Some remarks on exp, cosine, sine, cosh & sinh

The exp function, e^x, the trig functions cosine & sine, and the hyperbolic trig functions cosh and sinh are closely related. At first it might not seem so but if we look at the power series definitions of them we can extract some relationships.

Recall that $cosh(x) = \dfrac{e^x + e^{-x}}{2}$ and $sinh(x) = \dfrac{e^x - e^{-x}}{2}$.

(These are respectively the even and odd parts of e^x)

Power series definitions:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \ldots\ldots$$

$$cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} + \ldots\ldots$$

$$sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} + \ldots\ldots$$

$$cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \frac{x^{10}}{10!} + \frac{x^{12}}{12!} + \ldots\ldots$$

$$sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \frac{x^{11}}{11!} + \frac{x^{13}}{13!} + \ldots\ldots$$

Note that all terms in each of these look like $\dfrac{x^n}{n!}$ with signs alternating in cosine and sine. Note also that cos and cosh are both even functions so have only even powers in their series and sin and sinh are odd functions so have only odd powers in their series.

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All of these series converge absolutely for all reals. By using the series to define the functions we can extend the range of the functions to be all complex numbers. The properties of the functions also extend. For example $cos(z+w) = cos(z)cos(w) - sin(z)sin(w)$ for all complex z and w.

Replace x with $\boldsymbol{it}$ in the exponential series, rearrange the terms into the real and imaginary parts and discover that $e^{it} = cos(t) + isin(t)$. Now let $t = \pi$ to find that famous formula $e^{i\pi} = -1$ or $e^{i\pi} + 1 = 0$.

The reader is encouraged to verify the following for all complex z:

$$cos(z) = \frac{e^{iz} + e^{-iz}}{2} \qquad sin(z) = \frac{e^{1z} - e^{-iz}}{2i}$$

$$cosh(z) = \frac{e^{z} + e^{-z}}{2} \qquad sinh(z) = \frac{e^{z} - e^{-z}}{2}$$

In short, all trig functions, both circular and hyperbolic, can be defined in terms of the exponential function.

It is easy to show that $cos(iz) = cosh(z)$, $cosh(iz) = cos(z)$ and that $sin(iz) = isinh(z)$, $sinh(iz) = isin(z)$.

In particular, if x is a real number then both $cos(ix)$ and $cosh(ix)$ are both real numbers. Similarly $sin(ix)$ and $sinh(ix)$ are both purely imaginary.

I will use these various properties in the derivation of the phantom graphs for exp, cos, sin, cosh and sinh.

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For all complex z and w the following are true:

$exp(z + w) = exp(z)exp(w) \quad N.B.exp(t) = e^t$

$cos(z + w) = cos(z)cos(w) - sin(z)sin(w)$

$sin(z + w) = sin(z)cos(w) + cos(z)sin(w)$

$cosh(z + w) = cosh(z)cosh(w) - sinh(z)sinh(w)$

$sinh(z + w) = sinh(z)cosh(w) + cosh(z)sinh(w)$

We can use these to find the Phantom graphs of cos(x).

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$cos(x + iz) = cos(x)cos(iz) + sin(x)sin(iy) = cos(x)cosh(z) + isin(x)sinh(z)$

The imaginary part is 0 if $sin(x) = 0$ or if $sinh(z) = 0$. The latter is true only if z=0 so we can ignore that but $sin(x) = 0$ if $x = k\pi$ with k an integer. Substituting we have

$x = k\pi,\ \ y = cos(k\pi)cosh(z) = (-1)^k cosh(z)$

So we have a sequence of curves whose parametric forms are:

$$
\begin{aligned}
&x = k\pi \\
&y = (-1)^k cosh(t) \\
&z = t \\
&-\infty < t < \infty
\end{aligned}
$$