Lesson 9: Multiplying rational numbers

Goals

- Generalise (orally) that the product of two negative numbers is positive.
- Interpret directed numbers used to represent elapsed time before or after a chosen reference point.
- Use patterns to find the product of directed numbers, and explain (orally and using other representations) the reasoning.

Learning Targets

- I can explain what it means when time is represented with a negative number in a situation about speed and direction.
- I can multiply two negative numbers.

Lesson Narrative

The purpose of this lesson is to develop the rules for multiplying two negative numbers. Students use the familiar fact that distance $=$ velocity $\times$ time to make sense of this rule. They interpret negative time as time before a chosen starting time and then figure out what the position is of an object moving with a negative velocity at a negative time. An object moving with a negative velocity is moving from right to left along the number line. At a negative time it has not yet reached its starting point of zero, so it is to the right of zero, and therefore its position is positive. So a negative velocity times a negative time gives a positive position. Students connect reasoning about quantities with abstract properties of numbers.

There is also an optional activity where students can use another approach to understanding why the product of two negative numbers is positive, by examining patterns in an extended multiplication table that includes both positive and negative numbers.

Building On

- Solve unit rate problems including those involving unit pricing and constant speed. For example, if it took 7 hours to mow 4 lawns, then at that rate, how many lawns could be mowed in 35 hours? At what rate were lawns being mowed?

Addressing

- Apply and extend previous understandings of multiplication and division and of fractions to multiply and divide rational numbers.
- Understand that multiplication is extended from fractions to rational numbers by requiring that operations continue to satisfy the properties of operations, particularly the distributive property, leading to products such as $(-1)(-1) = 1$ and the rules for multiplying directed numbers. Interpret products of rational numbers by describing real-world contexts.

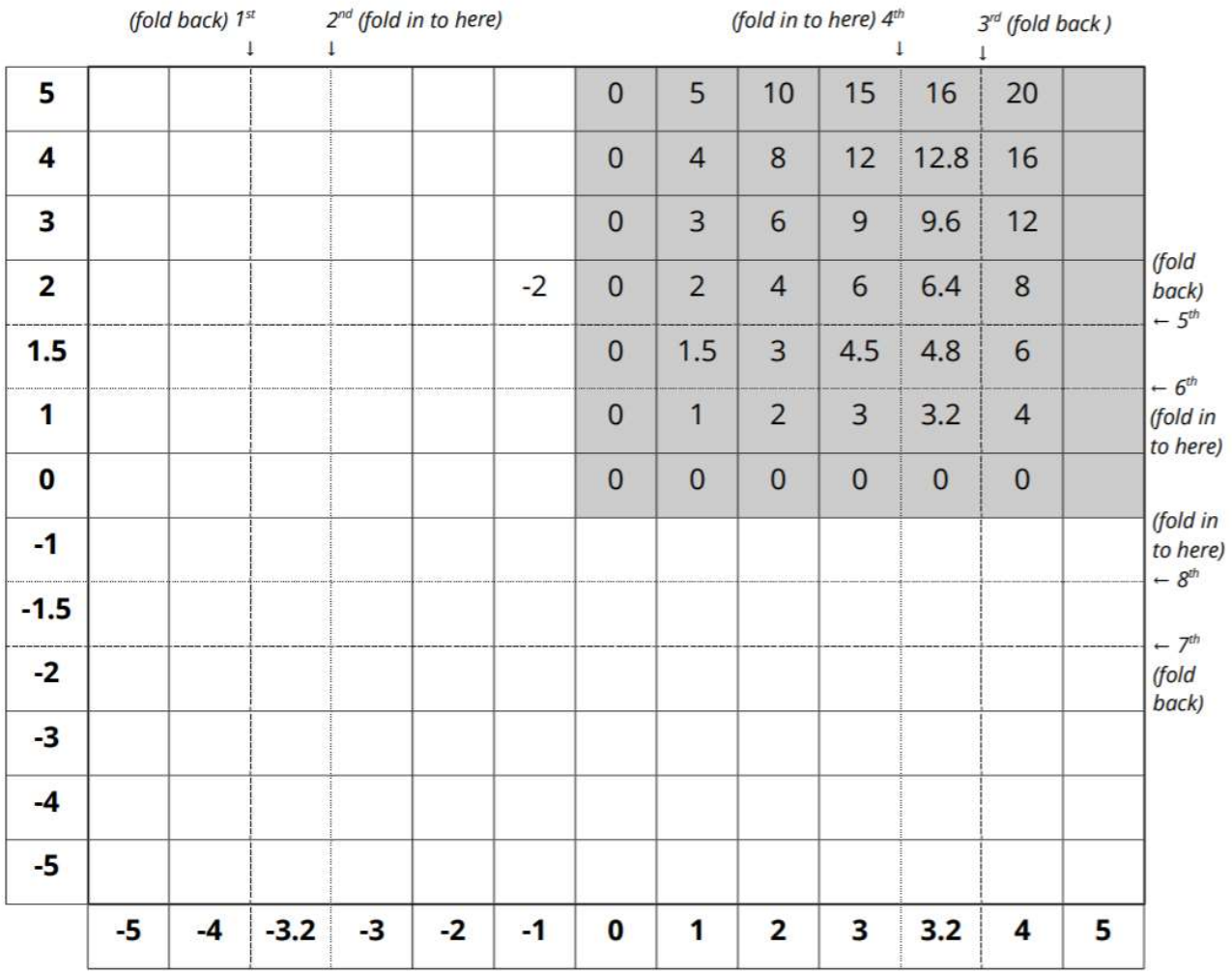
- Apply properties of operations as strategies to multiply and divide rational numbers.
- Recognise and represent proportional relationships between quantities.

Instructional Routines

- Collect and Display
- Co-Craft Questions
- Think Pair Share

Required Materials

Copies of blackline master



Required Preparation

It is optional to provide 1 copy of the Rational Numbers Multiplication Grid blackline master to each student.

Student Learning Goals

Let's multiply directed numbers.

9.1 Before and After

Warm Up: 5 minutes

In this lesson, students will interpret negative time in context. The warm-up primes them for those interpretations.

Instructional Routines

- Think Pair Share

Launch

Arrange students in groups of 2. Give students 30 seconds of quiet think time, followed by partner discussion.

Student Task Statement

Where was the girl:

1. 5 seconds *after* this picture was taken? Mark her approximate location on the picture.
2. 5 seconds *before* this picture was taken? Mark her approximate location on the picture.

Student Response

1. Students should mark a position after her current position.
2. Students should mark a position before her current position, about equally distant from her current position as the previous mark.

Activity Synthesis

Ask students to come to agreement with their partners, and help them to productively resolve any discrepancies. Point out that if she is walking at a constant speed, then her positions before and after will be equally far from her position in the picture.

9.2 Backwards in Time

15 minutes

Students use their earlier understanding of a chosen zero point and description of positive and negative velocity, and extend this to include negative values for time to represent a time before the time chosen as zero as directed. This will produce different end points depending on if the velocity or time is negative or positive. Students use the context to help make sense of the arithmetic problems. Looking at a number of different examples will help students describe rules for identifying the sign of the product of two negative numbers. Students may choose to use a number line to help them in their reasoning; this is an example of using appropriate tools strategically.

Instructional Routines

- Co-Craft Questions

Launch

Keep students in the same groups. Remind the students of movement east or west as positive or negative velocity.

This activity is the same context as one in the previous lesson, and the questions are related. So students should be able to get to work rather quickly. However, each question requires some careful thought, and one question builds on the other. Consider suggesting that students check in with their partner frequently and explain their thinking. Additionally, you might consider asking students to pause after each question for a quick whole-class discussion before continuing to the next question.

Action and Expression: Internalise Executive Functions. Chunk this task into more manageable parts to support students who benefit from support with organisational skills in problem solving. For example, present one question at a time and monitor students to ensure they are making progress throughout the activity.
Supports accessibility for: Organisation; Attention Conversing, Writing: Co-craft Questions. Display only the table of values from the task and ask students to write possible mathematical questions about the situation. Invite students to share their questions with a partner before selecting a few to share with the class. Highlight questions that connect the table to other representations, such as: "What equations can be written with the quantity's position and time?" or "How can each table entry be represented on a number line?" These questions help students look for and make use of structure to find the car's direction and velocity. This helps students produce the language of mathematical questions and talk about the relationships between the two quantities (e.g., position and time) prior to being

asked to determine the car's direction and speed.
Design Principle(s): Cultivate conversation; Support sense-making

Anticipated Misconceptions

If students struggle to calculate the velocity, ask them how fast is the car going after 1 second.

Student Task Statement

A traffic safety engineer was studying travel patterns along a motorway. She set up a camera and recorded the speed and direction of cars and trucks that passed by the camera. Positions to the east of the camera are positive, and to the west are negative.

1. Here are some positions and times for one car:

position (feet)	-180	-120	-60	0	60	120
time (seconds)	-3	-2	-1	0	1	2

   a. In what direction is this car travelling?

   b. What is its velocity?

   c. What does it mean when the time is zero?

   d. What could it mean to have a negative time?

2. Here are the positions and times for a different car whose velocity is -50 feet per second:

position (feet)				0	-50	-100
time (seconds)	-3	-2	-1	0	1	2

   a. Complete the table with the rest of the positions.

   b. In what direction is this car travelling? Explain how you know.

3. Complete the table for several different cars passing the camera.

	velocity (metres per second)	time after passing the camera (seconds)	ending position (metres)	equation
car C	+25	+10	+250	$25 \times 10 = 250$
car D	-20	+30		
car E	+32	-40		

car F	-35	-20		
car G	-15	-8		

a. If a car is travelling east when it passes the camera, will its position be positive or negative 60 seconds *before* it passes the camera?

b. If we multiply a positive number and a negative number, is the result positive or negative?

c. If a car is travelling west when it passes the camera, will its position be positive or negative 60 seconds *before* it passes the camera?

d. If we multiply two negative numbers, is the result positive or negative?

Student Response

1.

a. The car is travelling east.

b. Its velocity is 60 feet per second.

c. It is the time the car passed the camera.

d. It means time before the car passes the camera.

2. a.

position (feet)	-100	-50	0	50	100	150
time (seconds)	2	1	0	-1	-2	-3

b. Travelling west because velocity is negative.

3. Fill out the table.

	velocity (metres per second)	time after passing camera (seconds)	ending position (metres)	expression
car C	+25	+10	+250	$25 \times 10 = 250$
car D	-20	+30	-600	$-20 \times 30 = -600$
car E	+32	-40	-1280	$32 \times (-40) = -1\,280$
car F	-35	-20	+700	$-35 \times (-20) = +700$
car	-15	-8	+120	$-15 \times (-8) = +120$

G				

a. A car travelling east will be at a negative position before passing the camera.

b. If we multiply a positive and a negative number the result is negative.

c. A car travelling west will be at a positive position before passing the camera.

d. If we multiply two negative numbers the result is positive.

Activity Synthesis

The key thing for students to understand here is that a negative multiplied by another negative is a positive. The last two rows in the table and the final two questions are the keys to this so draw attention to the logical progression that movement in the negative direction will have a positive position when time is negative.

9.3 Cruising

Optional: 15 minutes (there is a digital version of this activity)

This is the first of two optional activities. The teacher may choose to implement either of the optional activities that would best reinforce the learning goals for their students.

In this optional activity, students find the position of a car travelling at a constant velocity at different positive and negative times, and plot the points in the coordinate grid. They see that just as with constant speed, the graph goes through (0,0), but because the velocity is negative it slants downward from left to right instead of passing through the first quadrant.

Instructional Routines

- Collect and Display
- Think Pair Share

Launch

Arrange students in groups of 2. Give them 4 minutes of quiet work time, followed by partner and then whole-class discussion.

Conversing, Reading: Collect and Display. While partners talk about their work, circulate and listen to students describe where the car is at -3 seconds and at 6.5 seconds. Record common or important phrases you hear students say about each location on a visual display connecting the phrases to the table or coordinate grid. This will help students read and use mathematical language during their paired and whole-group discussions.
Design Principle(s): Support sense-making

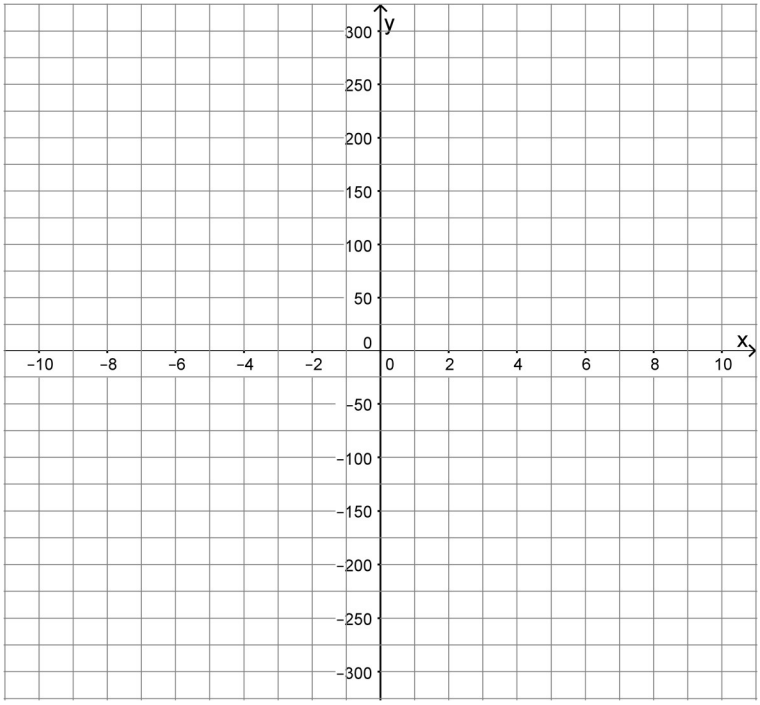
Student Task Statement

Around noon, a car was travelling -32 metres per second down a motorway. At exactly noon (when time was 0), the position of the car was 0 metres.

1. Complete the table.

time (s)	-10	-7	-4	-1	2	5	8	11
position (m)								

2. Graph the relationship between the time and the car's position.



3. What was the position of the car at -3 seconds?

4. What was the position of the car at 6.5 seconds?

Student Response

1.

time (s)	-10	-7	-4	-1	2	5	8	11
position (m)	320	224	128	32	-64	-160	-256	-352

2. A line through (0,0) with a slope of -32

3. 96 metres

4. -208 metres

Are You Ready for More?

Find the value of these expressions without using a calculator.

$(-1)^2$

$(-1)^3$

$(-1)^4$

$(-1)^{99}$

Student Response

1, -1, 1, -1

Activity Synthesis

Ask students how the graph is similar and different from the graph of a proportional relationship. Poll the class for the last two questions. Ask students if they can see an equation that relates the time and the position of the car. Record their ideas and make sure everyone comes to agreement that if d is the position and t is the time, then $d = -32t$.

9.4 Rational Numbers Multiplication Grid

Optional: 10 minutes (there is a digital version of this activity)

In this optional activity, students revisit the representation of a multiplication chart, which may be familiar from previous years; however, in this activity, the multiplication chart is extended to include negative numbers. Students identify and continue patterns to complete the chart and see that it fits the patterns in the chart for the product of two negative numbers to be a positive number.

The blackline master has a multiplication chart that also includes the factors 1.5, -1.5, 3.2, and -3.2, so that students can see how the patterns extend to rational numbers that are not integers. Encourage students to complete the rows and columns for the integers first and then come back to 1.5, -1.5, 3.2, and -3.2 later. Directions are included on the blackline master for a way that students can fold their papers to hide the non-integers while they fill in the integers. If you want students to do this, it would be good to demonstrate and walk them through the process of folding their paper.

Launch

Arrange students in groups of 3. If desired, distribute 1 copy of the blackline master to every student and instruct students to ignore the chart printed in their books or devices. (Also if desired, instruct students to fold their papers according to the directions on the top and right sides of the chart, so that the decimal rows and columns are temporarily hidden.) Give students 30 seconds of quiet think time. Have them share what patterns they notice about the numbers that are already filled in. Give the groups 5 minutes of work time followed by whole-class discussion.

If students have access to the digital materials, students can use the applet to complete the chart. The applet helps students focus on fewer of the numbers and patterns at a time, similar to the purpose of folding the blackline master. Also, the applet gives students

immediate feedback on whether their answers are correct which helps them test their theories about ramifications of multiplying by a negative number.

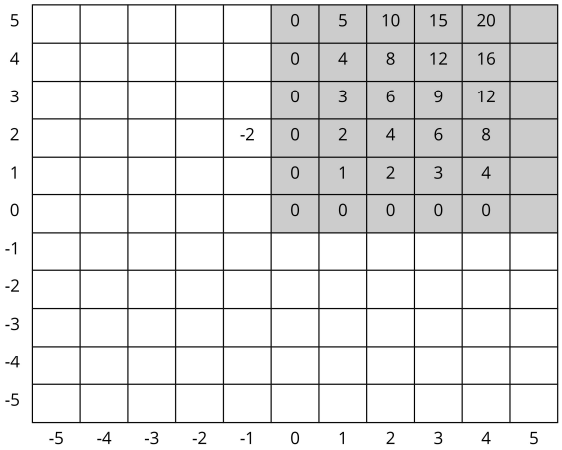
Representation: Internalise Comprehension. Activate or supply background knowledge. Allow students to use calculators to ensure inclusive participation in the activity.
Supports accessibility for: Memory; Conceptual processing

Anticipated Misconceptions

Some students may need a reminder of how a multiplication chart works: the factors are listed at the end of the rows and columns, and their products go in the boxes.

Student Task Statement

1. Complete the *shaded* boxes in the multiplication square.



5						0	5	10	15	20
4						0	4	8	12	16
3						0	3	6	9	12
2					-2	0	2	4	6	8
1						0	1	2	3	4
0						0	0	0	0	0
-1										
-2										
-3										
-4										
-5										
	-5	-4	-3	-2	-1	0	1	2	3	4	5

2. Look at the patterns along the rows and columns. Continue those patterns into the unshaded boxes.

3. Complete the whole table.

4. What does this tell you about multiplication with negative numbers?

Student Response

For the table printed in student books or devices:

5	-25	-20	-15	-10	-5	0	5	10	15	20	25
4	-20	-16	-12	-8	-4	0	4	8	12	16	20
3	-15	-12	-9	-6	-3	0	3	6	9	12	15
2	-10	-8	-6	-4	-2	0	2	4	6	8	10
1	-5	-4	-3	-2	-1	0	1	2	3	4	5
0	0	0	0	0	0	0	0	0	0	0	0
-1	5	4	3	2	1	0	-1	-2	-3	-4	-5
-2	10	8	6	4	2	0	-2	-4	-6	-8	-10
-3	15	12	9	6	3	0	-3	-6	-9	-12	-15
-4	20	16	12	8	4	0	-4	-8	-12	-16	-20
-5	25	20	15	10	5	0	-5	-10	-15	-20	-25
	-5	**-4**	**-3**	**-2**	**-1**	**0**	**1**	**2**	**3**	**4**	**5**

For the table on the blackline master:

5	-25	-20	-16	-15	-10	-5	0	5	10	15	16	20	25
4	-20	-16	-12.8	-12	-8	-4	0	4	8	12	12.8	16	20
3	-15	-12	-9.6	-9	-6	-3	0	3	6	9	9.6	12	15
2	-10	-8	-6.4	-6	-4	-2	0	2	4	6	6.4	8	10
1.5	-7.5	-6	-4.8	-4.5	-3	-1.5	0	1.5	3	4.5	4.8	6	7.5
1	-5	-4	-3.2	-3	-2	-1	0	1	2	3	3.2	4	5
0	0	0	0	0	0	0	0	0	0	0	0	0	0
-1	5	4	3.2	3	2	1	0	-1	-2	-3	-3.2	-4	-5
-1.5	7.5	6	4.8	4.5	3	1.5	0	-1.5	-3	-4.5	-4.8	-6	-7.5
-2	10	8	6.4	6	4	2	0	-2	-4	-6	-6.4	-8	-10
-3	15	12	9.6	9	6	3	0	-3	-6	-9	-9.6	-12	-15
-4	20	16	12.8	12	8	4	0	-4	-8	-12	-12.8	-16	-20
-5	25	20	16	15	10	5	0	-5	-10	-15	-16	-20	-25
	-5	**-4**	**-3.2**	**-3**	**-2**	**-1**	**0**	**1**	**2**	**3**	**3.2**	**4**	**5**

Answer vary. Sample response: A positive number multiplied by a negative number is negative, but a negative number multiplied by a negative number is positive.

Activity Synthesis

The most important takeaway is that it makes sense for the product of two negative numbers to be a positive number, whether or not the numbers are integers. This fits in with the patterns in the extended multiplication chart. Those patterns depend on the distributive property. For example, the reason the numbers in the top row go up by 5s is that $5(n+1) = 5n + 5$. So when students extend the pattern to negative numbers, they are extending the distributive property.

Display a complete chart for all to see, and ask students to explain the ways in which the chart shows that the product of a negative and a negative is a positive. The general argument involves assuming that a pattern observed in a row or column will continue on the other side of 0.

5	-25	-20	-16	-15	-10	-5	0	5	10	15	16	20	25
4	-20	-16	-12.8	-12	-8	-4	0	4	8	12	12.8	16	20
3	-15	-12	-9.6	-9	-6	-3	0	3	6	9	9.6	12	15
2	-10	-8	-6.4	-6	-4	-2	0	2	4	6	6.4	8	10
1.5	-7.5	-6	-4.8	-4.5	-3	-1.5	0	1.5	3	4.5	4.8	6	7.5
1	-5	-4	-3.2	-3	-2	-1	0	1	2	3	3.2	4	5
0	0	0	0	0	0	0	0	0	0	0	0	0	0
-1	5	4	3.2	3	2	1	0	-1	-2	-3	-3.2	-4	-5
-1.5	7.5	6	4.8	4.5	3	1.5	0	-1.5	-3	-4.5	-4.8	-6	-7.5
-2	10	8	6.4	6	4	2	0	-2	-4	-6	-6.4	-8	-10
-3	15	12	9.6	9	6	3	0	-3	-6	-9	-9.6	-12	-15
-4	20	16	12.8	12	8	4	0	-4	-8	-12	-12.8	-16	-20
-5	25	20	16	15	10	5	0	-5	-10	-15	-16	-20	-25
	-5	**-4**	**-3.2**	**-3**	**-2**	**-1**	**0**	**1**	**2**	**3**	**3.2**	**4**	**5**

Lesson Synthesis

Key takeaways:

- Positive times are after a chosen zero time, and negative times are times before the chosen zero time.
- A positive times a positive is always positive.
- A negative times a positive or a positive times a negative is always negative.
- A negative times a negative is always positive.

Discussion questions:

- How can we represent a time that came before a specific zero point?
- What kind of number do you get when you multiply a negative number by a positive number? Use a context from the lesson to explain why this makes sense.
- What kind of number do you get when you multiply a negative number by a negative number? Use a context from the lesson to explain why this makes sense.

9.5 True Statements

Cool Down: 5 minutes

Student Task Statement

Decide if each equation is true or false.

1. $7 \times 8 = 56$
2. $-7 \times 8 = 56$
3. $-7 \times -8 = -56$

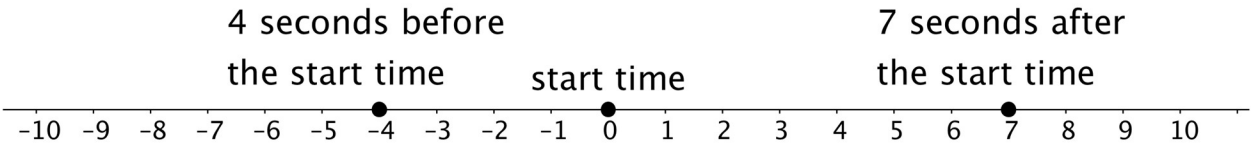
4. $-7 \times -8 = 56$
5. $3.5 \times 12 = 42$
6. $-3.5 \times -12 = -42$
7. $-3.5 \times -12 = 42$
8. $-12 \times \frac{7}{2} = 42$

Student Response

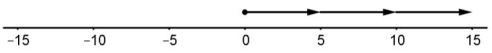
1. True
2. False
3. False
4. True
5. True
6. False
7. True
8. False

Student Lesson Summary

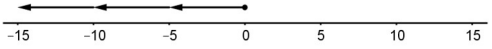
We can use directed numbers to represent time relative to a chosen point in time. We can think of this as starting a stopwatch. The positive times are after the watch starts, and negative times are times before the watch starts.



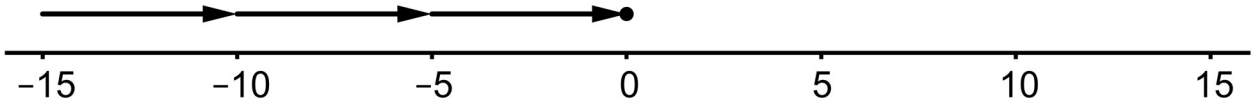
If a car is at position 0 and is moving in a positive direction, then for times after that (positive times), it will have a positive position. A positive times a positive is positive.



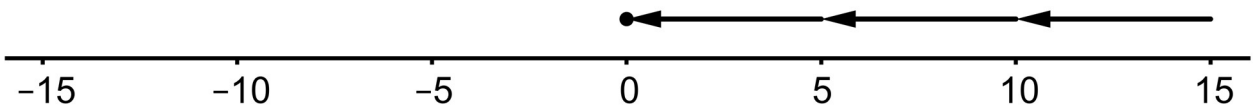
If a car is at position 0 and is moving in a negative direction, then for times after that (positive times), it will have a negative position. A negative times a positive is negative.



If a car is at position 0 and is moving in a positive direction, then for times *before* that (negative times), it must have had a negative position. A positive times a negative is negative.



If a car is at position 0 and is moving in a negative direction, then for times *before* that (negative times), it must have had a positive position. A negative times a negative is positive.



Here is another way of seeing this:

We can think of 3×5 as $5 + 5 + 5$, which has a value of 15.

We can think of $3 \times (-5)$ as $-5 + -5 + -5$, which has a value of -15.

We know we can multiply positive numbers in any order: $3 \times 5 = 5 \times 3$

If we can multiply directed numbers in any order, then $(-5) \times 3$ would also equal -15.

Now let's think about multiplying two negatives.

We can find $-5 \times (3 + -3)$ in two ways:

- Applying the distributive property:

$-5 \times 3 + -5 \times (-3)$

- Adding the numbers in parentheses:

$-5 \times (0) = 0$

This means that these expressions must be equal.

$-5 \times 3 + -5 \times (-3) = 0$

Multiplying the first two numbers gives

$-15 + -5 \times (-3) = 0$

Which means that

$-5 \times (-3) = 15$

There was nothing special about these particular numbers. This always works!

- A positive times a positive is always positive.

- A negative times a positive or a positive times a negative is always negative.
- A negative times a negative is always positive.

Lesson 9 Practice Problems

1. **Problem 1 Statement**

   Fill in the missing numbers in these equations

   a. $-2 \times (-4.5) = ?$

   b. $(-8.7) \times (-10) = ?$

   c. $(-7) \times ? = 14$

   d. $? \times (-10) = 90$

   Solution

   a. $-2 \times (-4.5) = 9$

   b. $(-8.7) \times (-10) = 87$

   c. $(-7) \times (-2) = 14$

   d. $(-9) \times (-10) = 90$

2. **Problem 2 Statement**

   A weather station on the top of a mountain reports that the temperature is currently 0°C and has been falling at a constant rate of 3°C per hour. If it continues to fall at this rate, find each indicated temperature. Explain or show your reasoning.

   a. What will the temperature be in 2 hours?

   b. What will the temperature be in 5 hours?

   c. What will the temperature be in half an hour?

   d. What was the temperature 1 hour ago?

   e. What was the temperature 3 hours ago?

   f. What was the temperature 4.5 hours ago?

   Solution

   a. The temperature will be -6°C, because $-3 \times 2 = -6$

   b. The temperature will be -15°C, because $-3 \times 5 = -15$

   c. The temperature will be -1.5°C, because $-3 \times 0.5 = -1.5$

d. The temperature was 3°C, because $-3 \times (-1) = 3$

e. The temperature was 9°C, because $-3 \times (-3) = 9$

f. The temperature was 13.5°C, because $-3 \times (-4.5) = 13.5$

3. **Problem 3 Statement**

Find the value of each expression.

a. $\frac{1}{4} \times (-12)$

b. $-\frac{1}{3} \times 39$

c. $\left(-\frac{4}{5}\right) \times (-75)$

d. $-\frac{2}{5} \times \left(-\frac{3}{4}\right)$

e. $\frac{8}{3} \times -42$

Solution

a. -3

b. -13

c. 60

d. $\frac{3}{10}$

e. -112

4. **Problem 4 Statement**

To make a specific hair dye, a hair stylist uses a ratio of $1\frac{1}{8}$ oz of red tone, $\frac{3}{4}$ oz of grey tone, and $\frac{5}{8}$ oz of brown tone.

a. If the stylist needs to make 20 oz of dye, how much of each dye colour is needed?

b. If the stylist needs to make 100 oz of dye, how much of each dye colour is needed?

Solution

a. 9 oz red, 6 oz grey, and 5 oz brown. The given amounts of dye make $2\frac{1}{2}$ oz so each quantity needs to be multiplied by 8 to get 20 oz.

b. 45 oz red, 30 oz grey, and 25 oz brown. 100 oz is 5 batches of 20 oz.

5. **Problem 5 Statement**

   a. Here are the vertices of rectangle $FROG$: (-2,5), (-2,1), (6,5), (6,1).
   Find the perimeter of this rectangle. If you get stuck, try plotting the points on a coordinate grid.

   b. Find the area of the rectangle $FROG$.

   c. Here are the coordinates of rectangle $PLAY$: (-11,20), (-11,-3), (-1,20), (-1,-3).
   Find the perimeter and area of this rectangle. See if you can figure out its side lengths without plotting the points.

   Solution

   a. 24 units. The short side of the rectangle has length 4 units, because $5 - 1 = 4$. The long side has length 8 units, because $6 - (-2) = 8$. The perimeter is 24 units, because $4 + 4 + 8 + 8 = 24$.

   b. 32 square units, because $8 \times 4 = 32$

   c. The perimeter is 66 units and the area is 230 square units. The short side has length 10 units, because $-1 - (-11) = 10$. The long side has length 23 units, because $20 - (-3) = 23$. The perimeter is 66 units, because $2(10 + 23) = 66$.