Lesson 3: More about constant of proportionality

Goals

- Compare, contrast, and critique (orally and in writing) different ways to express the constant of proportionality for a relationship.
- Explain (orally) how to determine the constant of proportionality for a proportional relationship represented in a table.
- Interpret the constant of proportionality for a relationship in the context of constant speed.

Learning Targets

- I can find missing information in a proportional relationship using a table.
- I can find the constant of proportionality from information given in a table.

Lesson Narrative

In this lesson, students continue to work with proportional relationships represented by tables using contexts familiar from previous years: unit conversion and constant speed. They recognise the constant of proportionality as the conversion factor or the speed, and use it to answer questions about the context. Although students might continue to reason with equivalent ratios to solve problems, the contexts are designed so that it is more efficient to use the constant of proportionality. For example, when converting length measurements from feet to inches, it is more convenient to know the rule "multiply by 12" than to use an equivalent ratio with a different scale factor every time: "1 foot is 12 inches, so multiplying both quantities by 3 I see that 3 feet is 36 inches, and multiplying both quantities by 5 I see that 5 feet is 60 inches."

Building On

- Convert among different-sized standard measurement units within a given measurement system (e.g., convert 5 cm to 0.05 m), and use these conversions in solving multi-step, real world problems.

Addressing

- Recognise and represent proportional relationships between quantities.
- Decide whether two quantities are in a proportional relationship, e.g., by testing for equivalent ratios in a table or graphing on a coordinate plane and observing whether the graph is a straight line through the origin.
- Identify the constant of proportionality (unit rate) in tables, graphs, equations, diagrams, and verbal descriptions of proportional relationships.

Instructional Routines

- Stronger and Clearer Each Time
- Discussion Supports

Student Learning Goals

Let's solve more problems involving proportional relationships using tables.

3.1 Equal Measures

Warm Up: 5 minutes

This warm-up prompts students to review work in KS2 of converting across different-sized standard units within a given measurement system.

Launch

Arrange students in groups of 2. Ask students to use the following numbers and units to record as many equivalent expressions as they can. Tell them they are allowed to reuse numbers and units. Give students 2 minutes of quiet think time followed by 1 minute to share their equations with their partner.

Student Task Statement

Use the numbers and units from the list to find as many equivalent measurements as you can. For example, you might write "30 minutes is $\frac{1}{2}$ hour."

You can use the numbers and units more than once.

1

12

0.4

60

50

$\frac{1}{2}$

40

0.01

$3\frac{1}{3}$

30

0.3

24

$\frac{1}{10}$

6

2

$\frac{2}{5}$

centimetre

metre

hour

feet

minute

inch

Student Response

Answers vary. Possible responses:

- 1 cm is 0.01 m, 30 cm is 0.3 m, 40 cm is 0.4 m, $\frac{1}{2}$ m is 50 cm
- $\frac{1}{2}$ hr is 30 min, 1 hr is 60 min, $\frac{1}{10}$ hr is 6 min, $\frac{2}{5}$ hr is 0.4 hr or 24 min
- 24 in is 2 ft, 12 in is 1 ft, 6 in is $\frac{1}{2}$ ft, 40 in is $3\frac{1}{3}$ ft

Activity Synthesis

Invite a few students to share equations that they had in common with their partner and ones that were different. Record these answers for all to see. After each equation is shared, ask students to give a signal if they had the same one recorded. Display the following questions for all to see and discuss:

- "What number(s) did you use the most? Why?"
- "If you could include two more cards to this selection, what would they be? Why?"

3.2 Centimetres and Millimetres

10 minutes

This activity has two purposes. First, it involves an important context that can be represented by proportional relationships, namely measurement conversion. Second, it introduces the idea that there are two constants of proportionality, and that they are reciprocals (also known as multiplicative inverses). Students understand why this is the case later when they use equations to represent proportional relationships. Students start to use "is proportional to" language to distinguish between the two constants of proportionality. During the discussion, students should be reminded that dividing by a number is equivalent to multiplying by its reciprocal. In principle, this is something students learn earlier in KS3, but they often need to be reminded. This will be needed in future lessons, so it is important to discuss it here.

Conversion of centimetres to millimetres has a constant of proportionality of 10 while conversion of millimetres to centimetres has a constant of proportionality of $\frac{1}{10}$. One way to explain why these two constants of proportionality are multiplicative inverses is to imagine starting with a measurement in centimetres, 15 cm for example. When we convert to millimetres we multiply by 10. 10×15 cm = 150 mm. If we convert the measurement in millimetres back to centimetres, we know it will be 15, so the constant of proportionality we need to multiply by is $\frac{1}{10}$.

Instructional Routines

- Discussion Supports

Launch

Tell students, "Let's look at how centimetres and millimetres are related and how it is related to what we have been doing recently."

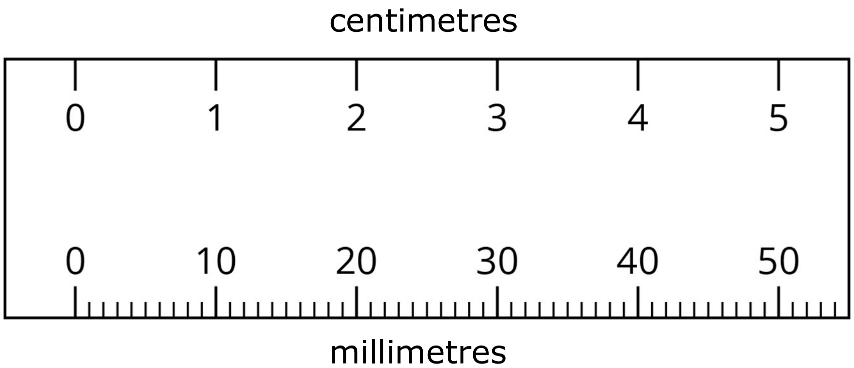
Representation: Internalise Comprehension. Activate or supply background knowledge. Create and review a display that includes 1–2 examples to remind students that dividing by a number is equivalent to multiplying by its reciprocal. Keep the display visible throughout the activity.
Supports accessibility for: Memory; Conceptual processing

Anticipated Misconceptions

Some students may say that the constants of proportionality are both 10 since you can divide by 10 in the second table. Tell students, "The constant of proportionality is what you *multiply* by. Can you find a way to *multiply* the numbers in the first column to get the numbers in second column?"

Student Task Statement

There is a proportional relationship between any length measured in centimetres and the same length measured in millimetres.



There are two ways of thinking about this proportional relationship.

1. If you know the length of something in centimetres, you can calculate its length in millimetres.

   a. Complete the table.

   b. What is the constant of proportionality?

length (cm)	length (mm)
9	
12.5	
50	
88.49	

2. If you know the length of something in millimetres, you can calculate its length in centimetres.

   a. Complete the table.

   b. What is the constant of proportionality?

length (mm)	length (cm)
70	
245	

4	
699.1	

3. How are these two constants of proportionality related to each other?

4. Complete each sentence:

   a. To convert from centimetres to millimetres, you can multiply by _______.

   b. To convert from millimetres to centimetres, you can divide by _______ *or* multiply by _______.

Student Response

1. a.

length (cm)	length (mm)
9	90
12.5	125
50	500
88.49	884.9

b. 10

2. a.

length (mm)	length (cm)
70	7
245	24.5
4	0.4
699.1	69.91

b. $\frac{1}{10}$ or 0.1.

3. 10 and $\frac{1}{10}$ are reciprocals.

4. a. To convert from centimetres to millimetres, you can multiply by 10.

b. To convert from millimetres to centimetres, you can divide by 10 *or* multiply by $\frac{1}{10}$.

Are You Ready for More?

1. How many square millimetres are there in a square centimetre?

2. How do you convert square centimetres to square millimetres? How do you convert the other way?

Student Response

1. 100

2. Divide by 100. Multiply by $\frac{1}{100}$

Activity Synthesis

The goal of this discussion is to help students see the connection between this situation and the earlier tasks, so they can use the structure of the table to find the constants of proportionality. Ask questions like, "Can you use any of the strategies we have been discussing in earlier problems to help you solve this problem?" as students work to help them make this connection.

If there is disagreement about the constants of proportionality, ask students to explain their thinking.

Look for students who have seen the connection between dividing by 10 and multiplying by $\frac{1}{10}$. Ask the students if there is a generalisation to be made here. Help them articulate the idea that when you divide by a number, that is the same as multiplying by the reciprocal.

Speaking, Listening, Representing: Discussion Supports. Use this routine to support whole-class discussion. As students share what they identified as the constant of proportionality and their strategies they used to identify the constant of proportionality, press for details in students' explanations by requesting students to challenge an idea, elaborate on an idea, or give an example. In this discussion, demonstrate uses of disciplinary language functions such as detailing steps, describing and justifying reasoning, and questioning strategies. If necessary, revoice student ideas to demonstrate mathematical language, and invite students to chorally repeat phrases that include relevant vocabulary in context, such as " constant of proportionality".
Design Principle(s): Support sense-making

3.3 Pittsburgh to Phoenix

15 minutes

This activity focuses on making connections between constant speed and proportional relationships, with special attention to the constant of proportionality. Earlier in KS3, students solved problems involving constant speed, but they need opportunities to make the connection to proportional relationships; students who successfully make this connection are reasoning abstractly about contexts with constant speed. Students who need support in understanding the context can trace the segments in the map, labelling the distances they know and putting question marks for unknown distances. An empty double number line could also be a useful tool in helping students reason about the context.

The numbers are chosen so that students are more likely to use unit rate rather than scale factors. The goal of the discussion is for students to understand that when speed is constant, time elapsed and distance travelled are proportional. The constant of proportionality indicates the magnitude of the speed. For example, if distance is given in miles and time in hours, then the constant of proportionality indicates the speed in miles per hour.

Students might wonder about the route. If they ask about it, teachers can share some reasons why airplane routes are complex, e.g., the need to avoid congested areas and the fact that the shortest distances on the curved surface of Earth do not always correspond to lines on a map.

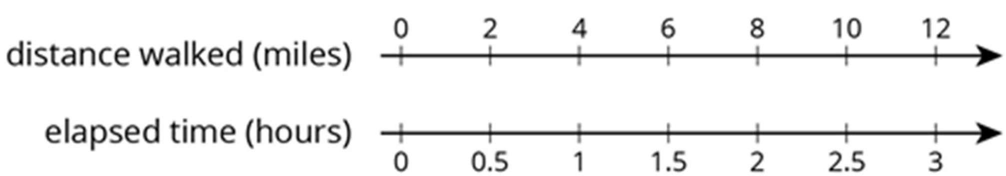
Instructional Routines

- Stronger and Clearer Each Time

Launch

It may have been some time since students reasoned about constant speed, so before they start on this activity, consider posing some simple problems about constant speed. For example:

- Someone walks at a constant speed of 4 miles per hour. How much time does it take them to walk 4 miles? 8 miles? 20 miles? 2 miles? $\frac{1}{2}$ mile? How far do they get in 3 hours? In 10 minutes?
- Someone rides a bike at a constant speed. They go 30 miles in 2 hours. What was their speed? If students did some work with double number lines when learning about constant speed earlier in KS3, they may be inclined to create one to help them think through these launch questions and when they get into the activity. If no one suggests it, it might be worth showing a double number line when talking through this launch, so that while working on the activity, students recall that these are useful tools. For example:



Representation: Internalise Comprehension. Activate or supply background knowledge. Allow students to use four-function calculators to ensure inclusive participation in the activity.
Supports accessibility for: Memory; Conceptual processing Speaking: Stronger and Clearer Each Time. Use this routine with successive pair shares to give students a structured opportunity to revise and refine their response to the final question. Ask each student to meet with 2-3 other partners in a row for feedback. Provide students with prompts for feedback that will help teams strengthen their ideas, such as, "How do the units affect your answer?" or "Why do you think that value is a constant of proportionality?" Students can borrow ideas and language from each partner to strengthen their final opinion and reasons.

Design Principle(s): Optimise output (for explanation)

Student Task Statement

On its way from New York to San Diego, a plane flew over Pittsburgh, Saint Louis, Albuquerque, and Phoenix travelling at a constant speed.

Complete the table as you answer the questions. Be prepared to explain your reasoning.

segment	time	distance	speed
Pittsburgh to Saint Louis	1 hour	550 miles	
Saint Louis to Albuquerque	1 hour 42 minutes		
Albuquerque to Phoenix		330 miles	

1. What is the distance between Saint Louis and Albuquerque?

2. How many minutes did it take to fly between Albuquerque and Phoenix?

3. What is the proportional relationship represented by this table?

4. Diego says the constant of proportionality is 550. Andre says the constant of proportionality is $9\frac{1}{6}$. Do you agree with either of them? Explain your reasoning.

Student Response

segment	time	distance	speed
Pittsburgh to Saint Louis	1 hour	550 miles	550 miles per hour
Saint Louis to Albuquerque	1 hour 42 minutes	935 miles	550 miles per hour
Albuquerque to Phoenix	36 minutes	330 miles	550 miles per hour

1. 935 miles. 42 minutes is $\frac{42}{60}$ hours or $\frac{7}{10}$ of an hour. $\frac{7}{10}$ of 550 is 385, and $385 + 550 = 935$.

2. 36 minutes. 330 is $\frac{3}{5}$ of 550, and $\frac{3}{5}$ of 60 minutes is 36 minutes.

3. The distance travelled is proportional to the elapsed time.

4. Answers vary. Possible responses: Diego uses miles per hour, and Andre uses miles per minute.

segment	time	distance	speed
Pittsburgh to Saint Louis	1 hour	550 miles	550 miles per hour
Saint Louis to Albuquerque	1.7 hours	935 miles	550 miles per hour
Albuquerque to Phoenix	0.6 hours	330 miles	550 miles per hour
segment	time	distance	speed
Pittsburgh to Saint Louis	60 minutes	550 miles	$9\frac{1}{6}$ miles per minute
Saint Louis to Albuquerque	102 minutes	935 miles	$9\frac{1}{6}$ miles per minute
Albuquerque to Phoenix	36 minutes	330 miles	$9\frac{1}{6}$ miles per minute

Activity Synthesis

The goal of the discussion is for students to understand that when speed is constant, then distance travelled is proportional to elapsed time. The constant of proportionality is the speed. For example, if distance is given in miles and time in hours, then the constant of proportionality indicates the speed in miles per hour.

Begin by having a student share a table that is all in hours, miles, and miles per hour. Use this example to point out that every number in the first column can be multiplied by the speed to get the number in the second column. Ask:

- "Which quantities are in a proportional relationship? How do you know?"
- "What is the constant of proportionality in this case?"

If a student wrote their times in minutes and speed in miles per minute, you can have the same discussion if the class seems ready for that leap, but it is not required. In any case, summarise by making it explicit that when time and distance are in a proportional relationship, the constant of proportionality is the speed.

Lesson Synthesis

Briefly revisit the two contexts, demonstrating the use of new terms. For example,

- In the first activity, we examined the proportional relationship between millimetres and centimetres from two different perspectives and found two constants of proportionality. What were they? What is the relationship between the two constants of proportionality? (They are reciprocals.)
- In the second activity, we examined a proportional relationship between the time a plane flies and the distance it travels. What was the constant of proportionality in this task? What does the constant of proportionality represent in terms of the context? (Magnitude of speed.)

3.4 Fish Tank

Cool Down: 5 minutes

Student Task Statement

Mai is filling her fish tank. Water flows into the tank at a constant rate. Complete the table as you answer the questions.

1. How many gallons of water will be in the fish tank after 3 minutes? Explain your reasoning.
2. How long will it take to fill the tank with 40 gallons of water? Explain your reasoning.
3. What is the constant of proportionality?

time (minutes)	water (gallons)
0.5	0.8
1	
3	
	40

Student Response

1. 4.8. If you double the first row (scale by 2), you get 1.6 gallons after 1 minute. If you triple the second row (scale by 3), you get 4.8 gallons after 3 minutes. Or you could scale the first row by 6 to get 4.8 gallons after 3 minutes.

2. 25 minutes. One way to find a scale factor to use is to divide 40 by 0.8: $\frac{40}{0.8} = 50$. That scale factor multiplied by the value in the first column of the first row is: $50 \times 0.5 = 25$.

3. 1.6 or equivalent. You can observe the amount of water that corresponds with 1 minute, or you can divide any value in the right column with its corresponding value in the left column.

Student Lesson Summary

When something is travelling at a constant speed, there is a proportional relationship between the time it takes and the distance travelled. The table shows the distance travelled and elapsed time for a bug crawling on a sidewalk.

Distance travelled (cm)	elapsed time (sec)
$\frac{3}{2}$	1
1	$\frac{2}{3}$
3	2
10	$\frac{20}{3}$

$\times \frac{2}{3}$

We can multiply any number in the first column by $\frac{2}{3}$ to get the corresponding number in the second column. We can say that the elapsed time is proportional to the distance travelled, and the constant of proportionality is $\frac{2}{3}$. This means that the bug's *pace* is $\frac{2}{3}$ seconds per centimetre.

This table represents the same situation, except the columns are switched.

elapsed time (sec)	Distance travelled (cm)
1	$\frac{3}{2}$
$\frac{2}{3}$	1
2	3
$\frac{20}{3}$	10

$\times \frac{3}{2}$

We can multiply any number in the first column by $\frac{3}{2}$ to get the corresponding number in the second column. We can say that the distance travelled is proportional to the elapsed time, and the constant of proportionality is $\frac{3}{2}$. This means that the bug's *speed* is $\frac{3}{2}$ centimetres per second.

Notice that $\frac{3}{2}$ is the reciprocal of $\frac{2}{3}$. When two quantities are in a proportional relationship, there are two constants of proportionality, and they are always reciprocals of each other. When we represent a proportional relationship with a table, we say the quantity in the second column is proportional to the quantity in the first column, and the corresponding constant of proportionality is the number we multiply values by in the first column to get the values in the second.

Lesson 3 Practice Problems

1. **Problem 1 Statement**

   Noah is running a portion of a marathon at a constant speed of 6 miles per hour.

   Complete the table to predict how long it would take him to run different distances at that speed, and how far he would run in different time intervals.

time in hours	miles travelled at 6 miles per hour
1	
$\frac{1}{2}$	
$1\frac{1}{3}$	
	$1\frac{1}{2}$
	9
	$4\frac{1}{2}$

Solution

time in hours	miles travelled at 6 miles per hour
1	6
$\frac{1}{2}$	3
$1\frac{1}{3}$	8
$\frac{1}{4}$	$1\frac{1}{2}$
$1\frac{1}{2}$	9
$\frac{3}{4}$	$4\frac{1}{2}$

2. **Problem 2 Statement**

One kilometre is 1000 metres.

a. Complete the tables. What is the interpretation of the constant of proportionality in each case?

metres	kilometres
1 000	1
250	
12	
1	

kilometres	metres
1	1 000
5	
20	
0.3	

The constant of proportionality tells us that:

The constant of proportionality tells us that:

b. What is the relationship between the two constants of proportionality?

Solution

metres	kilometres
1 000	1

250	0.25
12	0.012
1	0.001

0.001 kilometres per metre

kilometres	metres
1	1 000
5	5 000
20	20 000
0.3	300

1000 metres per kilometre

a. 0.001 and 1 000 are reciprocals of each other. This is easier to see if 0.001 is written as $\frac{1}{1\,000}$.

3. **Problem 3 Statement**

Jada and Lin are comparing inches and feet. Jada says that the constant of proportionality is 12. Lin says it is $\frac{1}{12}$. Do you agree with either of them? Explain your reasoning.

Solution

Jada is saying that there are 12 inches for every 1 foot. Lin is saying that there are $\frac{1}{12}$ foot for every 1 inch.

4. **Problem 4 Statement**

The area of the Mojave desert is 25 000 square miles. A scale drawing of the Mojave desert has an area of 10 square inches. What is the scale of the map?

Solution

1 inch to 50 miles

5. **Problem 5 Statement**

Which of these scales is equivalent to the scale 1 cm to 5 km? Select **all** that apply.

a. 3 cm to 15 km

b. 1 mm to 150 km

c. 5 cm to 1 km

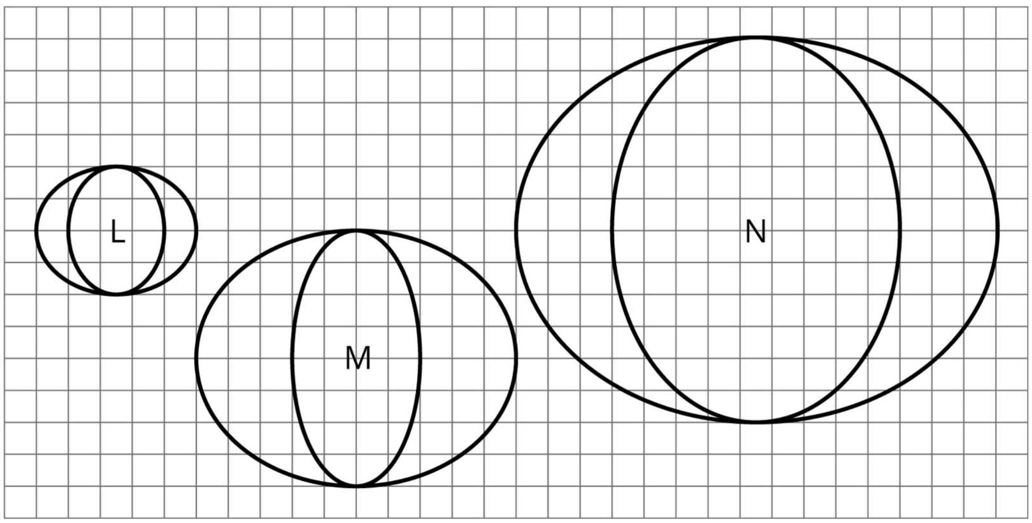
d. 5 mm to 2.5 km

e. 1 mm to 500 m

Solution ["A", "D", "E"]

6. **Problem 6 Statement**

Which one of these pictures is not like the others? Explain what makes it different using ratios.



Solution

M is different from L and N. The width:height ratios for the outsides of the pictures are all equivalent to 5:4. However, the width:height ratios of the insides of L and N both have a 3:4 ratio of width:height, while the inside of M has a width of 4 units and a height of 8 units, making its ratio 1:2.

Alternatively, the ratio of height to thickness at the widest part for L and N are both 4:1. But M has a height of 8 units and a thickness of 3 units, making that ratio 8:3.