Lesson 19: Estimating a hemisphere

Goals

- Calculate the volume of a cylinder and cone with the same radius and height, and justify (orally and in writing) that the volumes are an upper and lower bound for the volume of a hemisphere of the same radius.
- Estimate the volume of a hemisphere using the formulae for volume of a cone and cylinder, and explain (orally) the estimation strategy.

Learning Targets

- I can estimate the volume of a hemisphere by calculating the volume of shape I know is larger and the volume of a shape I know is smaller.

Lesson Narrative

The purpose of this lesson is to introduce students to working with spheres by using shapes they are now familiar with—cubes, cones, and cylinders—to estimate the volume of a hemisphere. In the previous lesson, students saw that changing the radius of a cone by a factor of a scales the volume by a factor of a^2. Here, the connection between spheres and cubes is made in the first activity to help them build understanding for why changing the radius of a sphere (or, in the case of this lesson, hemisphere) by a factor of a changes the volume by a factor of a^3. For example, think about a cube with side length 2 units that has a volume of $2^3 = 8$. If the length, width, and height are all scaled by a factor of a, then each edge length would be $2a$ units and the new volume would be $(2a)^3 = 8a^3$ cubic units, which is a^3 times the original volume. The first activity sets the expectation that spheres work the same way, so that the r^3 in the formula for the volume of a sphere of radius r, given in the next lesson, makes sense.

In the second activity, students fit a hemisphere inside a cylinder, and use the volume of the cylinder to make an estimate of the volume of the hemisphere. Then they do the same thing with a cone that fits inside the hemisphere. The volume of the hemisphere has to be between the volume of the cone and the volume of the cylinder, both of which students can calculate from work in previous lessons. So this activity gives a range of possibilities for volume of the hemisphere. In the next lesson, students will see the exact formula.

Addressing

- Solve real-world and mathematical problems involving volume of cylinders, cones, and spheres.
- Know the formulae for the volumes of cones, cylinders, and spheres and use them to solve real-world and mathematical problems.

Instructional Routines

- Stronger and Clearer Each Time

• Three Reads

• Notice and Wonder

• Think Pair Share

Required Materials

Spherical objects

Required Preparation

If possible, have some physical examples of hemispheres on hand for students to see. Examples could be glass paperweights or dome lids. Alternatively, have a sphere, such as a globe or basketball, with a marked equator to clearly divide it into two hemispheres.

Student Learning Goals

Let's estimate volume of hemispheres with shapes we know.

19.1 Notice and Wonder: Two Shapes

Warm Up: 5 minutes

The purpose of this warm-up is for students to review how to manipulate the formulae for volume of a cylinder and cone and consider what they look like when the height and radius are the same. Students will encounter these shapes again later in the lesson.

Instructional Routines

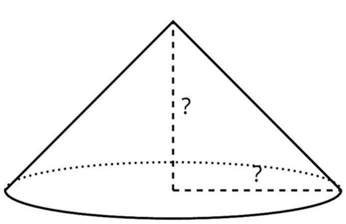
• Notice and Wonder

Launch

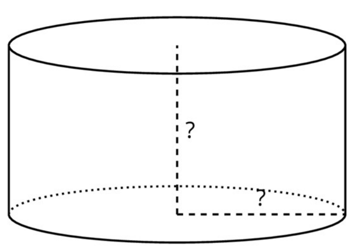
Arrange students in groups of 2. Tell students that they will look at an image, and their job is to think of at least one thing they notice and at least one thing they wonder. Display the image for all to see. Ask students to give a signal when they have noticed or wondered about something. Give students 1 minute of quiet think time, and then 1 minute to discuss the things they notice with their partner, followed by a whole-class discussion.

Student Task Statement

Here are two shapes.



$V = \frac{1}{3}\pi r^3$



$V = \pi r^3$

What do you notice? What do you wonder?

Student Response

Things students may notice:

- If the height and radius are the same for both the cylinder and cone, then the volume of the cone is one-third the volume of the cylinder.
- When the radius is the same as the height, the cone and cylinder seem much wider than tall.
- When the height and radius are the same, the volume acts as a function of one variable.

Things students may wonder:

- Why would we want a cone or cylinder where the height and radius are the same?
- Does the volume of this type of cone or cylinder change in the same way a cube's volume does?
- Could we put the cone inside the cylinder to create a new shape, since their dimensions match?

Activity Synthesis

Select students to share things they have noticed and things they have wondered. Encourage students to think about the relationship with the dimension r and the volume V of each of these shapes. Ensure students understand that the two equations have no variable h for height since the h was replaced by r due to the height and radius being the same for both shapes.

19.2 Hemispheres in Boxes

15 minutes

In this activity, students think about a hemisphere fitting inside the smallest possible box (or cuboid). Students reason that the smallest box in which a hemisphere can fit has a square base with edge length that is the same as the diameter of the hemisphere, and its height will be the radius of the hemisphere. Further, if we calculate the volume of the box, we have an upper bound for the volume of the hemisphere. This activity prepares students for the next, where they will calculate the upper and lower bounds for the volume of a hemisphere by considering cylinders and cones that fit outside or inside the hemisphere.

Instructional Routines

- Three Reads

Launch

Ask students if they are familiar with the word *sphere*, and if they can think of examples of spheres. Some examples they might come up with are: ping pong balls, soap bubbles, baseballs, volleyballs, a globe. If students don't come up with many examples, offer your own, or perhaps display an Internet image search for "sphere." If you brought in any examples of physical spheres, display these or allow students to hold them. Explain to students that the radius of a sphere is the distance from the centre of the sphere (the point in the exact middle) to any point on the sphere. If you brought in physical examples of spheres, ask students to rank them from smallest diameter to largest diameter.

Ask students if they are familiar with the word hemisphere, and if they can think of examples of hemispheres. Some examples are hemispherical (or 'half') balance boards, half of Earth, a dome or planetarium, and convex security mirrors. Display an example, illustration, or diagram of a hemisphere for all to see, pointing out how the radius is the distance from the centre of the flat side of the sphere to any point on the curved surface of the sphere.

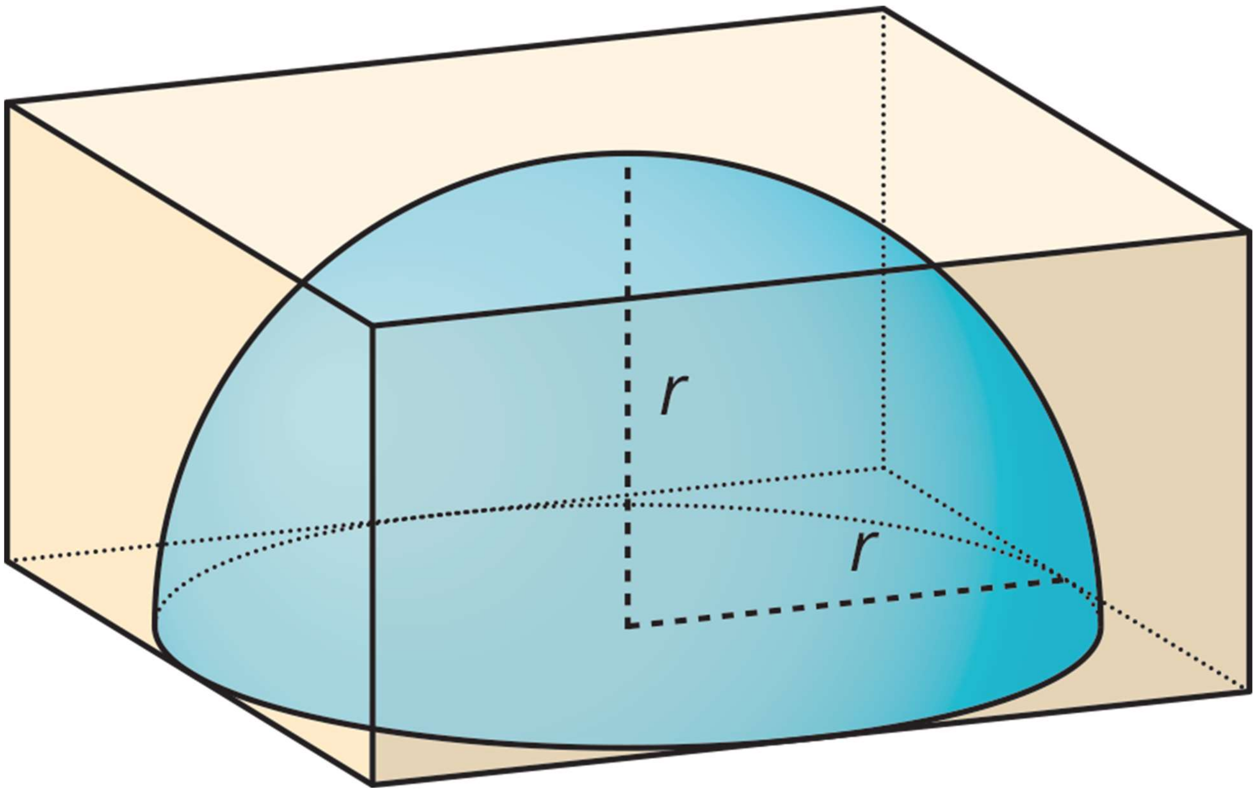
Tell students that in this activity, they will think about building a box (or cuboid) around a hemisphere. Arrange students in groups of 2. Ask students to quietly work through the first question and then share their reasoning with their partner. Pause for a whole-class discussion, and invite students to share their responses. The purpose of this problem is to see how the measurements of a hemisphere determine the dimensions of the cuboid. Ensure everyone understands the box must have edge lengths 6, 6, and 3 inches, and that the hemisphere must have a volume that is less than 108 cubic inches (since $6^2 \times 3 = 108$). Discuss how much less students *think* the volume of the sphere is and their reasoning, then have students move onto the second problem with their partner.

Representation: Develop Language and Symbols. Create a display of important terms and vocabulary. Invite students to suggest language or diagrams to include that will support their understanding of: sphere and hemisphere.
Supports accessibility for: Memory; Language Reading, Speaking, Listening: Three Reads. Use this routine to support comprehension of the situation before looking at the questions. In the first read, students read the problem with the goal of comprehending the situation (e.g., Mai has a paperweight that is a dome. She wants to design a box for it.). If needed, discuss the meaning of unfamiliar terms at this time. Use the second read to identify the important quantities by asking students what can be counted or measured (e.g., the radius of the dome, the area of the base of the dome, the side lengths of the box). In the third read, ask students to brainstorm possible strategies to answer the questions. This helps students connect the language in the word problem and the reasoning needed to solve the problem keeping the intended level of cognitive demand in the task.
Design Principle(s): Support sense-making

Anticipated Misconceptions

Students may believe that since this box is not a cube, but a cuboid, the volume will not increase by 8 when the side lengths are doubled. Since volume of a cuboid is the product of length, width, and height, the side lengths do not need to be equal; if *every* side length is doubled, then the volume will be increased by $2 \times 2 \times 2 = 8$.

Student Task Statement



1. Mai has a dome paperweight that she can use as a magnifier. The paperweight is shaped like a hemisphere made of solid glass, so she wants to design a box to keep it in so it won't get broken. Her paperweight has a radius of 3 cm.

   a. What should the dimensions of the inside of box be so the box is as small as possible?

   b. What is the volume of the box?

   c. What is a reasonable estimate for the volume of the paperweight?

2. Tyler has a different box with side lengths that are twice as long as the sides of Mai's box. Tyler's box is just large enough to hold a different glass paperweight.

   a. What is the volume of the new box?

   b. What is a reasonable estimate for the volume of this glass paperweight?

   c. How many times bigger do you think the volume of the paperweight in this box is than the volume of Mai's paperweight? Explain your thinking.

Student Response

1.

   a. 6 cm by 6 cm by 3 cm. The radius of the hemispherical paperweight is only half of the box's side length. 3×2 gives us the entire length of the side. The height is the same as the radius.

   b. 108 cubic centimetres. 108 is $6^2 \times 3$.

   c. Answers vary. Correct responses should be less than 108 cubic centimetres. The box's volume is $6^2 \times 3$, and since there is space in the box that the hemisphere does not take up, the volume of the hemisphere has to be less than the volume of the box.

2.

   a. 864 cubic centimetres. If every side length is doubled then the volume gets 8 times larger, and 864 is 108×8.

   b. Answers vary. Correct responses should be less than 864 cubic centimetres. There is still space in the box that the hemisphere does not take up, so the volume of the hemisphere has to be less than the volume of the box.

   c. Answers vary. Sample response: Eight times bigger. If the volume of a box gets 8 times bigger and the hemisphere has to fit inside the new box, then the hemisphere's volume must also get 8 times bigger.

Activity Synthesis

Invite students to share their answers to the second question. The purpose of this discussion is for students to see that since a cuboid's volume gets larger by a factor of 2^3 when edge lengths are doubled then it would make sense for a hemisphere's volume to do the same. Ensure that students understand the volume calculated for the box holding the hemisphere is greater than the actual volume of the hemisphere because of the space left around the paperweight in the box and that this will be the upper bound of the volume of a sphere.

Tell students that in the next activity they are going to investigate a better way to estimate the volume of a hemisphere. If time allows, ask students to suggest shapes they are already familiar with that they could use to find the volume of a hemisphere.

19.3 Estimating Hemispheres

15 minutes

In this activity, students use different solid figures to estimate an upper and lower bound for the volume of a hemisphere. For the upper bound, the hemisphere fits snugly inside a cylinder whose height and radius are equal to the radius of the hemisphere. For the lower bound, the cone fits snugly inside the hemisphere, and its radius and height also equal the radius and height of the hemisphere.

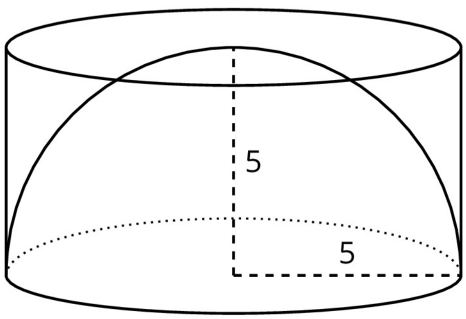
Select students who use the reasoning from the first problem to assist them in answering the second problem to share during the Activity Synthesis. For example, since the cylinder and cone have the same dimensions, the volume of the cone must be $\frac{1}{3}$ that of the cylinder.

Instructional Routines

- Stronger and Clearer Each Time
- Think Pair Share

Launch

Keep students in the same groups. Display the image of a hemisphere of radius 5 that fits snugly inside a cylinder of the same radius and height.



Ask "How does the hemisphere affect the height of the cylinder?" and then give students one minute of quiet think time, then one minute to discuss their response with a partner. Ask partners to share their responses. If not mentioned by students, point out that the height of the cylinder is equal to the radius of the hemisphere.

Give students work time for the activity followed by a whole-class discussion.

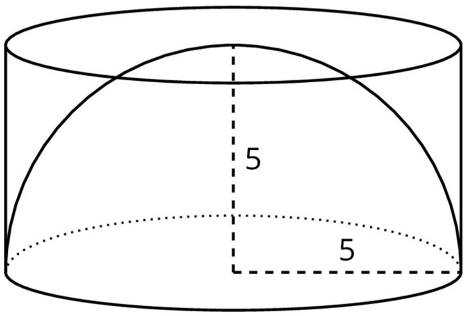
Representation: Internalise Comprehension. Provide examples of actual 3-D models of cylinders, hemispheres, and cones for students to view or manipulate. Ask students to use the 3-D models and the volumes calculated in each question to estimate the volume of the hemisphere.
Supports accessibility for: Visual-spatial processing; Conceptual processing

Anticipated Misconceptions

Students may not realise that the radius of the hemisphere determines the height of the cone and cylinder. If students struggle with identifying the needed information, remind them of the previous activity in which the radius of the cylinder was also the height of the box (cuboid). Ask students how that can help them determine the dimensions of the cone or cylinder.

Student Task Statement

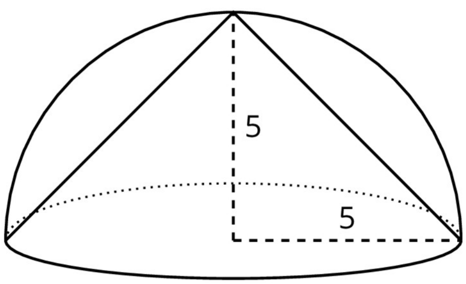
1. A hemisphere with radius 5 units fits snugly into a cylinder of the same radius and height.



   a. Calculate the volume of the cylinder.

   b. Estimate the volume of the hemisphere. Explain your reasoning.

2. A cone fits snugly inside a hemisphere, and they share a radius of 5.



   a. What is the volume of the cone?

   b. Estimate the volume of the hemisphere. Explain your reasoning.

3. Compare your estimate for the hemisphere with the cone inside to your estimate of the hemisphere inside the cylinder. How do they compare to the volumes of the cylinder and the cone?

Student Response

1

   a. 125π. The cylinder's volume can be calculated using $V = \pi(5)^2(5)$.

   b. Answers vary. Correct responses should be less than the volume of the cylinder.

2.

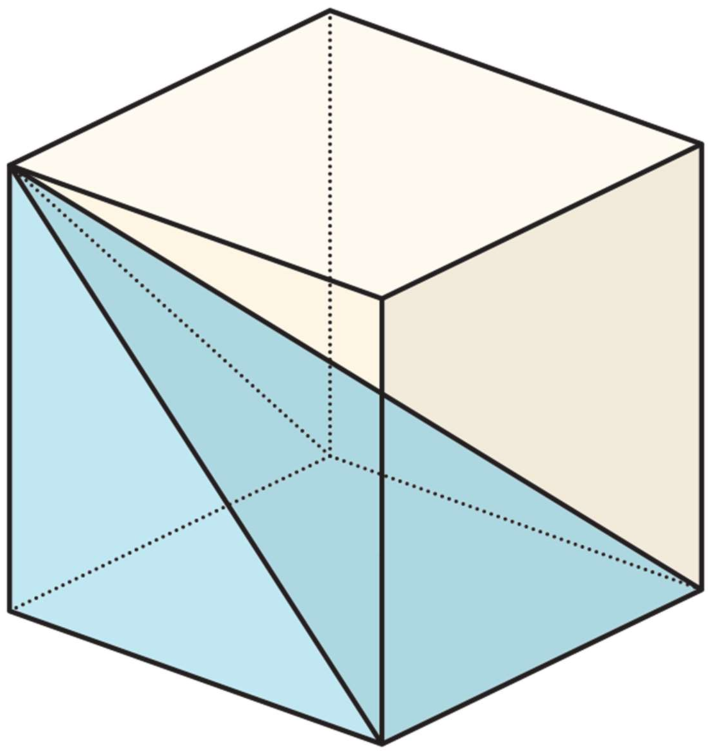
   a. $\frac{125}{3}\pi$. The cone's volume can be calculated using $V = \frac{1}{3}\pi(5)^2(5)$.

b. Answers vary. Correct responses should be greater than the volume of the cone and less than the volume of the cylinder from the previous question.

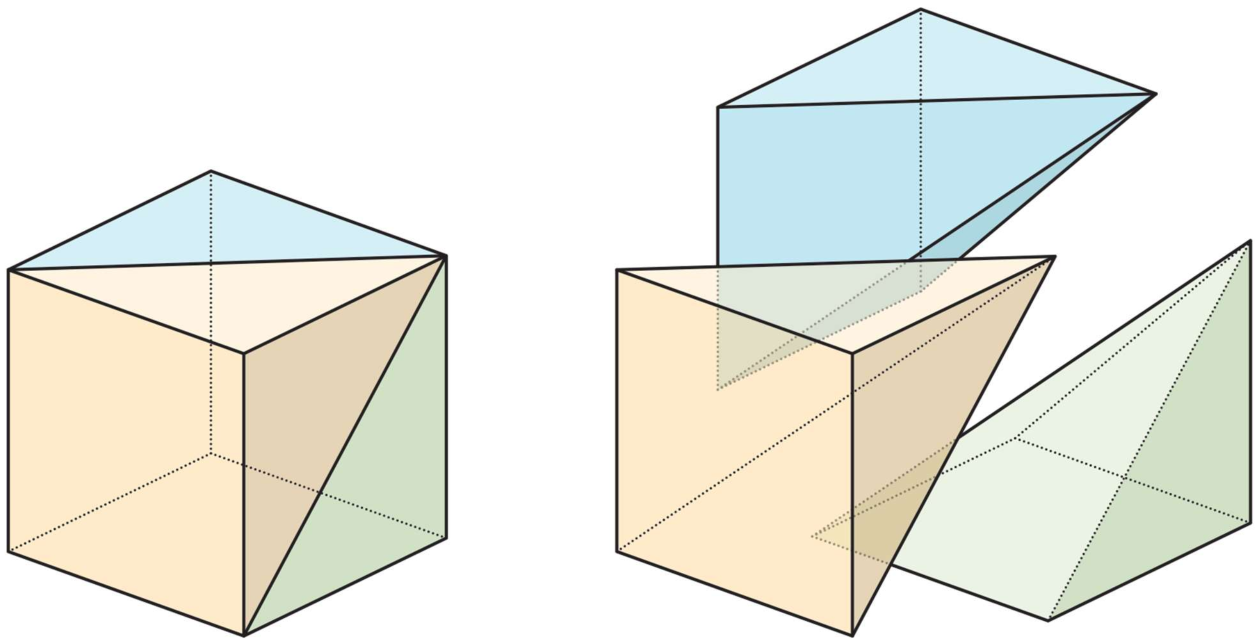
3. Answers vary. Sample response: The volume of the hemisphere will be greater than the volume of the cone but less than the volume of the cylinder.

Are You Ready for More?

Estimate what fraction of the volume of the cube is occupied by the pyramid that shares the base and a top vertex with the cube, as in the figure.



Student Response



In fact, the pyramid is precisely $\frac{1}{3}$ of the volume of the cube. One way to see this is by decomposing the cube into three pyramids each congruent to the original.

Activity Synthesis

Ask previously identified students to share their responses to the first two problems. Draw attention to any connections made between the two problems.

The purpose of this discussion is for students to recognise how the upper and lower bounds for the volume of a hemisphere are established by the cylinder and cone. Consider asking the following question:

- "What do the volumes of the cone and cylinder tell us about the volume of the hemisphere?" (The volume of the hemisphere has to be between the two volumes.)
- "Did anyone revise their original estimate for the hemisphere based on the calculation of the volume of the cone?" (Answers vary. If students estimated low values after calculating the volume of a cylinder, some may have needed to adjust their estimates.)
- "Compare the equations for volume of a cylinder and cone where radius and height are equal. If the volume of the hemisphere has to be between these two, what might an equation for the volume of a hemisphere look like?" (Cylinder volume: $V = \pi r^3$; Cone volume: $V = \frac{1}{3}\pi r^3$. A possible hemisphere volume might be the average of these two, or $V = \frac{2}{3}\pi r^3$.)

Writing, Speaking, Listening: Stronger and Clearer Each Time. Use this routine to give students a structured opportunity to revise and refine their response to the last question.

Ask students to meet with 2–3 partners in a row for feedback. Provide students with prompts for feedback that will help them to strengthen their ideas and clarify their language (e.g., "Why do you think…?", "How did you compare the volume of the two shapes?", and "How does the first/second question help?", etc.). Students can borrow ideas and language from each partner to strengthen their final version.
Design Principle(s): Optimise output (for explanation)

Lesson Synthesis

To have students describe some of the important highlights of the lesson, ask:

- "How did we use a cylinder to estimate a volume of a hemisphere that is an overestimate?"
- "How did we use a cone to estimate a volume of a hemisphere that is an underestimate?"
- "How did we get a closer estimate for the volume of a hemisphere?"
- "How did we use today's work to estimate the volume of a sphere?"

Explain to students that we used figures we know how to find the volume of (a cone and cylinder) to try and estimate the volume of a figure we do not know how to find the volume of (a hemisphere). In the next lesson, we will do something similar to learn how to find the volume of a sphere and see how close our reasoning today was to the actual calculation.

19.4 A Mirror Box

Cool Down: 5 minutes

Student Task Statement

A hemisphere-shaped security mirror fits exactly inside a cuboid box with a square base that has edge length 10 inches. What is a reasonable estimate for the volume of this mirror?

Student Response

Answers vary. Sample responses include:

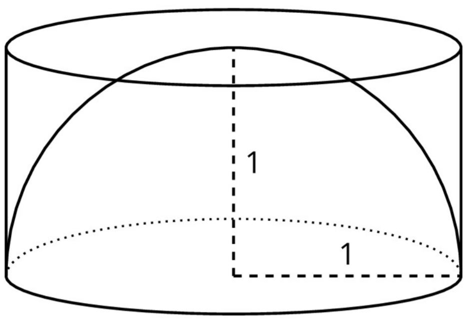
- Less than 500 cubic inches. The volume of the box that the mirror fits in is $10^2 \times 5$, and the mirror does not take up all the space in the box.
- Less than 125π cubic inches. The volume of the cylinder that the mirror fits in is $\pi(5)^2 \times 5$, and the mirror does not take up all the space in the cylinder.
- More than $\frac{125}{3}\pi$ cubic inches. The volume of the cone that fits in the mirror is $\frac{1}{3}\pi(5)^2 \times 5$, and the mirror is larger than the cone.

Student Lesson Summary

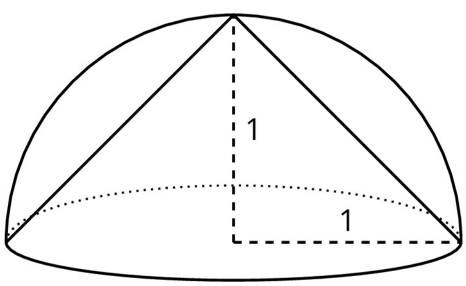
We can estimate the volume of a hemisphere by comparing it to other shapes for which we know the volume. For example, a hemisphere of radius 1 unit fits inside a cylinder with a radius of 1 unit and height of 1 unit.

Since the hemisphere is *inside* the cylinder, it must have a smaller volume than the cylinder making the cylinder's volume a reasonable over-estimate for the volume of the hemisphere.

The volume of this particular cylinder is about 3.14 units^3 since $\pi(1)^2(1) = \pi$, so we know the volume of the hemisphere is less than 3.14 cubic units.



Using similar logic, a cone of radius 1 unit and height 1 unit fits inside of the hemisphere of radius 1 unit.



Since the cone is *inside* the hemisphere, the cone must have a smaller volume than the hemisphere making the cone's volume a reasonable under-estimate for the volume of the hemisphere.

The volume of this particular cone is about 1.05 units^3 since $\frac{1}{3}\pi(1)^2(1) = \frac{1}{3}\pi \approx 1.05$, so we know the volume of the hemisphere is more than 1.05 cubic units.

Averaging the volumes of the cylinder and the cone, we can estimate the volume of the hemisphere to be about 2.10 units^3 since $\frac{3.14+\ .05}{2} \approx 2.10$. And, since a hemisphere is half of a sphere, we can also estimate that a sphere with radius of 1 would be double this volume, or about 4.20 units^3.

Lesson 19 Practice Problems

Problem 1 Statement

A baseball fits snugly inside a transparent display cube. The length of an edge of the cube is 2.9 inches.

Is the baseball's volume greater than, less than, or equal to 2.9^3 cubic inches? Explain how you know.

Solution

Less than 2.9^3 cubic inches. The baseball fits inside the cube, and the cube's volume is 2.9^3 cubic inches. Therefore, the baseball's volume is less than 2.9^3 cubic inches.

Problem 2 Statement

There are many possible cones with a height of 18 metres. Let r represent the radius in metres and V represent the volume in cubic metres.

a. Write an equation that represents the volume V as a function of the radius r.

b. Complete this table for the function, giving three possible examples.

r	V
2	

c. If you double the radius of a cone, does the volume double? Explain how you know.

d. Is the graph of this function a line? Explain how you know.

Solution

a. $V = 6\pi r^2$

b. Answers vary. Sample response:

r	V
2	24π
4	96π
5	150π

c. No, the volume does not double. It is multiplied by four. Explanations vary.

d. It is *not* a line. The three points in the table do not lie on a straight line.

Problem 3 Statement

A hemisphere fits snugly inside a cylinder with a radius of 6 cm. A cone fits snugly inside the same hemisphere.

a. What is the volume of the cylinder?

b. What is the volume of the cone?

c. Estimate the volume of the hemisphere by calculating the average of the volumes of the cylinder and cone.

Solution

a. 216π cm^3

b. 72π cm^3

c. 144π cm^3

Problem 4 Statement

a. Find the hemisphere's diameter if its radius is 6 cm.

b. Find the hemisphere's diameter if its radius is $\frac{1000}{3}$ m.

c. Find the hemisphere's diameter if its radius is 9.008 ft.

d. Find the hemisphere's radius if its diameter is 6 cm.

e. Find the hemisphere's radius if its diameter is $\frac{1000}{3}$ m.

f. Find the hemisphere's radius if its diameter is 9.008 ft.

Solution

a. 12 cm

b. $\frac{2000}{3}$ m (or equivalent)

c. 18.016 ft

d. 3 cm

e. $\frac{500}{3}$ m (or equivalent)

f. 4.504 ft

Problem 5 Statement

After almost running out of space on her phone, Elena checks with a couple of friends who have the same phone to see how many pictures they have on their phones and how much memory they take up. The results are shown in the table.

number of photos	2 523	3 148	1 875
memory used in MB	8 072	10 106	6 037

a. Could this information be reasonably modelled with a linear function? Explain your reasoning.

b. Elena needs to delete photos to create 1 200 MB of space. Estimate the number of photos should she delete.

Solution

a. Yes, all the points are close to $y = 3.2x$, where y represents the memory usage and x represents the number of photos.

b. About 375 photos ($1\,200 \div 3.2 = 375$)