Lesson 5: A new way to interpret a over b

Goals

- Comprehend that the notation $\frac{a}{b}$ can be used to represent division generally, and the numerator and denominator can include fractions, decimals, or variables.
- Describe (orally) a situation that could be represented by a given equation of the form $x + p = q$ or $px = q$.
- Express division as a fraction (in writing) when solving equations of the form $px = q$.

Learning Targets

- I understand the meaning of a fraction made up of fractions or decimals, like $\frac{2.1}{0.07}$ or $\frac{\frac{4}{5}}{\frac{3}{2}}$.
- When I see an equation, I can make up a story that the equation might represent, explain what the variable represents in the story, and solve the equation.

Lesson Narrative

In this lesson, students apply the general procedure they just learned for solving $px = q$ in order to define what $\frac{a}{b}$ means when a and b are not whole numbers. Up until now, students have likely only seen a fraction bar separating two whole numbers. This is because before KS3, they couldn't divide arbitrary rational numbers. Now an expression like $\frac{2.5}{8.9}$ or $\frac{\frac{1}{2}}{\frac{3}{5}}$ can be well-defined. But the definition is not the same as what they learned for, for example, $\frac{2}{5}$ in Year 3, where they learned that $\frac{2}{5}$ is the number you get by partitioning the interval from 0 to 1 into 5 equal parts and then marking off 2 of the parts. That definition only works for whole numbers. However, in Year 6, students learned that $2 \div 3 = \frac{2}{3}$, so in KS3 it makes sense to define $\frac{2.5}{8.9}$ as $2.5 \div 8.9$.

Alignments

Building On

- Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem. For example, create a story context for $(2/3) \div (3/4)$ and use a visual fraction model to show the quotient; use the relationship between multiplication and division to explain that $(2/3) \div (3/4) = 8/9$ because $3/4$ of $8/9$ is $2/3$. (In general, $(a/b) \div (c/d) = ad/bc$.) How much chocolate will each person get if 3 people share 1/2 kg of chocolate equally? How many 3/4-cup servings are in 2/3 of a cup of yogurt? How wide is a rectangular strip of land with length 3/4 mile and area 1/2 square miles?

- Fluently add, subtract, multiply, and divide multi-digit decimals using the standard algorithm for each operation.

Addressing

- Reason about and solve one-variable equations and inequalities.
- Understand solving an equation or inequality as a process of answering a question: which values from a specified set, if any, make the equation or inequality true? Use substitution to determine whether a given number in a specified set makes an equation or inequality true.
- Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number, or, depending on the purpose at hand, any number in a specified set.
- Solve real-world and mathematical problems by writing and solving equations of the form $x + p = q$ and $px = q$ for cases in which p, q and x are all nonnegative rational numbers.

Instructional Routines

- Stronger and Clearer Each Time
- Collect and Display
- Think Pair Share

Student Learning Goals

Let's investigate what a fraction means when the numerator and denominator are not whole numbers.

5.1 Recalling Ways of Solving

Warm Up: 5 minutes

The purpose of this warm-up is to apply what students have learned to some equations. Note that $0.07 \div 10$ and $10.1 - 7.2$ should be easy to evaluate given that work with fluently computing with decimals precedes this unit.

Launch

Ask students to summarise what they learned in the previous lessons before setting them to work on this warm-up. Allow 1-2 minutes quiet think time, followed by a whole-class discussion.

Student Task Statement

Solve each equation. Be prepared to explain your reasoning.

$0.07 = 10m$

$10.1 = t + 7.2$

Student Response

1. $0.007 = m$
2. $2.9 = t$

Activity Synthesis

At the conclusion of the previous lesson, students should have seen that we can approach solving any equation of the form $px = q$ (where p and q are rational numbers and x is unknown) by dividing each side by p. Also, we can approach solving any equation of the form $x + p = q$ by subtracting p from each side. Discussion should focus on given $0.07 = 10m$, we can write $0.07 \div 10 = 10m \div 10$ and then $0.007 = m$.

5.2 Interpreting $\frac{a}{b}$

15 minutes

Students solve more equations of the form $px = q$ while interpreting the division as a fraction.

Instructional Routines

- Collect and Display
- Think Pair Share

Launch

Arrange students in pairs. Give 5–10 minutes of quiet work time and time to share their responses with a partner, followed by a whole-class discussion.

Representation: Internalise Comprehension. Activate or supply background knowledge about division involving decimals and fractions. Allow students to use calculators to ensure inclusive participation in the activity.
Supports accessibility for: Memory; Conceptual processing Conversing, Representing, Writing: Collect and Display. As they share their responses with a partner, circulate and listen to their conversations. Collect and display any vocabulary or representations students use (e.g., reciprocal, dividing, multiplying) to describe how to solve each equation. Continue to update collected student language once students move on to the activity. Remind students to borrow language from the display as needed. This will help students to use academic mathematical language during paired and group discussions to connect fractions with

division.
Design Principle(s): Maximise meta-awareness; Support sense-making

Anticipated Misconceptions

Monitor for students who want to turn $\frac{35}{11}$ into a decimal, and reassure them that $\frac{35}{11}$ is a number.

Student Task Statement

Solve each equation.

1. $35 = 7x$
2. $35 = 11x$
3. $7x = 7.7$
4. $0.3x = 2.1$
5. $\frac{2}{5} = \frac{1}{2}x$

Student Response

1. 5
2. $\frac{35}{11}$
3. 1.1
4. 7
5. $\frac{4}{5}$

Are You Ready for More?

Solve the equation. Try to find some shortcuts.

$$\frac{1}{6} \times \frac{3}{20} \times \frac{5}{42} \times \frac{7}{72} \times x = \frac{1}{384}$$

Student Response

$x = 9$. Solution methods vary. One way is to factorise each denominator and notice that there are many numbers that occur in both numerator and denominator.

Activity Synthesis

Define what $\frac{a}{b}$ means when a and b are not whole numbers. Tell students, "In Year 3, when you saw something like $\frac{2}{5}$, you learned that that meant 'split up 1 into 5 equal pieces and take 2 of them.' But that definition only makes sense for whole numbers; it doesn't make sense for something like $\frac{2.1}{0.3}$ or $\frac{\frac{2}{5}}{\frac{1}{2}}$. From now on, when you see something like $\frac{2}{5}$, you'll know that that means the number $\frac{2}{5}$ that has a spot on the number line, but it also means '2 divided by 5.' The expression $\frac{2.1}{0.3}$ means 'the quotient of 2.1 and 0.3,' the expression $\frac{\frac{2}{5}}{\frac{1}{2}}$ means 'the quotient of two fifths and one half,' and generally, the expression $\frac{a}{b}$ means 'the quotient of a and b' or 'a divided by b.'"

5.3 Storytime Again

15 minutes

This is a continuation of the activities Storytime and More Storytime from previous lessons. Over time in this unit, we are reminding students of work they should have done in previous years with expressions that represent particular, concrete relationships. In KS3, students are working toward producing such expressions themselves to represent a context.

Instructional Routines

- Stronger and Clearer Each Time

Launch

Remind students of work they did previously to match a situation with an equation. For example, they matched the equation $x + 5 = 20$ with the situation "After Elena ran 5 miles on Friday, she had run a total of 20 miles for the week. How many miles did she run before Friday?" In this activity, they come up with their own situations that can be represented by equations.

Keep students in the same groups. Clarify that for each equation, each partner will come up with a story, and one of those stories is chosen. Give students 5–10 minutes to work with their partner, followed by a whole-class discussion.

Anticipated Misconceptions

For students with limited fraction and decimal understanding, coming up with a reasonable story where the numbers are not whole can be daunting. You might suggest that students imagine stories with similar structures that involve whole numbers, and then tweak the stories toward using the numbers given in the problems. Remind them that using fractions and decimals has to make sense in the situations, and encourage them to think about what kinds of situations those might be (measurement situations will usually work while those that involve counting discrete objects won't.)

Student Task Statement

Take turns with your partner telling a story that might be represented by each equation. Then, for each equation, choose one story, state what quantity x describes, and solve the equation. If you get stuck, consider drawing a diagram.

$0.7 + x = 12$

$\frac{1}{4}x = \frac{3}{2}$

Student Response

Answers vary. Sample responses:

1. $x = 11.3$ Diego went to the store to buy a box of crayons, which cost 70 pence. While there, he picked up some other items. The total amount he paid at checkout was £12. Solving the equation gives 11.3. Diego spent £11.30 on the other items he bought at the store.

2. $x = 6$ $\frac{1}{4}$ of a bottle of water contains $\frac{3}{2}$ cups of water. How many cups are in the whole bottle? The solution is $\frac{\frac{3}{2}}{\frac{1}{4}}$, which is 6. One bottle of water contains 6 cups.

Activity Synthesis

Invite students to share their stories. Ask each student to interpret the solution in terms of their situation.

Writing, Speaking: Stronger and Clearer Each Time. Use this routine with successive pair shares to give students a structured opportunity to revise and refine their writing. For this activity, students should use the story for the equation they chose to solve. Ask each student to meet with 2–3 other partners in a row for feedback. Provide students with prompts for feedback that will help teams strengthen their ideas and clarify their language (e.g., “How are the parts of the equation represented in your story?”, “Can you say more about how your solution fits in your story?”). Provide students with time to complete a final draft based on the feedback they receive about language and clarity.
Design Principle(s): Optimise output (for comparison)

Lesson Synthesis

Ask students to work with their partner. Each partner writes a number that is in fraction or decimal form. Have them choose one number to be the coefficient in an equation of the form $px = q$ and the second number the quantity on the other side of the equal sign. They then work together to write and evaluate the solution of the equation. Complete multiple rounds as time allows.

5.4 Choosing Solutions

Cool Down: 5 minutes

Student Task Statement

Select **all** the expressions that are solutions to $5 = \frac{2}{3}x$.

- $5 \times \frac{2}{3}$
- $\frac{5}{\frac{2}{3}}$
- $5 \div \frac{2}{3}$
- $\frac{15}{2}$
- $\frac{10}{3}$

Student Response

$\frac{5}{\frac{2}{3}}$, $5 \div \frac{2}{3}$, $\frac{15}{2}$

Student Lesson Summary

In the past, you learned that a fraction such as $\frac{4}{5}$ can be thought of in a few ways.

- $\frac{4}{5}$ is a number you can locate on the number line by dividing the section between 0 and 1 into 5 equal parts and then counting 4 of those parts to the right of 0.
- $\frac{4}{5}$ is the share that each person would have if 4 wholes were shared equally among 5 people. This means that $\frac{4}{5}$ is the result of *dividing* 4 by 5.

We can extend this meaning of *a fraction as a quotient* to fractions whose numerators and denominators are not whole numbers. For example, we can represent 4.5 pounds of rice divided into portions that each weigh 1.5 pounds as: $\frac{4.5}{1.5} = 4.5 \div 1.5 = 3$. In other words, $\frac{4.5}{1.5} = 3$ because the quotient of 4.5 and 1.5 is 3.

Fractions that involve non-whole numbers can also be used when we solve equations.

Suppose a road under construction is $\frac{3}{8}$ finished and the length of the completed part is $\frac{4}{3}$ miles. How long will the road be when completed?

We can write the equation $\frac{3}{8}x = \frac{4}{3}$ to represent the situation and solve the equation.

The completed road will be $3\frac{5}{9}$ or about 3.6 miles long.

$$\frac{3}{8}x = \frac{4}{3}$$
$$x = \frac{\frac{4}{3}}{\frac{3}{8}}$$
$$x = \frac{4}{3} \times \frac{8}{3}$$
$$x = \frac{32}{9} = 3\frac{5}{9}$$

Lesson 5 Practice Problems

1. **Problem 1 Statement**

   Select **all** the expressions that equal $\frac{3.15}{0.45}$.

   a. $(3.15) \times (0.45)$

   b. $(3.15) \div (0.45)$

   c. $(3.15) \times \frac{1}{0.45}$

   d. $(3.15) \div \frac{45}{100}$

   e. $(3.15) \times \frac{100}{45}$

   f. $\frac{0.45}{3.15}$

   Solution ["B", "C", "D", "E"]

2. **Problem 2 Statement**

   Which expressions are solutions to the equation $\frac{3}{4}x = 15$? Select **all** that apply.

   a. $\frac{15}{\frac{3}{4}}$

b. $\frac{15}{\frac{4}{3}}$

c. $\frac{4}{3} \times 15$

d. $\frac{3}{4} \times 15$

e. $15 \div \frac{3}{4}$

Solution ["A", "C", "E"]

3. **Problem 3 Statement**

Solve each equation.

$4a = 32$

$4 = 32b$

$10c = 26$

$26 = 100d$

Solution

a. $a = 8$

b. $b = \frac{1}{8}$

c. $c = 2.6$ (or equivalent)

d. $d = 0.26$ (or equivalent)

4. **Problem 4 Statement**

For each equation, write a story problem represented by the equation. For each equation, state what quantity x represents. If you get stuck, consider drawing a diagram.

a. $\frac{3}{4} + x = 2$

b. $1.5x = 6$

Solution

Answers vary. Sample response:

a. Jada ran for 2 miles. Elena ran for $\frac{3}{4}$ of a mile. How much further did Jada run than Elena? x represents the difference between the distance of Jada's run and Elena's run.

b. 1.5 times the amount a bucket holds makes 6 gallons. How many gallons does the bucket hold? x represents the volume in gallons that the bucket holds.

5. **Problem 5 Statement**

Write as many mathematical expressions or equations as you can about the image. Include a fraction, a decimal number, or a percentage in each.

Solution

Answers vary. Possible responses: $\frac{1}{5} \times 250\,000 = 50\,000$, 20% of 250 000 is 50 000, or $(0.2) \times 250\,000 = 50\,000$.

6. **Problem 6 Statement**

In a lilac paint mixture, 40% of the mixture is white paint, 20% is blue, and the rest is red. There are 4 cups of blue paint used in a batch of lilac paint.

a. How many cups of white paint are used?

b. How many cups of red paint are used?

c. How many cups of lilac paint will this batch yield?

If you get stuck, consider using a bar model.

Solution

a. 8

b. 8

c. 20

7. **Problem 7 Statement**

Triangle P has a base of 12 inches and a corresponding height of 8 inches. Triangle Q has a base of 15 inches and a corresponding height of 6.5 inches. Which triangle has a greater area? Show your reasoning.

Solution

Triangle Q has a larger area. The area of Triangle P is $\frac{1}{2} \times 12 \times 8$ or 48 square inches. The area of Triangle Q is $\frac{1}{2} \times 15 \times (6.5)$ or 48.75 square inches.