Lesson 10: Distinguishing circumference and area

Goals

- Critique (orally and in writing) claims about the radius, diameter, circumference, or area of a circle in a real-world situation.
- Decide whether to calculate the circumference or area of a circle to solve a problem in a real-world situation, and justify (orally) the decision.
- Estimate measurements of a circle in a real-world situation, and explain (orally and in writing) the estimation strategy.

Learning Targets

- I can decide whether a situation about a circle has to do with area or circumference.
- I can use formulae for circumference and area of a circle to solve problems.

Lesson Narrative

Students have spent several lessons investigating circumference, and then several lessons investigating area, separately. In this lesson, both types of problems are mixed together so students have to distinguish which measurement is called for in each problem situation. Also, in previous lessons students were always given one measurement of each circle, but in this lesson they must rely on their own estimations to solve the problems. Students continue working with answers expressed in terms of π, which was introduced in the previous lesson.

Addressing

- Know the formulae for the area and circumference of a circle and use them to solve problems; give an informal derivation of the relationship between the circumference and area of a circle.

Instructional Routines

- Group Presentations
- Collect and Display
- Compare and Connect
- Discussion Supports
- Take Turns

Required Materials

Pre-printed slips, cut from copies of the blackline master

Card sort: Circle problems Question 1 How much fabric is needed for a round tablecloth?	Card sort: Circle problems Question 2 How fast do you go when riding on a Ferris wheel?
Card sort: Circle problems Question 3 How much green space is there inside a traffic roundabout?	Card sort: Circle problems Question 4 How many square inches of cheese fit on a slice of pizza?
Card sort: Circle problems Question 5 How many times must a horse go round a horse walker to walk 1 mile?	Card sort: Circle problems Question 6 How many feet are travelled by a person riding once around a roundabout?
Card sort: Circle problems Question 7 How much room is there to spread frosting on a cookie?	Card sort: Circle problems Question 8 How far does a unicycle move when the wheel makes 5 full rotations?

Required Preparation

You will need the Card Sort: Circle Problems blackline master for this lesson. Prepare and cut one copy for every 2 students. These can be reused from one class to the next. If possible, copy each complete set of cards on a different colour of paper, so that a stray card can quickly be put back.

Be prepared to explain or show images of any of the examples of circles in the sorting activity that may be unfamiliar to your students.

Student Learning Goals

Let's contrast circumference and area.

10.1 Filling the Plate

Warm Up: 5 minutes

This warm-up prompts students to apply what they have learned about finding the area of a circle to estimating the area of a circular plate in terms of a smaller circle. Students see a plate with a single cheese puff and from this information need to make a reasoned estimate for the number of cheese puffs that will cover the plate. As students discuss their estimates with a partner, monitor the discussions. Select students who use different estimation strategies to share during the discussion.

This activity was inspired by one created by Andrew Stadel http://www.estimation180.com/day-207.html.

Launch

Arrange students in groups of 2. Show them the picture of the plate with one cheese puff and ask them what they notice and wonder.

Let students share their observation and question with one another and invite a few students to share with the class. If the question "How many cheese puffs will fit on the plate" does not come up, ask if any one wondered how many cheese puffs can fit on the plate. Give students two minutes to make an estimate.

Student Task Statement

About how many cheese puffs can fit on the plate in a single layer? Be prepared to explain your reasoning.

Student Response

Answers vary. Sample response: The radius of the plate is about 7 cheese balls, so its area is about 3×7^2 times the area of a cheeseball. So about 150 cheeseballs should fit (as long as there is not too much space left between them).

Activity Synthesis

Invite selected students to share their estimates and any information in the image that informs their estimates. After each explanation, solicit questions from the class that could help the student clarify his or her reasoning. Record the estimates and strategies and display them for all to see.

10.2 Card Sort: Circle Problems

15 minutes

The purpose of this activity is for students to think about how circumference and area of circles apply to real-world situations. First, students sort slips based on whether the question is related to the circumference or area of a circle. Next, each group focuses on one of the questions (#1 through 5). They estimate appropriate measurements for the context, and use these measurements to calculate a reasonable answer. Questions 6 through 8 will be examined more closely in a future activity.

You will need the blackline master for this activity.

Instructional Routines

- Collect and Display

- Take Turns

Launch

Arrange students in groups of 2. Explain or show images of any of the contexts that may be unfamiliar to your students.

Distribute question slips. Give students 3–4 minutes of partner work time to sort the question slips. Pause to poll the class on how they sorted each card. After students have come to an agreement on the sorting, assign each group one card from #1–5 to investigate further. Give students quiet work time to make their estimates and calculations, followed by small-group and whole-class discussion.

Representation: Internalise Comprehension. Chunk this task into more manageable parts to differentiate the degree of difficulty or complexity by beginning with fewer cards. For example, give students a subset of the cards to start with, and introduce the remaining cards once students have completed their initial round of sorting.
Supports accessibility for: Conceptual processing; Organisation Conversing, Reading: Collect and Display. As students sort the cards into two groups, write down the language students use to decide whether to use the circumference or area of the circle to answer the question. Sort the language into two columns labelled "circumference" and "area" of a circle. Listen for students who clarify that the circumference measures distance around a circle and uses linear units, while area measures the inside of a circle and uses square units. Display the language collected and encourage students to clarify the meaning of a word or phrase. For example, a phrase such as "We should use area because we want to know how big the pizza is" can be clarified by rephrasing it as "We should use area because we want to know how many square inches of cheese fit on the pizza." This routine will provide feedback to students in a way that supports sense-making while simultaneously increasing meta-awareness of mathematical language.
Design Principle(s): Support sense-making; Maximise meta-awareness

Anticipated Misconceptions

If students are confused about the difference between circumference and area, remind them that circumference measures distance around a circle and uses linear units, while area measures the inside of a circle and uses square units.

Students might think they need to solve the problems on all 8 cards. Point out that the first question is only asking them to think about *how* they would solve the problems, not to do any actual calculations.

For the horse walker problem, students might not realise that they need to convert 1 mile to the same units as their estimated diameter, or that they need to divide by the circumference.

Student Task Statement

Your teacher will give you cards with questions about circles.

1. Sort the cards into two groups based on whether you would use the circumference or the area of the circle to answer the question. Pause here so your teacher can review your work.

2. Your teacher will assign you a card to examine more closely. What additional information would you need in order to answer the question on your card?

3. Estimate measurements for the circle on your card.

4. Use your estimates to calculate the answer to the question.

Student Response

1. Area: 1, 3, 4, 7; Circumference: 2, 5, 6, 8

2. For every card, knowing the radius or diameter of the circle would help solve the problem because the circumference and area can both be calculated from the radius or diameter. Additionally, for the Ferris wheel problem you would need to know the time it takes to go around the Ferris wheel once. For the pizza problem, you would need to know the number of slices the pizza is cut into (and that they are roughly the same size). For the horse walker problem, you would need to know how to convert 1 mile into a smaller unit of measure.

3. Answers vary. Possible solutions:

    - Question 1: A radius between 8 and 40 inches
    - Question 2: A diameter between 10 and 160 metres and about 1 to 30 minutes per rotation
    - Question 3: A radius between 5 and 25 yards
    - Question 4: A radius between 4 and 9 inches and 8 slices per pizza
    - Question 5: A diameter between 9 and 33 yards and 1 760 yards per mile

4. Answers vary. Possible solutions:

    - Question 1: Between 64π and $1\,600\pi$ in^2
    - Question 2: From between 10π and 160π metres per minute (1 revolution per minute) to between $\frac{1}{3}\pi$ and $\frac{16}{3}\pi$ metres per minute (1 revolution per 30 minutes)
    - Question 3: Between 25π and 625π yd^2
    - Question 4: Between 2π and $10\frac{1}{8}\pi$ in^2
    - Question 5: Between $\frac{1\,760}{33\pi}$ and $\frac{1\,760}{9\pi}$ rotations

Activity Synthesis

The goal of this discussion is for students to articulate how they decide whether an answer is reasonable.

First, have the students who worked on the same question compare answers and strategies. Display these questions to guide their group discussions:

- Did you use the same units?
- How did you come up with your estimate for the size of the circle (radius or diameter)?
- Are your estimates very close? Are they reasonable?
- How did you calculate your answer to the question?
- Are your answers very close? Are they reasonable?

Next, invite students who have different answers to the same question to share their reasoning with the class. For each group, ask the rest of the class "Which of these answers do you think are reasonable? Why?" Make sure students understand that since estimates were called for, there is not one exact correct answer for each of these problems.

Some mistakes that could lead to an unreasonable answer include:

- Making too inaccurate of an initial estimate about the size of the circle
- Using the diameter as if it were the radius, or vice versa
- Using the wrong formula for circumference or area
- Forgetting to address an aspect of the question (such as finding the area of the entire pizza, not one slice)
- Labelling the units incorrectly, like using feet for a measure of area instead of square feet
- Reporting an answer with more decimals places than is reasonable given the level of precision of their initial estimates

If not mentioned by students, look for opportunities to bring these up, in preparation for a future activity in which students will analyse claims made about questions #6–8.

10.3 Visual Display of Circle Problem

Optional: 15 minutes

This activity asks students to create a visual display of the circle problem that they solved previously. They can practise explaining their reasoning more clearly on this display than they did in the previous activity. This gives them an opportunity to organise and record their information in a way that can be shared with others who worked on a different problem. The displays can also serve as a record of reasoning about circles which can be referred back to later in the year.

Instructional Routines

- Group Presentations
- Compare and Connect

Launch

Keep students in the same groups. Explain that they are going to create a visual display of the circle problem that they just worked on. Follow with discussion.

Engagement: Internalise Self-Regulation. Provide a project checklist that chunks the various steps of the project into a set of manageable tasks.
Supports accessibility for: Organisation; Attention

Student Task Statement

In the previous activity you estimated the answer to a question about circles.

Create a visual display that includes:

- The question you were answering
- A diagram of a circle labelled with your estimated measurements
- Your thinking, organised so that others can follow it
- Your answer, expressed in terms of π and also expressed as a decimal approximation

Student Response

Answers vary.

Activity Synthesis

Arrange for groups that are assigned the same problem to present their visual displays near one another. Give students a few minutes to visit the displays and to see the estimates others used to answer the question. Before students begin a gallery walk, ask them to be prepared to share a couple of observations about how their estimates and strategies are the same as or different from others'.

After the gallery walk, invite a couple of students to share their observations.

Representing, Speaking, Listening: Compare and Connect. After groups have prepared a visual display that shows how they solved the circle problem on the card, arrange pairs of groups with different estimated measurements for the same problem. As groups investigate each other's work, ask them to share what worked well in a particular approach or what is especially clear in a particular diagram. Listen for and amplify any comments about what might make an approach or diagram more complete or easy to understand. Then encourage students to make connections between the radius and circumference or area of the circle. Listen for and amplify language students use to describe how their estimation for the radius affects the circumference or area of the circle. This will foster students' meta-awareness and support constructive conversations as they compare strategies for estimating the measurements of a circle and make connections between the radius and circumference or area of a circle. *Design Principle(s): Cultivate conversation; Maximise meta-awareness*

10.4 Analysing Circle Claims

10 minutes

The purpose of this activity is for students to look more closely at the last three situations of the card sort activity (questions 6 through 8). They analyse and critique two claims about each situation, choosing or supplying the best response and explaining why. Students must recognise that in the first situation, one of the claims inaccurately estimates the size of the circle. In the second situation, one of the claims calculates the circumference instead of the area. In the third situation, both claims are inaccurate. One of the claims has the right number but uses square units, and the other has the right units but the wrong number.

As students work, monitor and select students who can explain why they agree with the correct claim and others who can explain why they disagree with the incorrect claim. For the third situation, select students who can explain why they disagree with both of the claims, even if they are unsure they are allowed to disagree with both.

Instructional Routines

- Discussion Supports

Launch

Keep students in same groups. Tell students they are going to look at how some other students solved the questions on cards 6, 7, and 8. For the first situation, make sure students realise we are referring to the type of roundabout at a playground (as pictured in their books or devices), not the larger type of carousel they might see at a fair.

Action and Expression: Develop Expression and Communication. To help get students started, display sentence frames such as: "I agree/disagree because….", "That could/couldn't be true because….", and "_____'s idea reminds me of…."
Supports accessibility for: Language; Organisation

Anticipated Misconceptions

Students might multiply by a decimal approximation, without recognising that the answers in the claims are all given in terms of π.

Students might not realise there is an error with both of the claims in the third question.

Finally, students might not realise that they are supposed to analyse the reasonableness of the estimates, not just the mathematical correctness of the calculations.

Student Task Statement

Here are two students' answers for each question. Do you agree with either of them? Explain or show your reasoning.

1. How many feet are travelled by a person riding once around the roundabout?

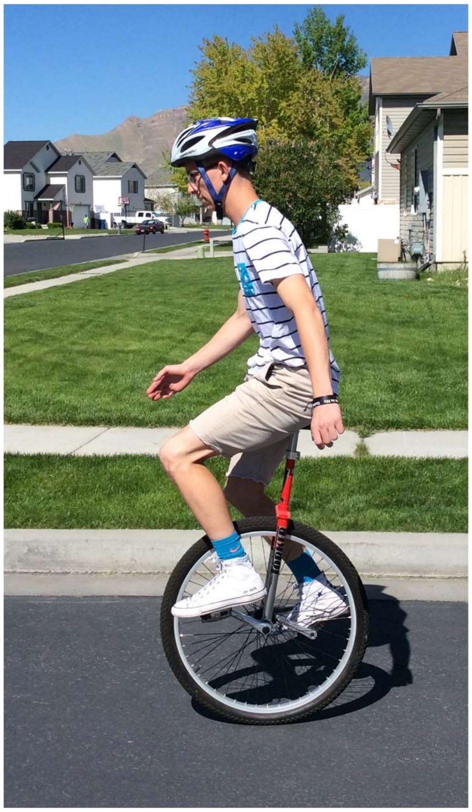
   - Clare says, "The radius of the roundabout is about 4 feet, so the distance around the edge is about 8π feet."
   - Andre says, "The diameter of the roundabout is about 4 feet, so the distance around the edge is about 4π feet."

2. How much room is there to spread frosting on the cookie?

   - Clare says "The radius of the cookie is about 3 centimetres, so the space for frosting is about 6π cm^2."

– Andre says "The diameter of the cookie is about 3 inches, so the space for frosting is about 2.25π in^2."

3. How far does the unicycle move when the wheel makes 5 full rotations?



– Clare says, "The diameter of the unicycle wheel is about 0.5 metres. In 5 complete rotations it will go about $\frac{5}{2}\pi$ m^2."

– Andre says, "I agree with Clare's estimate of the diameter, but that means the unicycle will go about $\frac{5}{4}\pi$ m."

Student Response

1. Clare's claim is more reasonable. Both people correctly calculated the circumference, given their estimated dimension for the circle. However, Andre's estimated diameter of 4 feet is too small, given the relative size of the child.

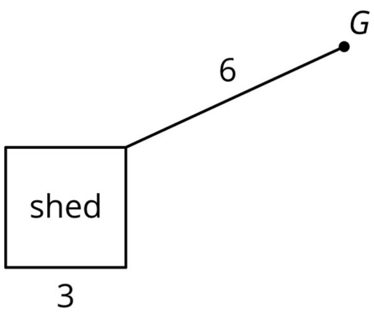
2. Andre's claim is more reasonable. Both estimated measurements for the circle are reasonable. However, Clare applied the circumference formula when the problem called for the area.

3. Neither claim is accurate. Clare has the correct number but answered with square units when the problem called for linear units. Andre has the correct units, but he squared the diameter when calculating the numerical value, as if he were using the

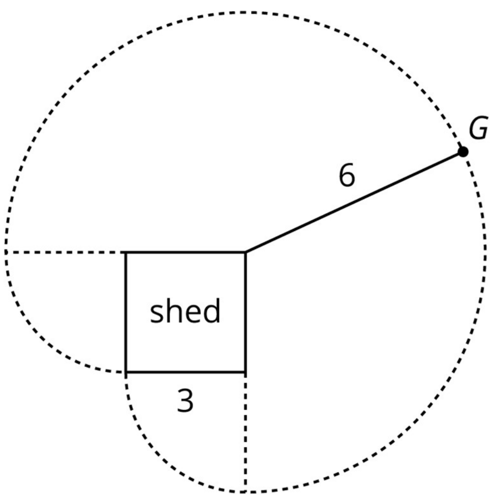
radius to find the area. (It is also possible that Andre mistakenly used the radius instead of the diameter in the circumference formula.)

Are You Ready for More?

A goat (point G) is tied with a 6-foot rope to the corner of a shed. The floor of the shed is a square whose sides are each 3 feet long. The shed is closed and the goat can't go inside. The space all around the shed is flat, grassy, and the goat can't reach any other structures or objects. What is the area over which the goat can roam?



Student Response



31.5π square feet (or approximately 99 square feet). The edge of the goat's roaming area is three quarters of a circle with radius 6 feet, until the rope gets caught on the corner of the shed, at which point the goat has two quarter-circles with radius 3 feet. Adding $\frac{3}{4}\pi \times 6^2$ and $2 \times \frac{1}{4}\pi \times 3^2$ gives 31.5π square feet.

Activity Synthesis

For each situation, poll the class on which person’s claim is more accurate. Ask selected students to share their reasoning for each claim. If it does not come out during the discussion, point out that the formula for the area of a circle has a squared term and the units of the answer are square units, while the formula for the circumference does not have a squared term and the units of the answer are linear units.

Note: It is not possible to know for certain what Clare or Andre were thinking when they made their calculations. For example, it is likely in the second problem that Clare found the circumference of the cookie instead of its area, but it is not possible to know. A wide range of interpretations need to be considered, always keeping an open mind.

Speaking: Discussion Supports. As students share whether they agree or disagree with each claim, press for details in students' reasoning by asking whether they should find the circumference or area of the circle in the problem. Also, ask students whether the estimates for the circumference or area are accurate given the estimated radius or diameter of the circle. Listen for and amplify comments that clarify when it is appropriate to use the formula for the circumference or area of a circle. Also, listen for language students use to describe Clare's or Andre's estimation errors or use of the incorrect formula. This will support rich and inclusive discussion about strategies for applying the formulae for the circumference and area of a circle to solve problems.
Design Principle(s): Support sense-making

Lesson Synthesis

Discussion Questions:

- When would we need to calculate the circumference of a circle?
- When would we need to calculate the area?
- What do you need to know to estimate or calculate the circumference of a circle? (radius or diameter)
- What do you need to know in order to estimate or calculate the area of a circle? (radius or diameter)

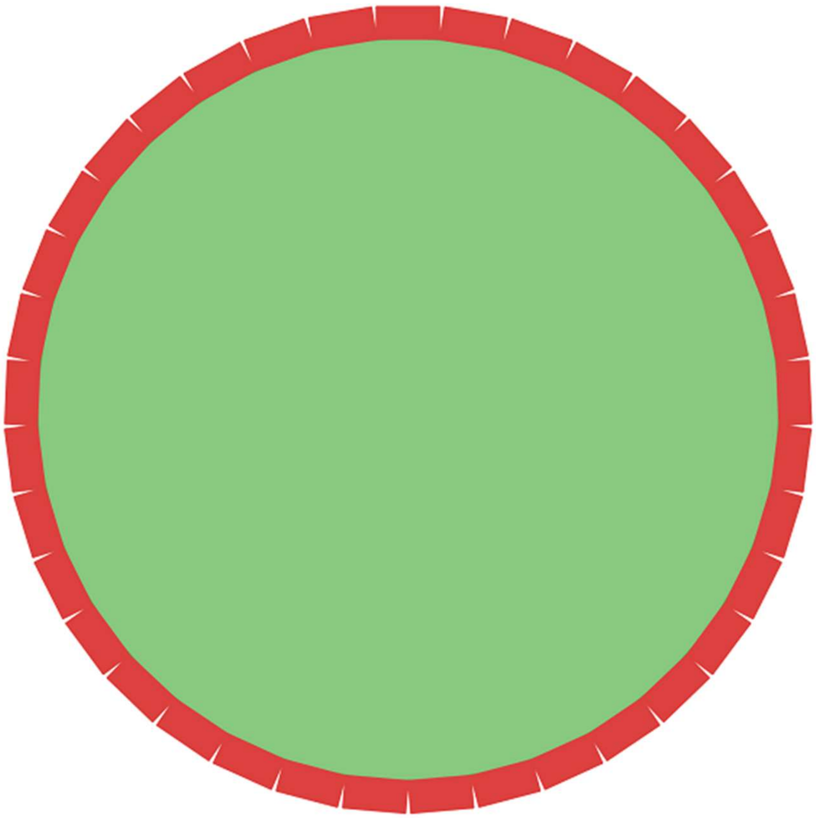
Consider posting the students' displays from the card sorting activity, grouped by circumference or area, so students can refer to them later.

10.5 Measuring a Circular Lawn

Cool Down: 5 minutes

Student Task Statement

A circular lawn has a row of bricks around the edge. The diameter of the lawn is about 40 feet.



1. Which is the best estimate for the amount of grass in the lawn?

    a. 125 feet

    b. 125 square feet

    c. 1 250 feet

    d. 1 250 square feet

2. Which is the best estimate for the total length of the bricks?

    a. 125 feet

    b. 125 square feet

    c. 1 250 feet

    d. 1 250 square feet

Student Response

1. D. 1 250 square feet

2. A. 125 feet

Student Lesson Summary

Sometimes we need to find the circumference of a circle, and sometimes we need to find the area. Here are some examples of quantities related to the circumference of a circle:

- The length of a circular path.
- The distance a wheel will travel after one complete rotation.
- The length of a piece of rope coiled in a circle.

Here are some examples of quantities related to the area of a circle:

- The amount of land that is cultivated on a circular field.
- The amount of frosting needed to cover the top of a round cake.
- The number of tiles needed to cover a round table.

In both cases, the radius (or diameter) of the circle is all that is needed to make the calculation. The circumference of a circle with radius r is $2\pi r$ while its area is πr^2. The circumference is measured in linear units (such as cm, in, km) while the area is measured in square units (such as cm^2, in^2, km^2).

Lesson 10 Practice Problems

1. **Problem 1 Statement**

   For each problem, decide whether the circumference of the circle or the area of the circle is most useful for finding a solution. Explain your reasoning.

   a. A car's wheels spin at 1 000 revolutions per minute. The diameter of the wheels is 23 inches. You want to know how fast the car is travelling.

   b. A circular kitchen table has a diameter of 60 inches. You want to know how much fabric is needed to cover the table top.

   c. A circular puzzle is 20 inches in diameter. All of the pieces are about the same size. You want to know about how many pieces there are in the puzzle.

   d. You want to know about how long it takes to walk around a circular pond.

   Solution

   a. Circumference. The circumference of the wheels and the number of revolutions per minute tells you how far the car is travelling and this can be used to calculate the speed.

   b. Area. The fabric covers the surface of the table and it is this area that is needed.

c. Area. The area of the puzzle divided by the area of a puzzle piece will give an estimate of the number of pieces.

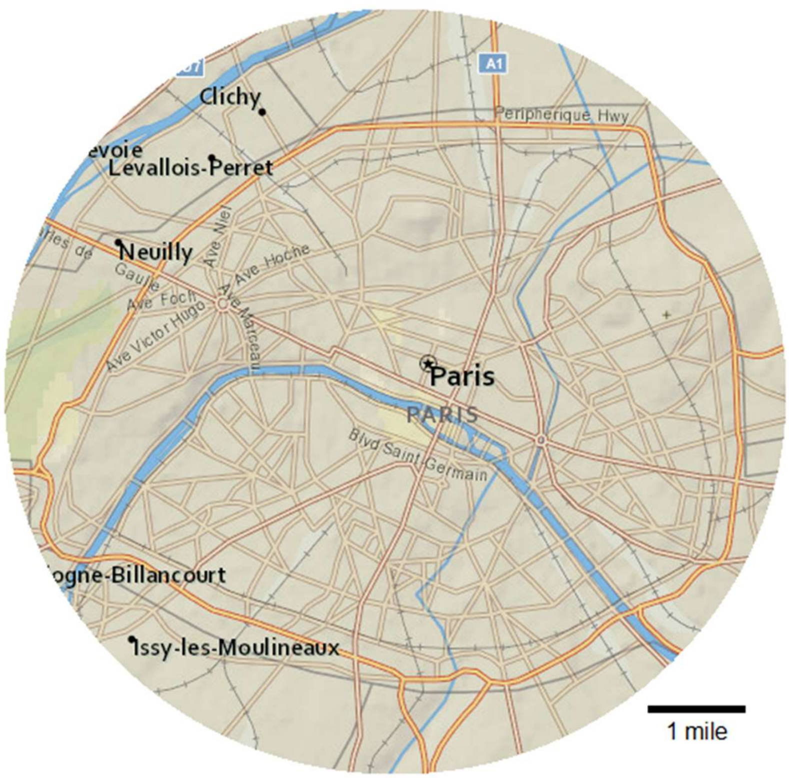
d. Circumference. You need to know the distance around the pond which is its circumference.

2. **Problem 2 Statement**

The city of Paris, France is completely contained within an almost circular road that goes around the edge. Use the map with its scale to:

a. Estimate the circumference of Paris.

b. Estimate the area of Paris.



Solution

Answers vary. Sample response:

a. About 6π miles (or about 20 miles)

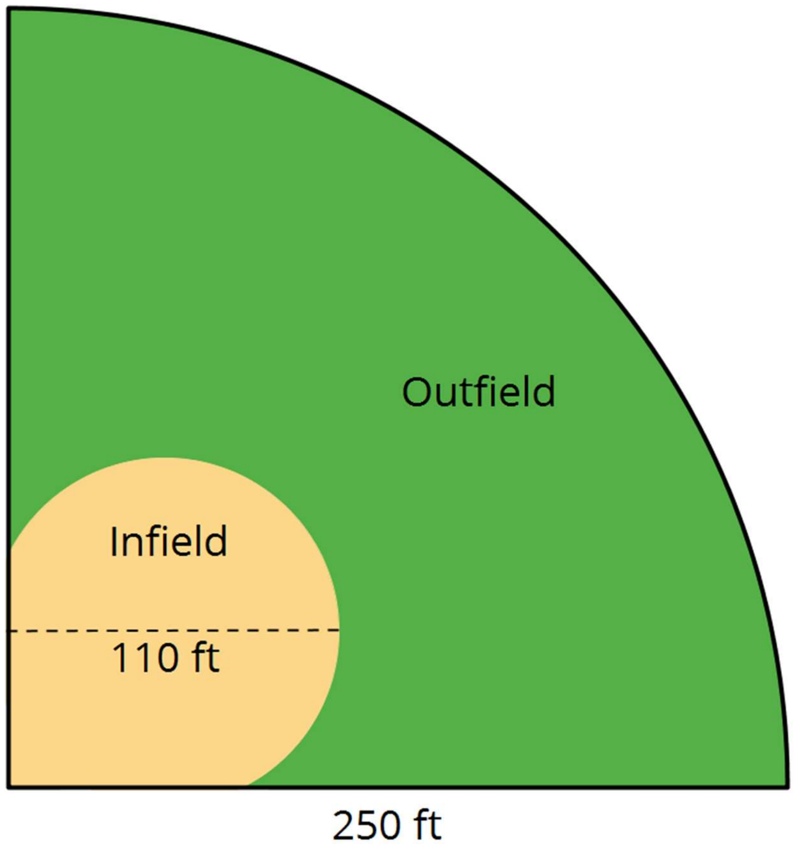
b. About $(3)^2\pi$ mi^2 (or about 30 mi^2)

3. **Problem 3 Statement**

Here is a diagram of a softball field:

a. About how long is the fence around the field?

b. About how big is the outfield?



Solution

Answers vary. Sample responses:

a. $500 + 125\pi$ (or about 893 ft): This estimate assumes that the curved boundary of the outfield is modelled by a quarter circle.

b. $12\,600\pi$ (or about 39 600 ft^2): The area of the full softball field, modelled by a quarter circle, is $\frac{1}{4} \times \pi \times 250^2$ or $15\,625\pi$ square feet. The infield, which needs to be subtracted, has about the same area as a circle of radius 55 feet or $3\,025\pi$ square feet. The difference is $12\,600\pi$ square feet. Note that if we draw a circle with diameter 110 feet (where the 110 foot measurement is marked), it misses some of the lower left part of the infield but also contains some extra area below the softball field so this is a good estimate.

4. **Problem 4 Statement**

While in a maths lesson, Priya and Kiran come up with two ways of thinking about the proportional relationship shown in the table.

x	y
2	?
5	1750

Both students agree that they can solve the equation $5k = 1\,750$ to find the constant of proportionality.

– Priya says, "I can solve this equation by dividing 1 750 by 5."

– Kiran says, "I can solve this equation by multiplying 1 750 by $\frac{1}{5}$."

a. What value of k would each student get using their own method?

b. How are Priya and Kiran's approaches related?

c. Explain how each student might approach solving the equation $\frac{2}{3}k = 50$.

Solution

a. 350

b. Priya divided each side of the equation by the same number. Seeing that 5 and k were multiplied in the equation, she used division to get k by itself. Meanwhile, Kiran multiplied by the reciprocal of 5.

c. Priya divides by $\frac{2}{3}$ since k is being multiplied by $\frac{2}{3}$. Her equation is $k = 50 \div \frac{2}{3}$. Kiran multiplies by the reciprocal of $\frac{2}{3}$. His equation is $k = \frac{3}{2} \times 50$.