their top choices win. For how many, and what percentage, of people was the winning option:
    - a. their first choice?
    - b. their second choice?
    - c. their third choice?
    - d. their last choice?
  4. After the second round of voting, did any choice get a majority? If so, is it the same choice that got a plurality in voting system #1?
  5. Which choice won?
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6. How satisfied were the voters by the election results? For how many, and what percentage, of people was the winning option:
  - a. their first choice?
  - b. their second choice?
  - c. their third choice?
  - d. their last choice?
7. Compare the satisfaction results for the plurality voting rule and the runoff rule. Did one produce satisfactory results for more people than the other?

### Student Response

For the actual class vote, numbers will vary.

Here is a scenario for a class with 50 students (for details see next activity):

Voting system #1. Plurality:

First choice vote: Meat: 21 votes, 42%; Veg: 13 votes, 26%; Ev: 9 votes, 18%; Con: 7 votes, 14%

1. 50 students are voting. It takes 26 votes to win a majority.
2. Meat won a plurality, but not a majority.
3. 21 students got their first choice, 29 students got their last choice.

Voting system #2. Runoff:

4. No majority after second round.
  5. So Something for Everyone wins.
  6. How satisfied voters were:
    - 9 voters got their first choice.
    - 7 voters got their second choice.
    - 34 voters got their third choice.
    - Nobody had to put up with their last choice winning.
  7. With the runoff voting fewer students got their first choice but nobody got their last choice.
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## Activity Synthesis

Ask students which system seems more fair, plurality or runoff. (The plurality system doesn't take second, third, etc., choices into account, while the runoff system does.)

In an election in Oakland, California, a candidate won by campaigning to ask voters to vote for her for first choice, and if she was not their first choice, then put her as their second choice. Is this like what happened with one of the votes we analysed?

## 5.3 School Lunch (Part 2)

### Optional: 20 minutes

In this activity students revisit the situation from the previous activity but they analyse the votes of a different class. In this case the members of different student clubs all voted for the same lunch option. Students repeat the process of the run-off election on the provided data and compare it to a plurality vote. They use quantitative reasoning to analyse and compare the two different voting rules.

Students must think through the voting process and determine which choice is eliminated at each round, and what votes the club presidents will turn in at every round of voting.

### Launch

Arrange students in groups of 4. Tell students that they analyse the results of the vote from a different class for the same lunch caterer situation from the previous activity. Tell them, "There are four clubs in this other class, and everyone in each club agrees to vote exactly the same way, as shown in the table."

Have each group of four act out the voting for this class: each person is the president of a club, and delivers the votes for all the club members. Demonstrate with a group, "This person is the president of the barbecue club. Tell us how many votes you are turning in, and for which choice."

Give students 10 minutes to work through the questions with their group, followed up with whole-class discussion.

*Representation: Internalise Comprehension.* Activate or supply background knowledge about reasoning quantitatively. Allow students to use calculators to ensure inclusive participation in the activity.

*Supports accessibility for: Memory; Conceptual processing*

### Anticipated Misconceptions

















The voting rules are somewhat complicated. Acting out the voting process should make things more clear.

### Student Task Statement

Let's analyse a different election.

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In another class, there are four clubs. Everyone in each club agrees to vote for the lunch menu exactly the same way, as shown in this table.

	Barbecue Club (21 members)	Garden Club (13 members)	Sports Boosters (7 members)	Film Club (9 members)
A. Meat Lovers				
B. Vegetarian				
C. Something for Everyone				
D. Concession Stand				

1. Figure out which option won the election by answering these questions.
  - a. On the first vote, when everyone voted for their first choice, how many votes did each option get? Did any choice get a majority?
  - b. Which option is removed from the next vote?
  - c. On the second vote, how many votes did each of the remaining three menu options get? Did any option get a majority?
  - d. Which menu option is removed from the next vote?
  - e. On the third vote, how many votes did each of the remaining two options get? Which option won?
2. Estimate how satisfied all the voters were.
  - a. For how many people was the winner their first choice?
  - b. For how many people was the winner their second choice?
  - c. For how many people was the winner their third choice?
  - d. For how many people was the winner their last choice?
3. Compare the satisfaction results for the plurality voting rule and the runoff rule. Did one produce satisfactory results for more people than the other?

### Student Response

1.
  - a. Meat: 21 votes, 42%; Veg: 13 votes, 26%; Ev: 9 votes, 18%; Con: 7 votes, 14%. Meat won a plurality, but not a majority. You don't really need to calculate a

percentage for this. There were 50 votes in all, so 26 votes are needed to win a majority.

- b. Eliminate Concession Stand, since it only got 7 votes.
  - c. In the new vote, everyone still votes for their first choice, except the 7 people in the Sports Booster Club. They originally voted for Concession Stand, so now their votes go to their second choice, which is Something for Everyone. Meat: 21 votes, 42%; Veg: 13 votes, 26%; Ev:  $9+7=16$  votes, 32%. Meat still has the most votes, but still not a majority.
  - d. Second runoff vote: eliminate Vegetarian. The Garden Club members' first choice is gone. But their second choice, Concession Stand, is also gone. So their 13 votes go to their third choice, Something for Everyone. Meat: 21 votes, 42%; Ev:  $9+7+13=29$  votes, 58%.
  - e. So Something for Everyone wins.
- 2.
- a. 9 voters in the Film Club got their first choice.
  - b. 7 voters in the Sports Booster Club got their second choice.
  - c. 21 members of the Barbecue Club and 13 members of the Garden Club got their third choice.
  - d. Nobody had to put up with their last choice winning.
3. With the plurality voting rule, 21 students would have gotten their first choice and 29 students would have gotten their last choice. So even though fewer students got their first choice with the runoff voting rule, more students got a higher choice than with the plurality voting rule.

### Activity Synthesis

Ask for results from the fictitious class.

Notice that after the first round, Meat seemed to be winning. After the second round, Vegetarian seemed to be winning. But the actual winner was Something for Everyone.

Ask students:

- How did the results of this class compare to our own class?
  - What are some advantages and disadvantages of plurality and runoff voting? (Plurality takes less effort but could be less fair.)
  - Which system seems more fair, plurality or runoff? (The plurality system doesn't take second, third, etc., choices into account, while the runoff system does.)
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## 5.4 Just Vote Once

### Optional: 30 minutes

This activity presents another method for choosing among three or more choices when none wins a majority, instant runoff voting. Voters again rank their choices. Each choice is given points, with 0 for the last choice, 1 for the next to last, and so on. The choice with the most total points wins, and no runoff elections are needed. Students use quantitative reasoning to compare two models of fairness in voting.

#### Instructional Routines

- Discussion Supports

#### Launch

Arrange students in groups of 2–4.

Introduce another method of voting: "The runoff system sometimes needs more than one election if no choice got a majority. If we are all here together, that's not a big problem. But if it were a vote with everyone in a city, or county, or country, it would be too complicated and expensive to have more than one vote. The instant runoff system gives each vote points 0 for the last choice, 1 for the next to last, and so on. The choice with the most total points wins, and no runoff elections are needed. Let's redo our election using this system, and then see what the other class's votes would choose. Remember what your choices were for the first time we voted for school lunch providers. Write down the points for each of the choices."

Then either ask for votes by hand-raising or ask students to come to the board to record their choices.

Raising hands: "Raise your hand if you gave Meat 3 points (count and record in table). Raise your hand if you gave Meat 2 points, etc." There will be 16 categories. Everyone should raise their hand 4 times.

choice	number of votes for top choice (star)	number of votes for second choice (smiley)	number of votes for third choice (square face)	number of votes for last choice (X)
A				
B				
C				
D				

Come to the board: Have students come up and record their numbers. Each student should have a 0, a 1, a 2, and a 3, one in each category. The table below shows the points for two students' choices.

	points
A. Meat Lovers	3, 0, ...
B. Vegetarian	2, 3, ...
C. Something for Everyone	1, 1, ...
D. Concession stand	0, 2, ...

After the results are recorded for all to see and students understand the presented information, students work in groups and answer the questions in the activity statement.

*Action and Expression: Internalise Executive Functions.* Chunk this task into more manageable parts. After students have solved the first 2–3 problems, check in with either select groups of students or the whole class. Invite students to share the strategies they have used so far, as well as inviting them to ask any questions they have before continuing.  
*Supports accessibility for: Organisation; Attention*

### Anticipated Misconceptions

Students may not remember how to do the satisfaction survey. Remind them of the work done in the previous activity. Ask them to fill out a similar table:

what choice did you rank the winner?	number of people	% of people
top choice (star)		
second choice (smiley)		
third choice (square face)		
last choice (X)		

### Student Task Statement

Your class just voted using the *instant runoff* system. Use the class data for following questions.

1. For our class, which choice received the most points?
2. Does this result agree with that from the runoff election in an earlier activity?
3. For the other class, which choice received the most points?
4. Does this result agree with that from the runoff election in an earlier activity?

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5. The runoff method uses information about people's first, second, third, and last choices when it is not clear that there is a winner from everyone's first choices. How does the instant runoff method include the same information?
  6. After comparing the results for the three voting rules (plurality, runoff, instant runoff) and the satisfaction surveys, which method do you think is fairest? Explain.

### Student Response

1. Results vary for the actual voting in class.
2. Answer vary.
3. The Vegetarian option won with 97 points. A got 21 first place votes and 29 last place votes:  $21 \times 3 + 29 \times 0 = 63$  points. B got 21 second place votes, 13 first place votes, and 7+9 third place votes:  $21 \times 2 + 13 \times 3 + 7 \times 1 + 9 \times 1 = 97$  points. C got 21+13 third place votes, 7 second place votes, and 9 first place votes:  $(21 + 13) \times 1 + 7 \times 2 + 9 \times 3 = 75$  points. D got 21 last place votes, 13+9 second place votes, and 7 first place votes:  $21 \times 0 + (13 + 9) \times 2 + 7 \times 3 = 65$  points.
4. This is a different result than we got from the other two voting systems. Meat (A) won under the plurality system, and Something for Everyone (C) won under the runoff system. Even though A had the largest group vote for it, it was last choice for everyone else, so it could not pick up any extra points for being second or third choice. Vegetarian (B) won by having a bigish number of first place points, and a bigger number of second place points, and some more third place points.
5. The instant runoff method includes the information about people's choices in the number of points that are assigned to each vote.
6. Answers vary. In the plurality method, most people get their first choice. In the runoff method, nobody got their last choice. In the instant runoff method, more people got a higher choice, even if it was not their first one.

### Are You Ready for More?

Numbering your choices 0 through 3 might not really describe your opinions. For example, what if you really liked A and C a lot, and you really hated B and D? You might want to give A and C both a 3, and B and D both a 0.

1. Design a numbering system where the size of the number accurately shows how much you like a choice. Some ideas:
    - The same 0 to 3 scale, but you can choose more than one of each number, or even decimals between 0 and 3.
    - A scale of 1 to 10, with 10 for the best and 1 for the worst.
  2. Try out your system with the people in your group, using the same school lunch options for the election.
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3. Do you think your system gives a more fair way to make choices? Explain your reasoning.

### Student Response

Answers vary.

### Activity Synthesis

Poll students about the results of the instant runoff vote. Ask:

- How do the three voting methods we have seen compare?
- Which method should we use the next time our class has to make a decision? Why?

We have seen several methods for fairly deciding between more than two choices. There is no single fairest method. Some methods give one winner, others a different winner with the same vote.

*Conversing: Discussion Supports.* Use this routine to support small-group discussion. Display the following prompts: “I think the \_\_\_ method is most fair because . . .”, “I agree/disagree because . . .”, “Does anyone else have something to add to this explanation?”, “How can we justify that more students were represented in the final results?” These prompts will help students summarise the results of each type of voting system.

*Design Principle(s): Cultivate conversation; Maximise meta-awareness*

## 5.5 Weekend Choices

### Optional: 10 minutes

This voting activity helps students summarise the voting systems for more than two choices that were discussed in the previous lessons. Five students vote on three choices of weekend activities. In this activity, students engage in quantitative reasoning to compare two mathematical models for fairness in voting.

### Launch

Arrange students in groups of 2–4.

### Student Task Statement

Clare, Han, Mai, Tyler, and Noah are deciding what to do on the weekend. Their options are cooking, hiking, and bowling. Here are the points for their instant runoff vote. Each first choice gets 2 points, the second choice gets 1 point, and the last choice gets 0 points.

	cooking	hiking	bowling
Clare	2	1	0
Han	2	1	0

Mai	2	1	0
Tyler	0	2	1
Noah	0	2	1

1. Which activity won using the instant runoff method? Show your calculations and use expressions or equations.
2. Which activity would have won if there was just a vote for their top choice, with a majority or plurality winning?
3. Which activity would have won if there was a runoff election?
4. Explain why this happened.

### Student Response

1. Hiking won under the instant runoff (points) system. Cooking got three first place votes. Hiking got two first place votes and three second place votes. Bowling got two second place votes.

The calculation of points for cooking is  $3 \times 2 = 6$ .

The calculation of points for hiking is  $2 \times 2 + 3 \times 1 = 7$ .

The calculation of points for bowling is  $2 \times 1 = 2$ .

2. With a plurality wins system, cooking would win, since it got a majority of the first place votes.
3. With a runoff system cooking would win since it already got the majority of first place votes in one round of voting.
4. Hiking won by getting several first place votes and more second place votes. The 3 points from second place votes gave more points than the third first place vote that cooking got.



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