Lesson 17: Scaling one dimension

Goals

- Create a graph and an equation to represent the function relationship between the volume of a cylinder and its height, and justify (orally) that the relationship is linear.
- Interpret (in writing) a point on a graph representing the volume of a cone as a function of its height, and explain (orally) how changing one dimension affects the other.

Learning Targets

- I can create a graph of the relationship between volume and height for all cylinders (or cones) with a fixed radius.
- I can explain in my own words why changing the height by a scale factor changes the volume by the same scale factor.

Lesson Narrative

This lesson is optional. This is the first of a series of lessons building up to the formula for the volume of a sphere. In order to understand why that formula has the radius raised to the third power, students start studying how the volume of a three-dimensional figure changes when you scale one or more of its dimensions (length, width, height, radius). In this lesson they consider just one of the dimensions.

In the warm-up, students graph a proportional relationship and recall that in a proportional relationship the two quantities change by the same scale factor: when you multiply one of them by a scale factor the other one gets multiplied by the same scale factor. In the first activity, students consider a cuboid with two edges of constant length and one edge of variable length. They graph the volume of the cuboid as a function of the length and see that the volume is proportional to the length. They conclude that when you double the length the volume doubles. Then they investigate the volume of a cone as a function of its height when you keep the radius constant. Again they see that the volume is proportional to the height, and that when you halve the height you halve the volume. In the final activity they use a graph of this proportional relationship to find the radius.

The main purpose of the lesson is to understand that when you scale just one of the dimensions of a three-dimensional figure by a factor, the volume scales by the same factor. A secondary purpose is to see some examples of linear functions arising out of geometry. (A proportional relationship is a particular kind of linear function.)

Addressing

- Understand that a function is a rule that assigns to each input exactly one output. The graph of a function is the set of ordered pairs consisting of an input and the corresponding output. Function notation is not required at KS3.

- Use functions to model relationships between quantities.
- Solve real-world and mathematical problems involving volume of cylinders, cones, and spheres.
- Know the formulas for the volumes of cones, cylinders, and spheres and use them to solve real-world and mathematical problems.

Instructional Routines

- Stronger and Clearer Each Time
- Co-Craft Questions
- Compare and Connect
- Think Pair Share

Student Learning Goals

Let's see how changing one dimension changes the volume of a shape.

17.1 Driving the Distance

Warm Up: 5 minutes

The purpose of this warm-up is for students to jump back into recognising functions and determining if two quantities are a function of each other. Students are asked questions similar to these throughout this lesson, and the discussion of this warm-up is meant to get students using the language of functions, which continues throughout the rest of the activities.

Launch

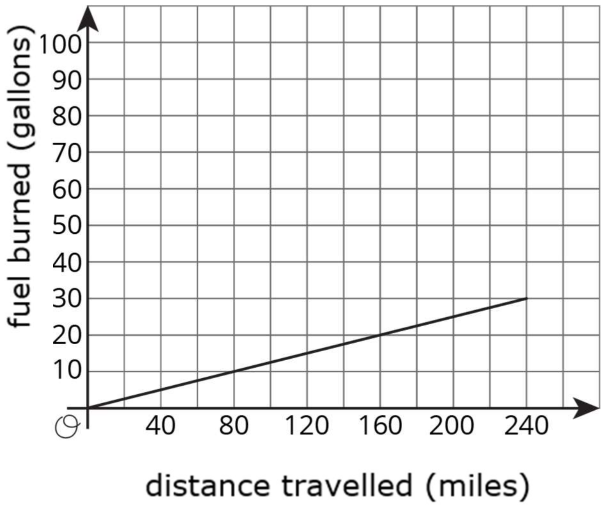
Give students 1–2 minutes of quiet work time, followed by a whole-class discussion.

Anticipated Misconceptions

If students struggle with the second and third questions, ask them if this is an example of a proportional relationship. Remind them what they learned previously about proportional relationships and how that can be used to answer these questions.

Student Task Statement

Here is a graph of the amount of fuel burned during a trip by a tractor-trailer truck as it drives at a constant speed down a road:



1. At the end of the trip, how far did the truck drive, and how much fuel did it use?
2. If a truck travelled half this distance at the same rate, how much fuel would it use?
3. If a truck travelled double this distance at the same rate, how much fuel would it use?
4. Complete the sentence: __________ is a function of ___________.

Student Response

1. The truck drove 240 miles and used 30 gallons of fuel.
2. 15 gallons. Since it is a proportional relationship, if the miles are halved then the gallons are also halved.
3. 60 gallons. Since it is a proportional relationship, if the miles are doubled then the gallons are also doubled.
4. Gallons of fuel burned is a function of miles travelled. The number of miles travelled is also a function of the gallons of fuel burned.

Activity Synthesis

Invite students to share their answers and their reasoning for why fuel burned is a function of distance travelled. Questions to further the discussion around functions:

- "Looking at the graph, what information do you need in order to determine how much fuel was used?" (We need to know the number of miles travelled.)
- "What is the independent variable? Dependent variable? How can you tell from the graph?" (The independent value is the distance travelled. The dependent value is the fuel burned. By convention, the independent is on the x-axis and the dependent is on the y-axis.)

- "What are some ways that we can tell from the graph that the relationship between fuel burned and distance travelled is proportional?" (The graph is a line that goes through the origin, we can see a constant ratio between y and x in some points like (80,10), (160,20), and (240,30).)

17.2 Double the Edge

Optional: 10 minutes (there is a digital version of this activity)

This activity is optional. The purpose of this activity is for students to apply what they know about functions and their representations in order to investigate the effect of a change in one dimension on the volume of a cuboid. Students use graphs and equations to represent the volume of a cuboid with one unknown edge length. They then use these representations to investigate what happens to the volume when one of the edge lengths is doubled. Groups make connections between the different representations by pointing out how the graph and the equation reflect an edge length that is doubled. Identify groups who make these connections and ask them to share during the discussion.

Instructional Routines

- Stronger and Clearer Each Time
- Think Pair Share

Launch

Arrange students in groups of 2. Give students 2–3 minutes of quiet work time, followed by time to discuss the last question with their partner. Follow with a whole-class discussion.

Students using the digital activity can generate the graph using the digital applet.

Action and Expression: Provide Access for Physical Action. Provide access to tools and assistive technologies such as a graphing software or applet. Some students may benefit from a checklist or list of steps to be able to use the software or applet.
Supports accessibility for: Organisation; Conceptual processing; Attention

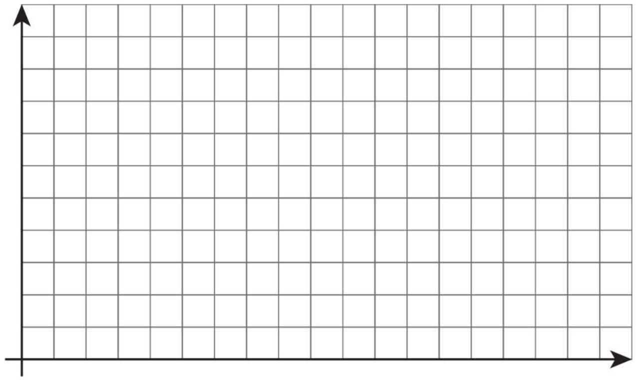
Anticipated Misconceptions

If students struggle to see how the change in volume is reflected in the equation, have them start with values from the graph and put those into the equation to see how the volume changes.

Student Task Statement

There are many cuboids with one edge of length 5 units and another edge of length 3 units. Let s represent the length of the third edge and V represent the volume of these cuboids.

1. Write an equation that represents the relationship between V and s.
2. Graph this equation and label the axes.

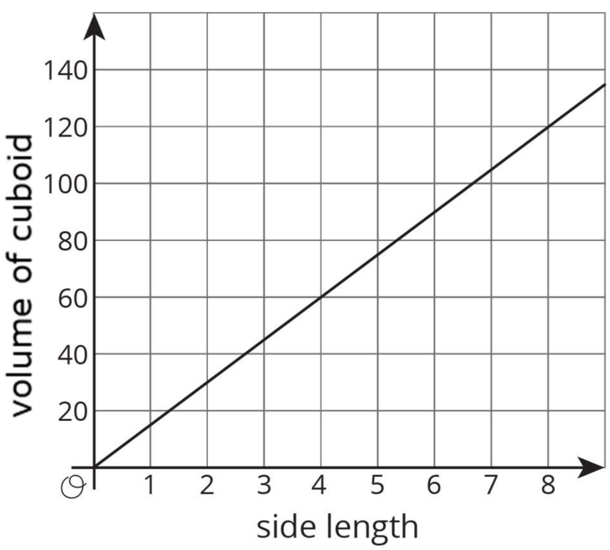


3. What happens to the volume if you double the edge length s? Where do you see this in the graph? Where do you see it algebraically?

Student Response

1. $V = 15s$

2.



3. When the edge length s is doubled, the volume is also doubled. Other answers vary. Sample response: In the graph, it can be seen that when the edge length is 4 cm, the volume is 60 cm^3. When the edge length doubles to 8 cm, then the volume doubles to be 120 cm^3 as the graph shows a proportional relationship. Algebraically, if the edge length is doubled from s to $2s$, then the volume goes from $15s$ to $15 \times 2s$, or $30s$, which is double the original volume.

Activity Synthesis

The goal of the discussion is to ensure that students use function representations to support the idea that the volume doubles when s is doubled.

Select previously identified groups to share what happens to the volume when you double s. Display the graph and equation for all to see and have students point out where they see the effect of doubling s in the graph (by looking at any two edge lengths that are double each other, their volume will be double also). Ask students:

- "Which of your variables is the independent? The dependent?" (The side length, s, is the independent variable, and volume, V, is dependent variable.)
- "Which variable is a function of which?" (Volume is a function of side length.)

If it has not been brought up in students' explanations, ask what the volume equation looks like when we double the edge length s. Display volume equation $V = 15(2s)$ for all to see. Ask, "How can we write this equation to show that the volume doubled when s doubled?" (Using algebra, we can rewrite $V = 15(2s)$ as $V = 2(15s)$. Since the volume for s was $15s$, this shows that the volume for $2s$ is twice the volume for s).

Writing, Conversing: Stronger and Clearer Each Time. Use this routine to give students a structured opportunity to refine their response to the last question. Ask each student to meet with 2–3 other partners for feedback. Provide students with prompts for feedback that will help students strengthen their ideas and clarify their language (e.g., "What part of the graph supports your reasoning?" or "Can algebraic calculations be applied to tripling the side length?"). Students can borrow ideas and language from each partner to strengthen the final product.
Design Principle(s): Optimise output (for explanation)

17.3 Halve the Height

Optional: 10 minutes (there is a digital version of this activity)

This activity is optional. This activity is similar to the previous (optional) one from a function point of view, but now students investigate the volume of a cylinder instead of a cuboid. Students continue working with functions to investigate what happens to the volume of a cylinder when you halve the height. The exploration and representations resemble what was done in the previous activity, and students continue to identify the effect of the changing dimension on the graph and the equation of this function. As groups work on the task, identify those who make the connection between the graph and equation representations, and encourage groups to look for similarities and differences between what they see in this activity and what they saw in the previous activity. Invite these groups to share during the whole-class discussion.

Instructional Routines

- Compare and Connect
- Think Pair Share

Launch

Keep students in the same groups. Tell students that this activity is similar to the previous one, but they will work with a cylinder instead of a cuboid. Give students 2–3 minutes of quiet work time followed by time to discuss the last question with their partner. Follow with a whole-class discussion.

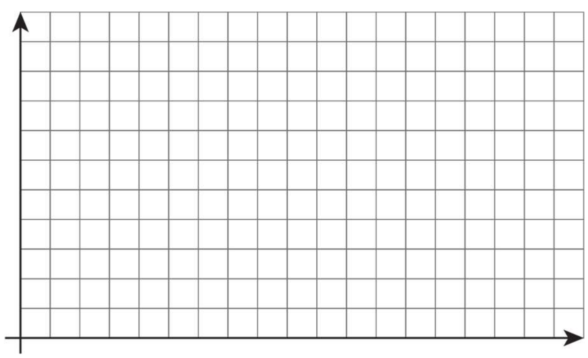
Students using the digital activity can graph their equations using the applet.

Student Task Statement

There are many cylinders with radius 5 units. Let h represent the height and V represent the volume of these cylinders.

1. Write an equation that represents the relationship between V and h. Use 3.14 as an approximation of π.

2. Graph this equation and label the axes.

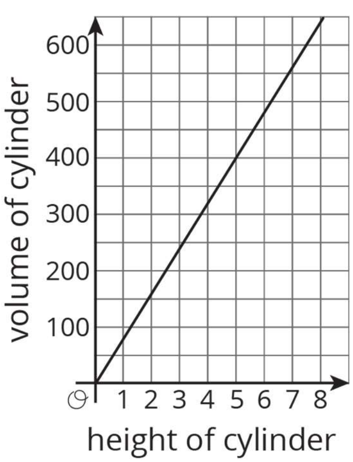


3. What happens to the volume if you halve the height, h? Where can you see this in the graph? How can you see it algebraically?

Student Response

1. The equation that expresses the relationship between V and h is $V = 78.5h$.

2.



3. When the height h is halved, the volume is also halved. Other answers vary. Sample response: In the graph, it can be seen that the graph shows a proportional relationship and when the height is $\frac{1}{2}$ the corresponding point on the line is half as high as when the height is 1. Algebraically, if the height is halved from h to $\frac{1}{2}h$, then the volume goes from $78.5 \times h$ to $78.5 \times \frac{1}{2}h$, or $39.25h$, which is half the original volume.

Are You Ready for More?

Suppose we have a cuboid with dimensions 2 units by 3 units by 6 units, and we would like to make a cuboid of volume 216 cubic units by stretching *one* of the three dimensions.

- What are the three ways of doing this? Of these, which gives the cuboid with the smallest surface area?
- Repeat this process for a starting cuboid with dimensions 2 units by 6 units by 6 units.
- Can you give some general tips to someone who wants to make a box with a certain volume, but wants to save cost on material by having as small a surface area as possible?

Student Response

- The volume of the starting box is 36 cubic units, so to get to 216 cubic units we need to increase any one of the dimensions by a factor of 6. This gives the following three possibilities:
  - 12 units by 3 units by 6 units, with a surface area of 252 square units.
  - 2 units by 18 units by 6 units, with a surface area of 312 square units.
  - 2 units by 3 units by 36 units, with a surface area of 372 square units.
- The volume of the starting box is 72 cubic units, so to get to 216 cubic units we need to increase any one of the dimensions by a factor of 3. This gives the following two possibilities (since scaling either side of length 6 would give a box of the same dimensions:
  - 6 units by 6 units by 6 units, with a surface area of 216 square units.
  - 2 units by 6 units by 18 units, with a surface area of 504 square units.
- All five boxes we built have a volume of 216 cubic units, but have surfaces areas ranging from 216 to 504 square units. This could make quite a difference to the cost of enclosing 216 cubic units! In general, it seems that there is a lot more surface area when there is a lot of imbalance between the side lengths -- one side significantly shorter or longer than the other two. The best result we found, and indeed the best result possible, is to make all three sides the same length.

Activity Synthesis

Ask previously identified groups to share their graphs and equations.

If students completed the optional activity, display student graphs from both activities and ask:

- "Compare the graph in this activity to the graph in the last activity. How are they alike? How are they different?"
- "Compare the equation in this activity to the equation in the last activity. How are they alike? How are they different?"
- "How can you tell that this is a linear relationship?"

For the last question, make sure students understand that "looks like a line" is insufficient evidence for saying a relationship is linear since the scale of the axes can make non-linear graphs look linear. It is important students connect that the equation is linear (of the form $y = mx + c$) or relate back to the relationship between the height and volume: that the volume is 25π, or about 78.5, times the height.

If students did not complete the optional activity, ask:

- "Which of your variables is the independent? The dependent?" (The height, h, is the independent variable, and volume, V, is dependent variable.)
- "Which variable is a function of which?" (Volume is a function of height.)
- "How can you tell that this is a linear relationship?"

See notes above on the last question in this list. If it has not been brought up in students' explanations, ask what the volume equation looks like when we double the height h. Display volume equation $V = 78.5(2h)$ for all to see. Ask how we can write this equation to show that the volume doubled when h doubled (using algebra, we can rewrite $V = 78.5(2h)$ as $V = 2(78.5h)$. Since the volume for h was $78.5h$, this shows that the volume for $2h$ is twice the volume for h).

Representing, Listening, Speaking: Compare and Connect. Invite students to create a visual representation that shows what is the same and what is different between this activity and the previous one. If students did not complete the optional activity, students can create multiple representations that illustrate why volume is a function of height. Give students time to do a gallery walk of the displays. Look for opportunities to highlight approaches that compare the multiple representations of the function of volume. Lead a whole-class discussion that connects features on the displays, such as comparing the equations and graphs. This will help students communicate about the multiple representations of volume functions while using mathematical language.
Design Principle(s): Optimise output; Maximise meta-awareness

17.4 Figuring Out Cone Dimensions

Optional: 10 minutes

This activity is optional. The purpose of this activity is for students to use representations of functions to explore more about the volume of a cone. This activity differs from the previous ones because students are given a graph that shows the relationship between the volume of a cone and the height of a cone. They use the graph to reason about the coordinates and their meaning. Students use the equation for the volume of a cone and coordinates from the graph to determine the radius of this cone and discuss the answer to this last question with their partners. Identify students to share their strategies during the discussion.

Instructional Routines

- Co-Craft Questions

Launch

Arrange students in groups of 2. Give students 2–3 minutes of quiet work time followed by time to discuss the answer and their strategies for the last question with their partner. Follow with a whole-class discussion.

Representation: Internalise Comprehension. Demonstrate and encourage students to use colour coding and annotations to highlight connections between representations in a problem. For example, ask students to use different colours to represent the height and volume of the cone in the graph. Ask students to use these same colours and a third colour for the radius when writing an equation.
Supports accessibility for: Visual-spatial processing Writing, Conversing: Co-Craft Questions. Display the graph and the context without revealing the questions that follow. Ask pairs of students to write possible questions that could be answered by the information contained in the graph. Then, invite pairs to share their questions with the class. Look for questions that ask students to interpret the relationship between the two quantities represented in the graph and create an equation to represent that relationship. Next, reveal the questions of the activity. This helps students produce the language of mathematical questions and talk further about the relationships between the height and volume of cones.
Design Principle(s): Maximise meta-awareness; Support sense-making

Anticipated Misconceptions

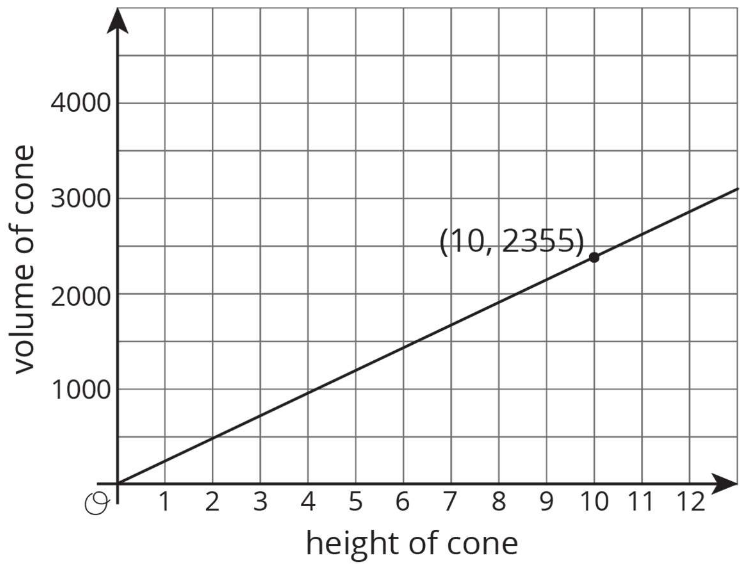
Students may round π at different points during their work, leading to slightly different answers.

Students can solve the equation $2\,355 = \frac{1}{3}\pi r^2 10$ with different series of steps. If they choose to first multiply the numbers on the side with r, they may round the expression $\frac{1}{3}\pi r^2 10$ to $10.47r^2$. This would give $224.9 = r^2$, making r slightly less than 15, though it rounds to 15. Solving by multiplying each side by $3 \times \frac{1}{\pi} \times \frac{1}{10}$ yields $r^2 = 225$, making r

equal 15 exactly (if we accept 3.14 as the value of π). This is an opportunity to discuss how rounding along the way in a solution can introduce imprecision.

Student Task Statement

Here is a graph of the relationship between the height and the volume of some cones that all have the same radius:



1. What do the coordinates of the labelled point represent?

2. What is the volume of the cone with height 5? With height 30?

3. Use the labelled point to find the radius of these cones. Use 3.14 as an approximation for π.

4. Write an equation that relates the volume V and height h.

Student Response

1. The volume of a particular cone with a height of 10 units.

2. The approximate volume of the cone with a height of 5 units is 1 177.5 cubic units. The approximate volume of the cone with a height of 30 units is 7 065 cubic units.

3. 15 units. Substituting the values of the labelled point $(10, 2\,355)$ into the volume equation, we know that $2\,355 = \frac{1}{3}\pi r^2 10$, which means $225 \approx r^2$.

4. $V = 75\pi h$

Activity Synthesis

Ask previously identified students to share their answers and their strategies for finding the radius in the last question.

To further the discussion, consider asking some of the following questions:

- "How did you deal with the $\frac{1}{3}$ in the equation?"
- "How did you deal with π when you were trying to solve for the radius?"
- "How do you know that the relationship between volume and height for these cones is a function? How is this shown in the graph? In the equation?"
- "Identify the independent and dependent variables in this relationship. If they were switched, would we still have a function? Explain how you know."

Lesson Synthesis

Tell students to imagine a cylindrical water tank with a radius r units and a height h units. Display the following prompts and ask students to respond to them in writing, encouraging them to include sketches:

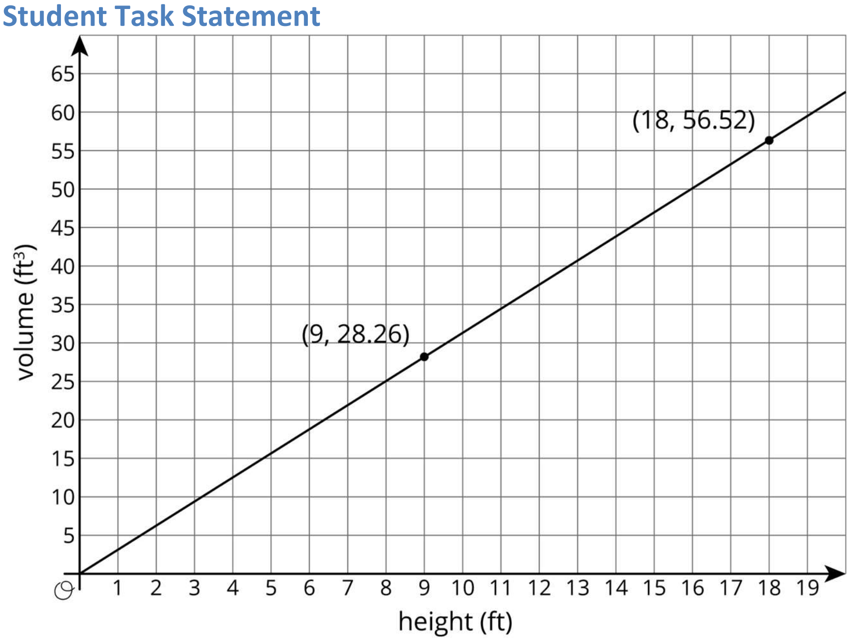
- "How can you change a dimension of the water tank so that the volume of the tank increases by a factor of 2?" (Increase the height by a factor of 2)
- "How can you change a dimension of the water tank so that the volume of the tank increases by a factor of a?" (Increase the height by a factor of a)

After quiet work time, invite students to share their responses. If students made a sketch, display them for all to see.

17.5 A Missing Radius

Cool Down: 5 minutes

Student Task Statement



Here is a graph of the relationship between the height and volume of some cylinders that all have the same radius, R. An equation that represents this relationship is $V = \pi R^2 h$ (use 3.14 as an approximation for π).

What is the radius of these cylinders?

Student Response

The radius is 1 foot. Substituting the information (9, 28.26) into the volume of a cylinder equation produces $28.26 = 3.14R^2 9$. This is equivalent to $1 = R^2$, which means $R = 1$ since $1^2 = 1$.

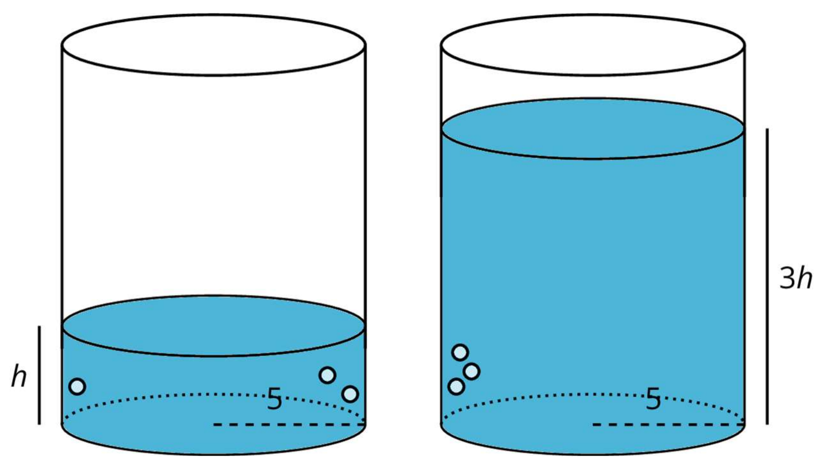
Student Lesson Summary

Imagine a cylinder with a radius of 5 cm that is being filled with water. As the height of the water increases, the volume of water increases.

We say that the volume of the water in the cylinder, V, depends on the height of the water h. We can represent this relationship with an equation: $V = \pi \times 5^2 h$ or just

$V = 25\pi h$

This equation represents a *proportional relationship* between the height and the volume. We can use this equation to understand how the volume changes when the height is tripled.



The new volume would be $V = 25\pi(3h) = 75\pi h$, which is precisely 3 times as much as the old volume of $25\pi h$. In general, when one quantity in a proportional relationship changes by a given factor, the other quantity changes by the same factor.

Remember that proportional relationships are examples of linear relationships, which can also be thought of as functions. So in this example V, the volume of water in the cylinder, is a function of the height h of the water.

Lesson 17 Practice Problems

Problem 1 Statement

A cylinder has a volume of 48π cm^3 and height h. Complete this table for volume of cylinders with the same radius but different heights.

height (cm)	volume (cm^3)
h	48π
$2h$	
$5h$	
$\frac{h}{2}$	
$\frac{h}{5}$	

Solution

height (cm)	volume (cm^3)
h	48π
$2h$	96π
$5h$	240π
$\frac{h}{2}$	24π
$\frac{h}{5}$	$\frac{48}{5}\pi$

Problem 2 Statement

A cylinder has a radius of 3 cm and a height of 5 cm.

a. What is the volume of the cylinder?

b. What is the volume of the cylinder when its height is tripled?

c. What is the volume of the cylinder when its height is halved?

Solution

a. 45π cm^3

b. 135π cm^3

c. $\frac{45}{2}\pi$ cm^3

Problem 3 Statement

A graduated cylinder that is 24 cm tall can hold 1 L of water. What is the radius of the cylinder? What is the height of the 500 ml mark? The 250 ml mark? Recall that 1 litre (L) is equal to 1000 millilitres (ml), and that 1 litre (L) is equal to 1 000 cm^3.

Solution

The radius of the cylinder is about 3.64 cm since $\frac{1000}{24\pi} \approx 13.26$ and $3.64^2 \approx 13.26$. The height of the 500 ml mark is 12 cm. The height of the 250 ml mark is 6 cm.

Problem 4 Statement

An ice cream shop offers two ice cream cones. The waffle cone holds 12 ounces and is 5 inches tall. The sugar cone also holds 12 ounces and is 8 inches tall. Which cone has a larger radius?

Solution

The waffle cone (Since its height is smaller, the radius must be larger in order to have the same volume as the sugar cone.)

Problem 5 Statement

A 6 oz paper cup is shaped like a cone with a diameter of 4 inches. How many ounces of water will a plastic cylindrical cup with a diameter of 4 inches hold if it is the same height as the paper cup?

Solution

18 oz (Since the cups are the same height and radius, the cylindrical cup must have 3 times the volume of the conical cup.)

Problem 6 Statement

Lin's smart phone was fully charged when she started school at 8:00 a.m. At 9:20 a.m., it was 90% charged, and at noon, it was 72% charged.

a. When do you think her battery will die?

b. Is battery life a function of time? If yes, is it a linear function? Explain your reasoning.

Solution

a. Answers vary. Sample response: Approximately 9:20 p.m. (Since 10 percent of battery was lost in 80 minutes, it would take 800 minutes to lose all of the battery. This would give a prediction of 13 hours and 20 minutes after the start time, so approximately 9:20 p.m.)

b. Battery remaining is a function of time but not a linear function. Explanations vary. Sample response: It took 80 minutes for her phone to lose the first 10 percent of the battery, and then then 160 minutes for her phone to lose another 18 percent. If the function were linear, it would lose exactly twice as much in 160 minutes as it did in 80 minutes. It might, however, be reasonably well-modelled by a linear function.