Phantom Graphs
Complex solutions to real equations using GeoGebra

Background:
Many functions that are ordinarily thought of as functions from the reals to the reals can also produce a real number when certain complex are substituted. For example, if $f(x)=x^2$ then $f(-2i)=-4$. Similarly, if $g(x)=x^3$ then $g(-1+i\sqrt{3})=(-1+i\sqrt{3})^3=8$. We will try to show graphically, using GeoGebra 3D, both the real and complex numbers that produce real numbers when substituted into a function. Specifically, for many functions *f* we will find all solutions to *f(x)=c* where *c* ranges over all reals, and not only find them but show them graphically.

Technique:
In GeoGebra 3D the *x*-axis and *y*-axis will be real as usual. The *z*-axis will be the imaginary axis and will therefore be perpendicular to the *x-y* plane. Consequently, the usual graph of *y=f(x)* will be in the *x-y* plane but the graph of complex numbers that produce real numbers will be perpendicular to the *x-y* plane. In general these graphs will be curves. More on this later.

Using GeoGebra vs Solving Algebraically:
If the original function is a polynomial then the GeoGebra command ComplexRoot can be exploited to find all complex solutions that give real numbers. For the polynomial *f* I create a slider, which I will call *c*, that ranges over a sufficiently large interval and the enter ComplexRoot(*f(x)-c*). This will create a set of complex numbers that vary with *c* which we can use to generate the graph. This technique can also be used for rational function. If $f(x)=\dfrac{n(x)}{d(x)}$ then we use ComplexRoot(*n(x)-c*d(x)*) to find the roots of *f(x)-c*.

For many functions, though not all, one can solve for these curves explicitly. To do so, substitute *u+iv* for *x* and then separate into the real and imaginary parts. Since we are looking for complex inputs that produce real outputs, set the imaginary part equal to zero. Hopefully, we can solve for either *u* or *v* in terms of the other and then substitute that into the real part. Two examples:

1. $f(x)=x^2, \quad f(u+iv)=(u+iv)^2=u^2-v^2+i(2uv)$

   The imaginary part is 0 if *u=0* or *v=0.* The latter can be ignored as this would just mean that we are substituting a real number into *f*. Hence we substitute *u=0* into the real part and get $-v^2$. Since *u* is the real part and *v* the imaginary, we get $y=-z^2$, $x=0$ for the phantom graph.

This is entered as $\text{Curve}(0,-v^2,v,v,-20,20)$. One could also enter $\text{Curve}(0,-t^2,t,t,-20,20)$. The range -20 to 20 is a bit arbitrary and can be adjusted.

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2. $f(x)=\dfrac{x^2}{x-1}$ We have $f(u+iv)=\dfrac{(u+iv)^2}{u+iv-1}=\dfrac{u^2-v^2+i(2uv)}{u-1+iv}$

Multiply by the conjugate of the bottom to get

$$\frac{u^2-v^2+i(2uv)}{u-1+iv}\cdot\frac{u-1-iv}{u-1-iv}=\frac{(u^2-v^2)(u-1)+2uv^2+i(2uv(u-1)-v(u^2-v^2)}{(u-1)^2+v^2}=f(u+iv)$$

Set the imaginary part of the denominator equal to zero. (We can ignore the numerator for this.)

$$2uv(u-1)-v(u^2-v^2)=v\left(2u^2-2u-u^2+v^2\right)=v(u^2-2u+v^2)$$

Again we can ignore $v=0$ and consider $u^2-2u+v^2=0$ which is equivalent to $(u-1)^2+v^2=1$. Substitute $v^2=2u-u^2$ into the real part, simplify and end up with $2u$ after some algebra. So letting u be the parameter, the curve is given by Curve(u,$2u$,sqrt$(2u-u^2)$,u,0,2) and Curve(u,$2u$,-sqrt$(2u-u^2)$,u,0,2). We need both +/- in front of the sqrt to get the complete graph. Also, since $(u-1)^2+v^2=1$ we must have $0\le u\le 2$. You could also use Curve($c/2$,c,sqrt$(c-c^2/4)$,u,0,4) if you want the y-value to be the parameter.

In more standard mathematical notation we have a parametric curves in 3-space defined as

$$\begin{aligned}x&=t\\ y&=2t\\ z&=\sqrt{2t-t^2}\\ 0&\le t\le 2\end{aligned}\qquad\text{and}\qquad\begin{aligned}x&=t\\ y&=2t\\ z&=-\sqrt{2t-t^2}\\ 0&\le t\le 2\end{aligned}$$

Many of the phantom graphs will have parametric equations with x, y, z being a function of a parameter.

Finally, we can also look at the path in the plane of the complex solutions. In this case the non-real solutions live in the u-v plane on the circle $(u-1)^2+v^2=1$ or in x-y notation $(x-1)^2+y^2=1$.

This second example is given to show that things can end up fairly simply after some algebraic work.

In the following GeoGebra applets, the phantom graphs are in blue and the regular graphs in green. Sometimes I have been unable to find the parametric equations of the phantom graphs. In those cases I either used the locus command to show the curve or the user will need to trace the points to see the curve. This latter is used when the locus command is less than robust.