Mathematics® Education Innovation

Lesson 11: Filling containers

Goals

- Create a graph of a function from collected data, and interpret (in writing) a point on the graph.
- Draw a container for which the height of water as a function of volume would be represented as a piecewise linear function, and explain (orally) the reasoning.
- Interpret (orally and in writing) a graph of heights of certain cylinders as a function of volume, and compare the rates of change of the functions.

Learning Targets

- I can collect data about a function and represent it as a graph.
- I can describe the graph of a function in words.

Lesson Narrative

This lesson is the beginning of a sequence of lessons that interweaves the development of the function concept with the development of formulae for volumes of cylinders and cones. Because students have not yet learned these formulae, the context of filling a cylindrical container with water is useful for developing the abstract concept of function. It makes physical sense that the height of the water is a function of its volume even if we cannot write down an equation for the function. At the same time, considering how changing the diameter of the cylinder changes the graph of the function helps students develop a geometric understanding of how the volume is related to the height and the diameter. In later lessons they will learn a formula for that relation.

In this lesson, students fill a graduated cylinder with different amounts of water and draw the graph of the height as a function of the volume. They next consider how their data and graph would change if their cylinder had a different diameter. The following activity turns the situation around: when given a graph showing the height of water in a container as a function of the volume of water in the container, can students create a sketch of what the container must look like?

Addressing

- Use functions to model relationships between quantities.
- Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from two (x, y) values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values.

Building Towards
- Solve real-world and mathematical problems involving volume of cylinders, cones, and spheres.

Instructional Routines
- Stronger and Clearer Each Time
- Discussion Supports
- Which One Doesn't Belong?

Required Materials
Graduated cylinders

Required Preparation

Students work in groups of 3–4 for the activity Height and Volume. Each group needs 1 graduated cylinder and water.

Student Learning Goals

Let's fill containers with water.

11.1 Which One Doesn't Belong: Solids

Warm Up: 5 minutes

The purpose of this warm-up is for students to compare different objects that may not be familiar and think about how they are similar and different from objects they have encountered in previous activities and grade levels. To allow all students to access the activity, each object has one obvious reason it does not belong. Encourage students to move past the obvious reasons (e.g., shape A has a point on top) and find reasons based on geometrical properties (e.g., shape D when looked at from every side is a rectangle). During the discussion, listen for important ideas and terminology that will be helpful in upcoming work of the unit.

Instructional Routines
- Which One Doesn't Belong?

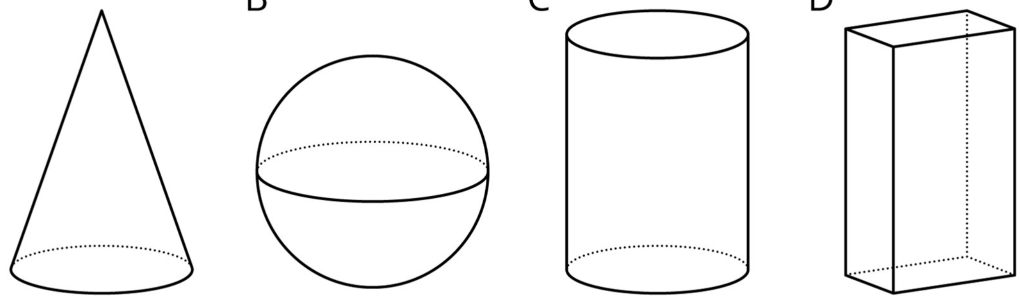
Launch

Arrange students in groups of 2. Display the image for all to see. Ask students to indicate when they have noticed one object that does not belong and can explain why. Give students 2 minutes of quiet think time and then time to share their thinking with their partner.

Student Task Statement

These are drawings of three-dimensional objects. Which one doesn't belong? Explain your reasoning.

A B C D



Student Response

Answers vary. Sample responses:

Shape A doesn't belong because:

- It's the only object with exactly two surfaces.
- It's the only object with exactly one base.

Shape B doesn't belong because:

- It's the only object that has no edges or planar (flat) faces.
- It's the only object that isn't stable if you set it down (i.e., it would roll around).

Shape C doesn't belong because:

- It's the only object in which a side is a rectangle in two dimensions but curved in three dimensions.
- It's the only object with exactly two bases.

Shape D doesn't belong because:

- It's the only object in which all the faces are flat planes.
- It's the only object with rectangular faces.

Activity Synthesis

Ask students to share one reason why a particular object might not belong. Record and display the responses for all to see. After each response, ask the rest of the class if they agree or disagree. Since there is no single correct answer to the question of which one does not belong, attend to students' explanations and ensure the reasons given are correct. During the discussion, prompt students to explain the meaning of any terminology they use, such as diameter, radius, vertex, edge, face, or specific names of the figures: sphere, cylinder, cone, cuboid.

11.2 Height and Volume

20 minutes (there is a digital version of this activity)

In this activity, students investigate how the height of water in a graduated cylinder is a function of the volume of water in the graduated cylinder. Students make predictions about how the graph will look and then test their prediction by filling the graduated cylinder with different amounts of water, gathering and graphing the data.

Instructional Routines

- Discussion Supports

Launch

Arrange students in groups of 3–4. Be sure students know how to measure using a graduated cylinder. If needed, display a graduated cylinder filled to a specific measurement for all to see and demonstrate to students how to read the measurement. Give each group access to a graduated cylinder and water.

Give groups 8–10 minutes to work on the task, follow with a whole-class discussion.

For classrooms with access to the digital materials or those with no access to graduated cylinders, an applet is included here. Physical measurement tools and an active lab experience are preferred.

Speaking: Discussion Supports. Display sentence frames to support small-group discussion. For example, "I think ____ , because _____ ." or "I (agree/disagree) because ______ ."
Design Principle(s): Support sense-making; Optimise output for (explanation)

Student Task Statement

Your teacher will give you a graduated cylinder, water, and some other supplies. Your group will use these supplies to investigate the height of water in the cylinder as a function of the water volume.

1. Before you get started, make a prediction about the shape of the graph.

2. Fill the cylinder with different amounts of water and record the data in the table.

volume (ml)						
height (cm)						

3. Create a graph that shows the height of the water in the cylinder as a function of the water volume.

4. Choose a point on the graph and explain its meaning in the context of the situation.

Student Response

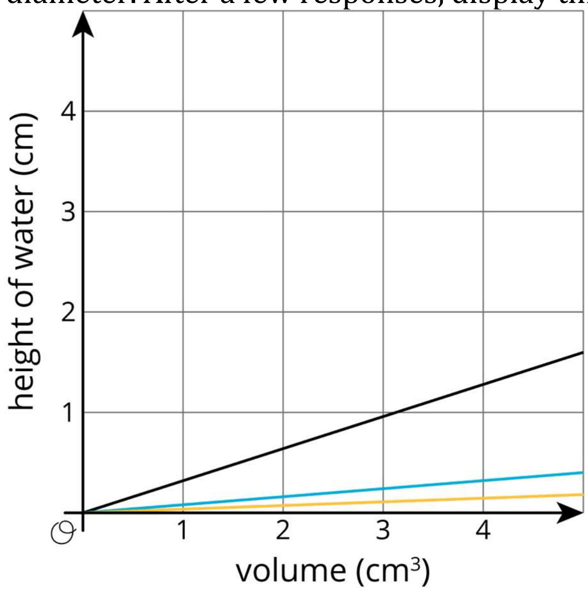
Answers vary according to the shape of the cylinder and the specific measurements taken. Sample response: For interpreting points, the point (150,3) on the graph would signify that when 150 millilitres of water is poured into the cylinder, the height of the water would be 3 centimetres.

Activity Synthesis

Select groups to share the graph for the third question and display it for all to see. Consider asking students the following questions:

- "What do you notice about the shape of your graph?"
- "What is the independent variable of your graph? Dependent variable?"
- "How does this graph differ from what you predicted the shape would be?"
- "For the last question, what point did you choose, and what does that point mean in the context of this activity?"
- "What would the endpoint of the graph be?" (There is a maximum possible volume for the cylinder. Once it's filled, any extra water will spill out and not raise the water height.)

Ask students to predict how the graph would change if their cylinder had double the diameter. After a few responses, display this graph for all to see:



Explain that each line represents the graph of a cylinder with a different radius. One cylinder has a radius of 1 cm, another has a radius of 2 cm, and another has a radius of 3 cm. Have students consider which line must represent which cylinder. Ask, "how did the gradient of each graph change as the radius increased?" (As the radius is larger, the gradient is less steep. This is because for a cylinder with a larger base, the same volume of water will not fill as high up the side of the cylinder.)

11.3 What Is the Shape?

10 minutes

In the previous activity, students were given a container and asked to draw the graph of the height as a function of the volume. In this activity, students are given the graph and asked to draw a sketch of the container that could have generated that height function. Since students have worked on the two previous activities, they have an idea of what the data for a graduated cylinder and a graduated cylinder with twice the diameter looks like and can use that information to compare to while working on this task.

Instructional Routines

- Stronger and Clearer Each Time

Launch

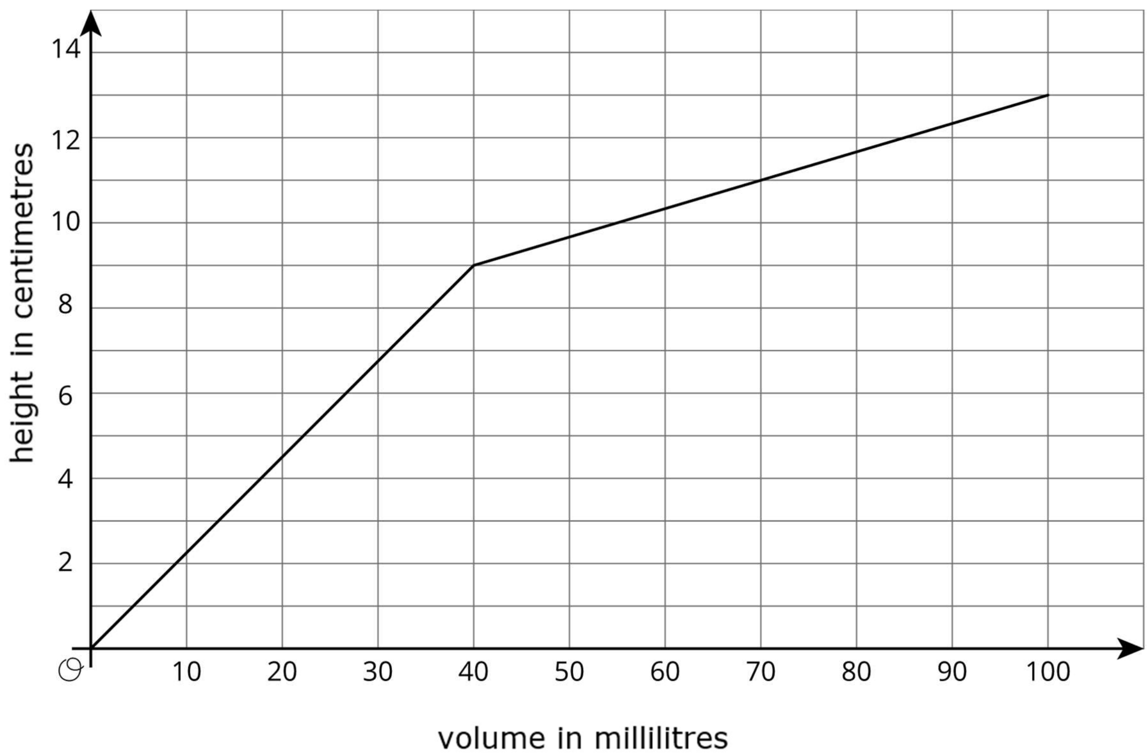
Arrange students in groups of 2. Give students 3–5 minutes of quiet work time and then time to share their drawings with their partner. Follow with a whole-class discussion.

If time is short, consider having half of the class work on the first question and the other half work on the second question and then complete the last question as part of the Activity Synthesis.

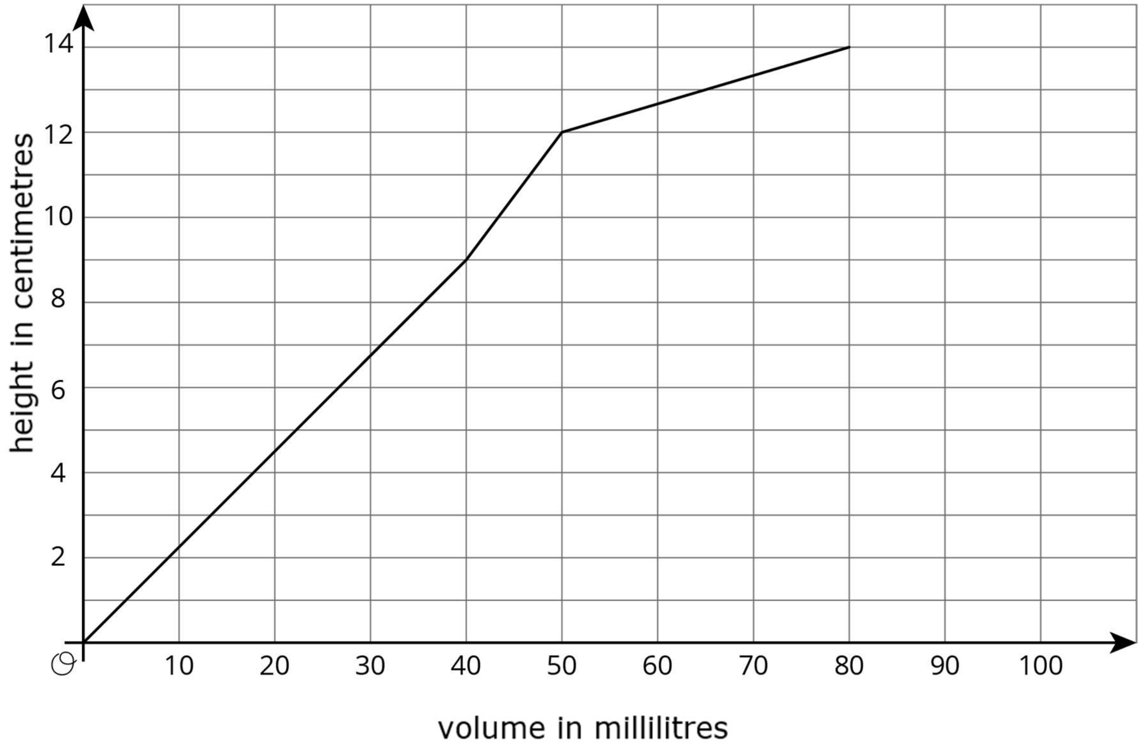
Representation: Internalise Comprehension. Begin by providing students with a range of different sized containers or students to test to determine if their volume and height could be represented by the given graphs.
Supports accessibility for: Conceptual processing Reading, Writing, Conversing: Stronger and Clearer Each Time. Use this routine to provide students with a structured opportunity to revise and refine their response to the last question. Ask students to meet with 2–3 partners for feedback. Provide students with prompts for feedback that will help students strengthen their ideas and clarify their language (e.g., “Why do you think...?”, “What in the graph makes you think that?”, “Can you give an example?”, etc.). Students can borrow ideas and language from each partner to strengthen their final version.
Design Principle(s): Optimise output (for explanation)

Student Task Statement

1. The graph shows the height vs. volume function of an unknown container. What shape could this container have? Explain how you know and draw a possible container.



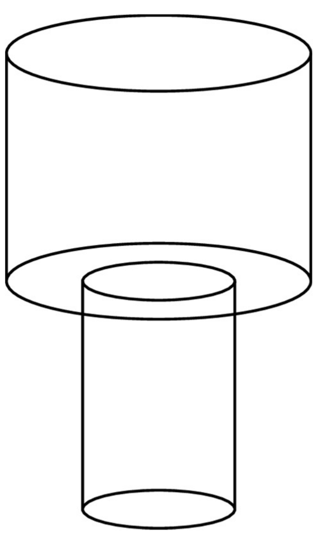
2. The graph shows the height vs. volume function of a different unknown container. What shape could this container have? Explain how you know and draw a possible container.



3. How are the two containers similar? How are they different?

Student Response

1. Answers vary. Sample response: A shape in the form of two cylinders stacked on top of each other, with the upper cylinder having a greater radius. The height grows linearly with the volume in each cylinder, but as the water level rises into the second container, the height will begin to grow less quickly (since it takes more volume to achieve the same increase in height).

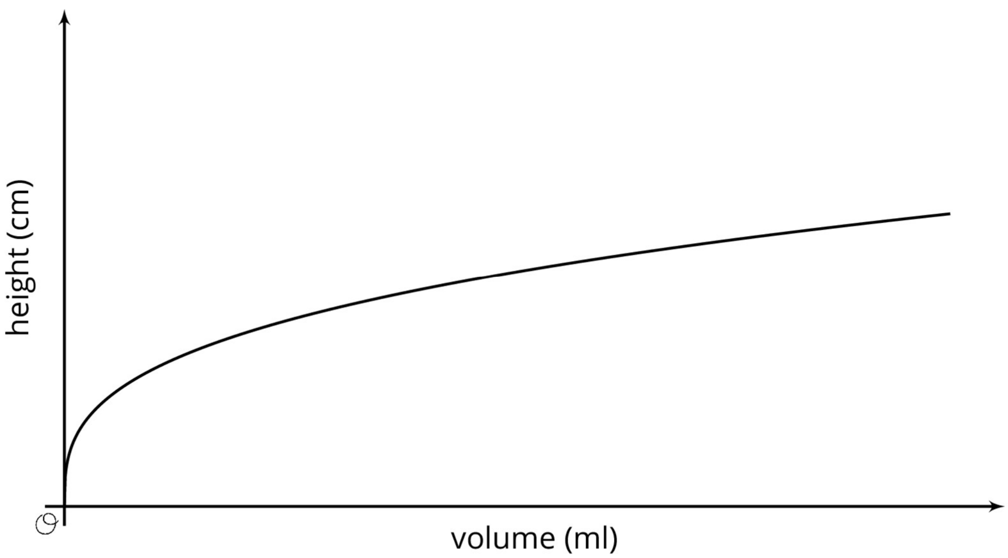


2. Answers vary. Sample response: 3 cylinders stacked on top of each other. The bottom cylinder should be the tallest. The middle cylinder should be shorter and have a smaller radius than the bottom. The top cylinder should be the shortest but have the largest radius.

3. Answers vary. Sample response: Both containers are made up of cylinders stacked on top of each other. The containers are different because the first container is made up of two parts, while the second is made up of three parts.

Are You Ready for More?

The graph shows the height vs. volume function of an unknown container. What shape could this container have? Explain how you know and draw a possible container.



Student Response

This graph in particular was made using the shape of a cone with its vertex at the bottom. Any shape that was very thin at the bottom and gradually got wider as you go up would be a reasonable answer.

Activity Synthesis

Select students to share the different containers they drew. Display their drawings and the graph for all to see. Ask students to explain how they came up with their drawing and refer to parts in the graph that determined the shape of their container.

Lesson Synthesis

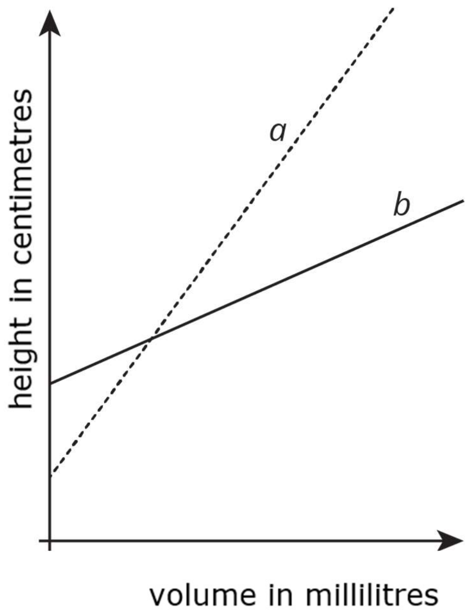
Have students make their own graph showing the height and volume of a container. Tell students to use 2–5 lines for their container. Once the graphs are made, have students swap with a partner and try to draw the shape of their partner's container. Ask a few groups to share their graphs and container drawings.

11.4 Which Cylinder?

Cool Down: 5 minutes

Student Task Statement

Two cylinders, a and b, each started with different amounts of water. The graph shows how the height of the water changed as the volume of water increased in each cylinder. Which cylinder has the larger radius? Explain how you know.

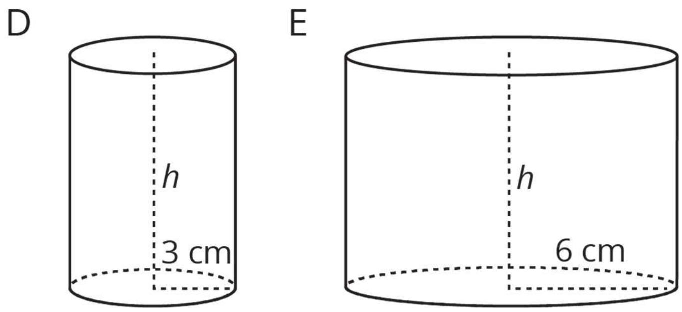


Student Response

Cylinder B. Sample reasoning: a cylinder with a large radius would have a smaller change in height (gradient) for the same volume of water added when compared to a cylinder with a smaller radius. Since the line for B has the smaller gradient, it must be the cylinder with the larger radius.

Student Lesson Summary

When filling a shape like a cylinder with water, we can see how the dimensions of the cylinder affect things like the changing height of the water. For example, let's say we have two cylinders, D and E, with the same height, but D has a radius of 3 cm and E has a radius of 6 cm.



If we pour water into both cylinders at the same rate, the height of water in D will increase faster than the height of water in E due to its smaller radius. This means that if we made graphs of the height of water as a function of the volume of water for each cylinder, we would have two lines and the gradient of the line for cylinder D would be greater than the gradient of the line for cylinder E.

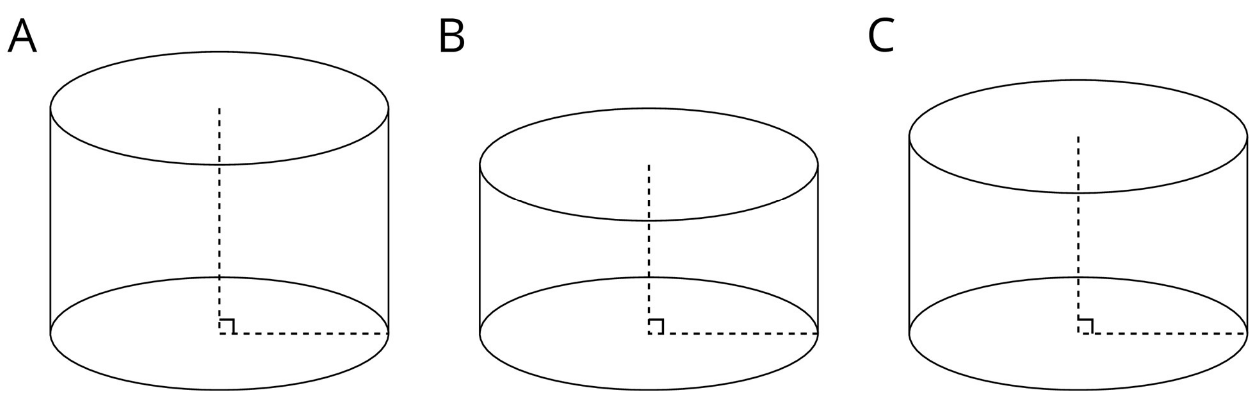
Glossary

- cylinder

Lesson 11 Practice Problems

Problem 1 Statement

Cylinder A, B, and C have the same radius but different heights. Put the cylinders in order of their volume from least to greatest.

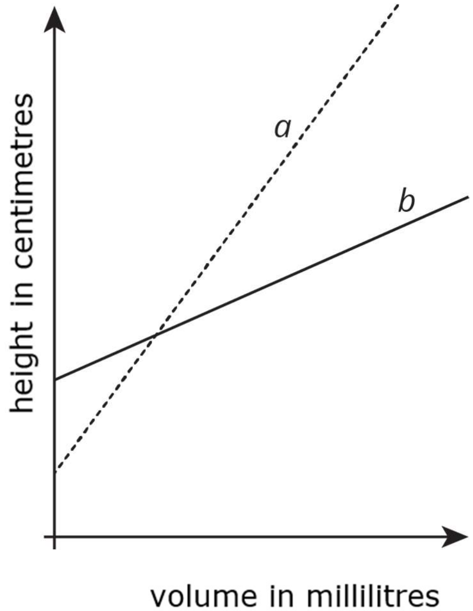


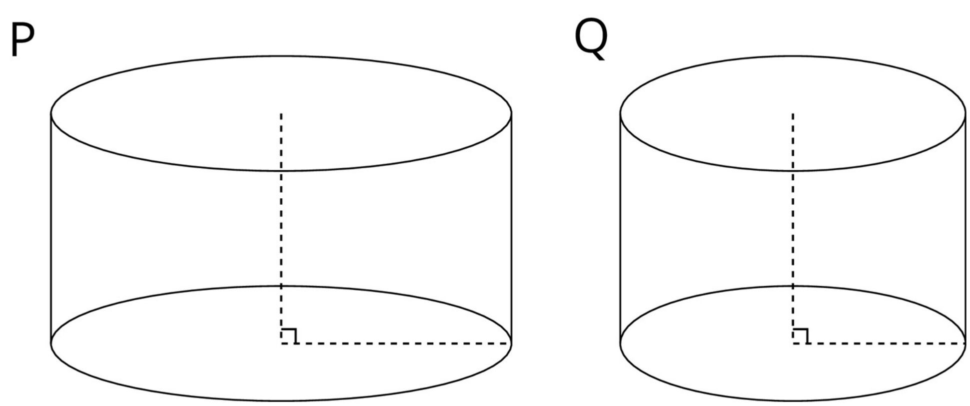
Solution

Cylinder B, cylinder C, cylinder A

Problem 2 Statement

Two cylinders, a and b, each started with different amounts of water. The graph shows how the height of the water changed as the volume of water increased in each cylinder. Match the graphs of a and b to cylinders P and Q. Explain your reasoning.



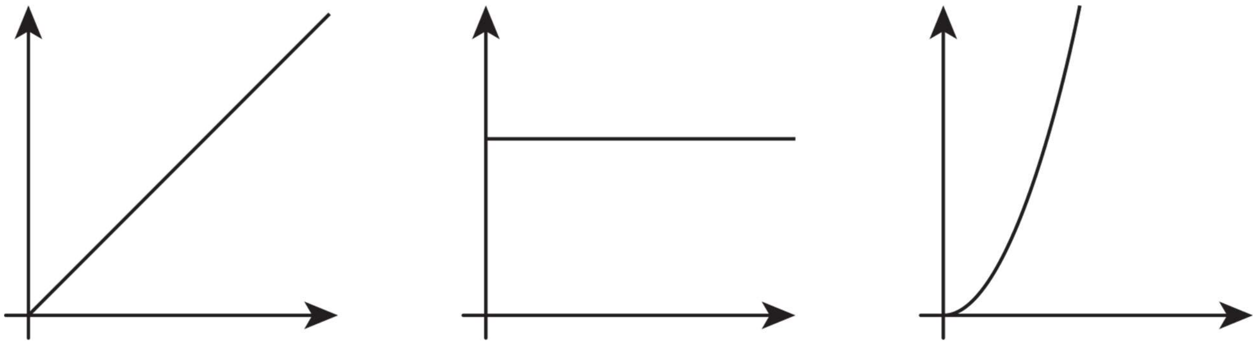


Solution

Line a matches cylinder Q and line b matches cylinder P. Sample reasoning: A cylinder with a large radius would have a smaller rate of change (gradient) for the same volume of water added when compared to a cylinder with a smaller radius. Since line b has the smaller gradient, it must be the cylinder with the larger radius.

Problem 3 Statement

Which of the following graphs could represent the volume of water in a cylinder as a function of its height? Explain your reasoning.

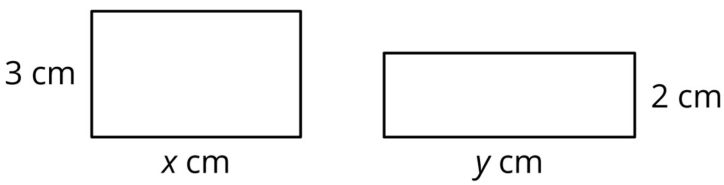


Solution

The linear, increasing graph. Sample reasoning: As the height of water in a cylinder increases, the volume increases by the same scale factor.

Problem 4 Statement

Together, the areas of the rectangles sum to 30 square centimetres.



a. Write an equation showing the relationship between x and y.

b. Fill in the table with the missing values.

x	3		8		12
y		5		10	

Solution

a. $3x + 2y = 30$

b.

x	3	$6\frac{2}{3}$	8	$3\frac{1}{3}$	12
y	10.5	5	3	10	not possible