# Lesson 8: Comparing relationships with equations

## Goals

- Compare and contrast (orally) equations that do and do not represent proportional relationships.
- Generalise that an equation equivalent to the form $y = kx$ can represent a proportional relationship.
- Use a table to determine whether a given equation represents a proportional relationship, and justify (in writing) the decision.

## Learning Targets

- I can decide if a relationship represented by an equation is proportional or not.

## Lesson Narrative

This lesson continues the work students did in the previous lesson on comparing proportional and nonproportional relationships. The focus is on students seeing the connection between the form of the equation and the kind of relationship it represents. Students should see by the end of this lesson that equations of the form $y = kx$ characterise proportional relationships.

### Building On

- Generate a number or shape pattern that follows a given rule. Identify apparent features of the pattern that were not explicit in the rule itself. For example, given the rule "Add 3" and the starting number 1, generate terms in the resulting sequence and observe that the terms appear to alternate between odd and even numbers. Explain informally why the numbers will continue to alternate in this way.
- Write, read, and evaluate expressions in which letters stand for numbers.

### Addressing

- Solve real-world and mathematical problems involving area, volume and surface area of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms.
- Compute unit rates associated with ratios of fractions, including ratios of lengths, areas and other quantities measured in like or different units. For example, if a person walks $\frac{1}{2}$ mile in each $\frac{1}{4}$ hour, compute the unit rate as the complex fraction $\frac{1/2}{1/4}$ miles per hour, equivalently 2 miles per hour.
- Recognise and represent proportional relationships between quantities.

### Instructional Routines

- Discussion Supports
- Notice and Wonder
- Think Pair Share

### Required Materials

**Four-function calculators**
**Snap cubes**

### Required Preparation

Calculators can optionally be made available to take the focus off computation.

### Student Learning Goals

Let's develop methods for deciding if a relationship is proportional.

## 8.1 Notice and Wonder: Patterns with Rectangles

**Warm Up: 5 minutes**

The purpose of this task is to elicit ideas that will be useful in the discussions in this lesson. While students may notice and wonder many things about these images, the relationship between the side lengths, perimeter, and area are the important discussion points.

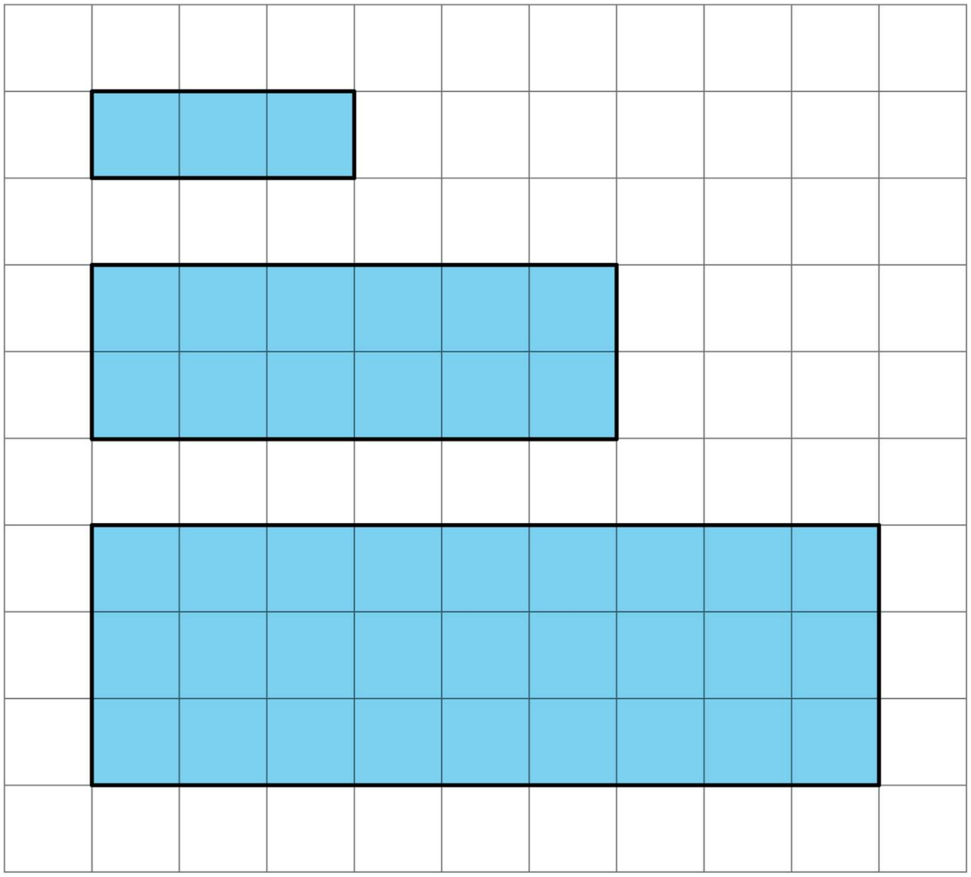
### Instructional Routines

- Notice and Wonder

### Launch

Arrange students in groups of 2. Tell students that they will look at a picture, and their job is to think of at least one thing they notice and at least one thing they wonder about the picture. Display the image for all to see. Ask students to give a signal when they have noticed or wondered about something and to think about the additional questions. Give students 1 minute of quiet think time, and then 1 minute to discuss the things they notice with their partner, followed by a whole-class discussion.

## Student Task Statement



Do you see a pattern? What predictions can you make about future rectangles in the set if your pattern continues?

## Student Response

Things (patterns) students may notice:

- The width increases by 3 each time.
- The height increases by 1 each time.

Things students may wonder:

- By how much does the area increase each time?
- By how much does the perimeter increase each time?
- Which rectangle will have a width of 10?

## Activity Synthesis

Invite students to share the things they noticed and wondered. Record and display the different ways of thinking for all to see. If possible, record the relevant reasoning on or near the images themselves. If the two questions below the image do not come up during the conversation, ask students to discuss them. After each response, ask the class if they agree or disagree and to explain alternative ways of thinking, referring back to what is happening in the images each time.

# 8.2 More Conversions

**15 minutes**

Students have already looked at measurement conversions that can be represented by proportional relationships, both earlier in KS3 and in previous lessons in this unit. This task introduces a measurement conversion that is not associated with a proportional relationship. The discussion should start to move students from determining whether a relationship is proportional by examining a table, to making the determination from the equation.

Note that some students may think of the two scales on a thermometer like a double number line diagram, leading them to believe that the relationship between degrees Celsius and degrees Fahrenheit is proportional. If not mentioned by students, the teacher should point out that when a double number line is used to represent a set of equivalent ratios, the tick marks for 0 on each line need to be aligned.

## Instructional Routines

- Discussion Supports
- Think Pair Share

## Launch

Arrange students in groups of 2. Provide access to calculators, if desired. Give students 5 minutes of quiet work time followed by students discussing responses with a partner, followed by whole-class discussion.

*Representation: Internalise Comprehension.* Activate or supply background knowledge. Allow students to use calculators to ensure inclusive participation in the activity.
*Supports accessibility for: Memory; Conceptual processing*

## Anticipated Misconceptions

Some students may struggle with the fraction $\frac{9}{5}$ in the temperature conversion. Teachers can prompt them to convert the fraction to its decimal form, 1.8, before trying to evaluate the equation for the values in the table.

### Student Task Statement

The other day you worked with converting metres, centimetres, and millimetres. Here are some more unit conversions.

1. Use the equation $F = \frac{9}{5}C + 32$, where $F$ represents degrees Fahrenheit and $C$ represents degrees Celsius, to complete the table.

| temperature (°C) | temperature (°F) |
|---|---|
| 20 | |
| 4 | |
| 175 | |

2. Use the equation $C = 2.54n$, where $C$ represents the length in centimetres and $n$ represents the length in inches, to complete the table.

| length (in) | length (cm) |
|---|---|
| 10 | |
| 8 | |
| $3\frac{1}{2}$ | |

3. Are these proportional relationships? Explain why or why not.

### Student Response

| temperature (°C) | temperature (°F) |
|---|---|
| 20 | 68 |
| 4 | 39.2 |
| 175 | 347 |
| length (in) | length (cm) |
| 10 | 25.4 |
| 8 | 20.32 |
| $3\frac{1}{2}$ | 8.89 |

1. The temperature conversion does not determine a proportional relationship because the number of degrees Fahrenheit per degree Celsius is not the same. The length conversion does determine a proportional relationship because the number of centimetres per inch is the same.

### Activity Synthesis

After discussing the work students did, ask, “What do you notice about the forms of the equations for each relationship?” After soliciting students’ observations, point out that the proportional relationship is of the form $y = kx$, while the nonproportional relationship is not.

*Speaking: Discussion Supports.* To provide support for students in producing statements when they compare and contrast equations with proportional and nonproportional relationships, provide sentence frames such as: “I noticed that _____,” “What makes _____ different from the others is _____,” or “The patterns show _____.” Invite students to use these sentence frames when students comprehend orally the patterns they notice with proportional equations.
*Design Principle(s): Support sense-making*

## 8.3 Total Edge Length, Surface Area, and Volume

**15 minutes**

This activity builds on the perimeter and area activity from the warm-up. Its goal is to use a context familiar from earlier in KS3 to compare proportional and nonproportional relationships. The units for the quantities are purposely not given in the task statement to avoid giving away which relationships are not proportional. However, discussion should raise the possible units of measurement for edge length, surface area, and volume.

The focus of this task is whether or not relationships between the quantities are proportional, but there is also an opportunity for students to reinforce their understanding of geometry. Students should not spend too much time figuring out the surface area and volume. Watch carefully as students work and be ready to provide guidance or equations as needed, so students can get to the central purpose of the task, which is noticing the correspondences between the nature of relationships and the form of their equations. Strategic pairing of students and having snap cubes on hand can help struggling students complete the tables and determine equations.

### Instructional Routines

- Think Pair Share

### Launch

Arrange students in groups of 2. Display a cube (e.g. cardboard box) for all to see and ask:

- “How many edges are there?”
- “How long is one edge?”
- “How many faces are there?”
- “How large is one face?”

Give students 5 minutes of quiet work time followed by students discussing responses with a partner, followed by whole-class discussion.

*Action and Expression: Internalise Executive Functions.* Chunk this task into more manageable parts to support students who benefit from support with organisational skills in problem solving. For example, present one question at a time and monitor students to ensure they are making progress throughout the activity.
*Supports accessibility for: Organisation; Attention*

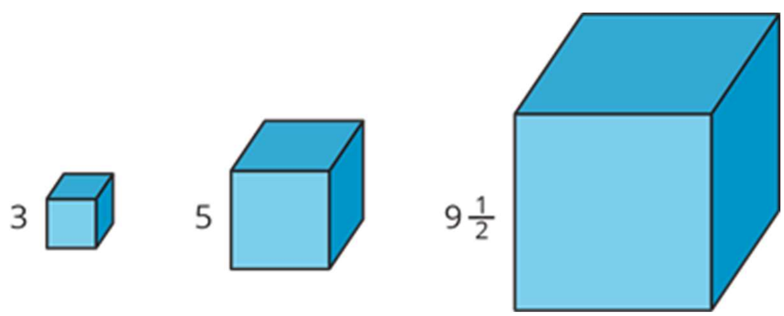
### Anticipated Misconceptions

Some students may struggle to complete the tables. Teachers can use nets of cubes (flat or assembled) partitioned into square units to reinforce the process for finding total edge length, surface area, and volume of the cubes in the task. Snap cubes would also be appropriate supports.

If difficulties with the fractional side length $9\frac{1}{2}$ keep students from being able to find the surface area and volume or write the equations, the teacher can tell those students to replace $9\frac{1}{2}$ with 10 and retry their calculations. Their answers for surface area and volume will be different for that row in the table, but their equations and proportionality decisions will be the same. That way they can still learn the connection between the form of the equations and the nature of the relationships.

### Student Task Statement

Here are some cubes with different side lengths. Complete each table. Be prepared to explain your reasoning.




1. How long is the total edge length of each cube?

| side length | total edge length |
|---|---|
| 3 | |
| 5 | |

| | |
|---|---|
| $9\frac{1}{2}$ | |
| $s$ | |

2. What is the surface area of each cube?

| side length | surface area |
|---|---|
| 3 | |
| 5 | |
| $9\frac{1}{2}$ | |
| $s$ | |

3. What is the volume of each cube?

| side length | volume |
|---|---|
| 3 | |
| 5 | |
| $9\frac{1}{2}$ | |
| $s$ | |

4. Which of these relationships is proportional? Explain how you know.

5. Write equations for the total edge length $E$, total surface area $A$, and volume $V$ of a cube with side length $s$.

### Student Response

1. A cube has 12 edges. Students may construct the third column below to reach this conclusion.

| side length | total edge length | $\frac{\text{total edge length}}{\text{side length}}$ |
|---|---|---|
| 3 | 36 | 12 |
| 5 | 60 | 12 |
| $9\frac{1}{2}$ | 114 | 12 |
| $s$ | $12s$ | 12 |

2. A cube has 6 faces each with an area of $s^2$ square units. Students may construct the third column below to reach this conclusion.

| side length | surface area | $\frac{\text{surface area}}{\text{side length}}$ |
|---|---|---|
| 3 | 54 | 18 |
| 5 | 150 | 30 |
| $9\frac{1}{2}$ | $541\frac{1}{2}$ | 57 |
| $s$ | $6s^2$ | $6s$ |

3. The bottom layer of a cube fits $s^2$ cubic units and s of these layers make up the cube. Students may construct the third column below to reach this conclusion.

| side length | volume | $\frac{\text{volume}}{\text{side length}}$ |
|---|---|---|
| 3 | 27 | 9 |
| 5 | 125 | 25 |
| $9\frac{1}{2}$ | $857\frac{3}{8}$ | $90\frac{1}{4}$ |
| $s$ | $s^3$ | $s^2$ |

4. The relationship between side length and total edge length is proportional because the ratios in the third column is 12 for every side length. The relationships between side length and surface area, and between side length and volume are not proportional because the ratios in the third column of tables 2 and 3 are not the same for each side length.

5. $E = 12s, A = 6s^2, V = s^3$

## Are You Ready for More?

1. A rectangular solid has a square base with side length $\ell$, height 8, and volume $V$. Is the relationship between $\ell$ and $V$ a proportional relationship?

2. A different rectangular solid has length $\ell$, width 10, height 5, and volume $V$. Is the relationship between $\ell$ and $V$ a proportional relationship?

3. Why is the relationship between the side length and the volume proportional in one situation and not the other?

## Student Response

1. no

2. yes

3. In one situation, there are two unknown dimensions and in the other only one. So even though these situations look very similar, the relationship is different.

### Activity Synthesis

Ask, “What do you notice about the equation for the relationship that is proportional?” Make sure students see that it is of the form $y = kx$, and the others are not. This realisation is the central purpose of this task.

Ask students:

- "What could be possible units for the side lengths?" (linear measurements: centimetres, inches)
- "Then what would be the units for the surface area?" (square units: square centimetres, square inches)
- "What would be the units for the volume?" (cubic units: cubic centimetres, cubic inches)

Connect the units of measurements with the structure of the equation for each quantity: The side length and the units are raised to the same power.

## 8.4 All Kinds of Equations

**Optional: 10 minutes**

This activity involves checking for a constant of proportionality in tables generated from simple equations that students should already be able to evaluate from their work with expressions and equations from earlier in KS3. The purpose of this activity is to generalise about the forms of equations that do and do not represent proportional relationships. The relationships in this activity are presented without a context so that students can focus on the structure of the equations without being distracted by what the variables represent.

### Instructional Routines

- Discussion Supports
- Think Pair Share

### Launch

Arrange students in groups of 2. Provide access to calculators. Give students 5 minutes of quiet work time followed by students discussing responses with a partner, followed by whole-class discussion.

*Engagement: Internalise Self Regulation.* Chunk this task into more manageable parts to differentiate the degree of difficulty or complexity. Invite students to choose and complete tables for four of the equations. Encourage students to complete tables for the equations $y = \frac{x}{4}$ and $y = 4x$.
*Supports accessibility for: Organisation; Attention*

## Anticipated Misconceptions

Students might struggle to see that the two proportional relationships have equations of the form $y = kx$ and to characterise the others as not having equations of that form. Students do not need to completely articulate this insight for themselves; this synthesis should emerge in the whole-class discussion.

## Student Task Statement

Here are six different equations.

$y = 4 + x$

$y = \frac{x}{4}$

$y = 4x$

$y = 4^x$

$y = \frac{4}{x}$

$y = x^4$

$y = 4 + x$

| $x$ | $y$ | $\frac{y}{x}$ |
|---|---|---|
| 2 | | |
| 3 | | |
| 4 | | |
| 5 | | |

$y = 4x$

| $x$ | $y$ | $\frac{y}{x}$ |
|---|---|---|
| 2 | | |
| 3 | | |
| 4 | | |
| 5 | | |

$y = \frac{4}{x}$

| $x$ | $y$ | $\frac{y}{x}$ |
|---|---|---|
| 2 | | |
| 3 | | |
| 4 | | |
| 5 | | |

$y = \frac{x}{4}$

| $x$ | $y$ | $\frac{y}{x}$ |
|---|---|---|
| 2 | | |
| 3 | | |
| 4 | | |
| 5 | | |

$y = 4^x$

| $x$ | $y$ | $\frac{y}{x}$ |
|---|---|---|
| 2 | | |
| 3 | | |
| 4 | | |
| 5 | | |

$y = x^4$

| $x$ | $y$ | $\frac{y}{x}$ |
|---|---|---|
| 2 | | |
| 3 | | |
| 4 | | |
| 5 | | |

1. Predict which of these equations represent a proportional relationship.
2. Complete each table using the equation that represents the relationship.

3. Do these results change your answer to the first question? Explain your reasoning.

4. What do the equations of the proportional relationships have in common?

**Student Response**

1. Answers vary. Possible response: $y = 4x$ and $y = \frac{x}{4}$ represent proportional relationships, but the others do not.

2.

$y = 4 + x$

| $x$ | $y$ | $\frac{y}{x}$ |
|---|---|---|
| 2 | 6 | 3 |
| 3 | 7 | $2\frac{1}{3}$ |
| 4 | 8 | 2 |
| 5 | 9 | $1\frac{4}{5}$ |

$y = 4x$

| $x$ | $y$ | $\frac{y}{x}$ |
|---|---|---|
| 2 | 8 | 4 |
| 3 | 12 | 4 |
| 4 | 16 | 4 |
| 5 | 20 | 4 |

$y = \frac{4}{x}$

| $x$ | $y$ | $\frac{y}{x}$ |
|---|---|---|
| 2 | 2 | 1 |
| 3 | $\frac{4}{3}$ | $\frac{4}{9}$ |
| 4 | 1 | $\frac{1}{4}$ |
| 5 | $\frac{4}{5}$ | $\frac{4}{25}$ |

$y = \frac{x}{4}$

| $x$ | $y$ | $\frac{y}{x}$ |
|---|---|---|
| 2 | $\frac{1}{2}$ | $\frac{1}{4}$ |

| | | |
|---|---|---|
| 3 | $\frac{3}{4}$ | $\frac{1}{4}$ |
| 4 | 1 | $\frac{1}{4}$ |
| 5 | $\frac{5}{4}$ | $\frac{1}{4}$ |

$y = 4^x$

| $x$ | $y$ | $\frac{y}{x}$ |
|---|---|---|
| 2 | 16 | 8 |
| 3 | 64 | $21\frac{1}{3}$ |
| 4 | 256 | 64 |
| 5 | 1 024 | $204\frac{4}{5}$ |

$y = x^4$

| $x$ | $y$ | $\frac{y}{x}$ |
|---|---|---|
| 2 | 16 | 8 |
| 3 | 81 | 27 |
| 4 | 256 | 64 |
| 5 | 625 | 125 |

3. Answers vary. Possible response: No, they just confirm that $y = 4x$ and $y = \frac{x}{4}$ represent proportional relationships, but the others do not.

4. Answers vary. Possible responses: The equations representing proportional relationships:

   - can be written in the form $y = kx$
   - do not contain any exponents or addition operations
   - do not involve dividing by $x$

### Activity Synthesis

Invite students to share what the equations for proportional relationships have in common, and by contrast, what is different about the other equations. At first glance, the equation $y = \frac{x}{4}$ does not look like our standard equation for a proportional relationship, $y = kx$. Suggest to students that they rewrite the equation using the constant of proportionality they found after completing the table: $y = 0.25x$ which can also be expressed $y = \frac{1}{4}x$. If students do not express this idea themselves, remind them that they can think of dividing by 4 as multiplying by $\frac{1}{4}$.

*Reading, Writing, Speaking: Clarify, Critique,* Correct. Before students share what they identified as similarities between the proportional and nonproportional relationships, present an incorrect explanation. For example, "There are no commonalities among the equations because they're all different. " Ask students to identify the error, critique the reasoning, and write a correct description. As students discuss with a partner, listen for students who clarify that equations $y = 4x$ and $y = \frac{1}{4}x$ are of the form $y = kx$ and both represent proportional relationships. Invite students to share their critiques and corrected responses with the class. Listen for and amplify the language students use to describe the similarities and differences between equations that do and do not represent proportional relationships. This helps students evaluate, and improve upon, the written mathematical arguments of others, as they clarify their understanding of proportional relationships and how they can be represented with equations.
*Design Principle(s): Maximise meta-awareness*

## Lesson Synthesis

Review the findings from the activities and make explicit the fact that proportional relationships are characterised by equations of the form $y = kx$. Be sure to point out that this includes equations with other variables. For example:

$y = 5.2x$ $\quad d = 58t$ $\quad a = 0.12B$ $\quad W = 205n$

This form characterises proportional relationships due to a property we examined in previous lessons: if a table represents a proportional relationship between $x$ and $y$, then the unit rates $\frac{y}{x}$ are always the same.

| $x$ | $y$ | $\frac{y}{x}$ |
|---|---|---|
| 3 | $3k$ | $k$ |
| 5 | $5k$ | $k$ |
| 400 | $400k$ | $k$ |

If $\frac{y}{x} = k$, then $y = kx$ (as long as $x \neq 0$, but you don't need to mention this now unless a student brings it up.)

## 8.5 Tables and Chairs

**Cool Down: 5 minutes**

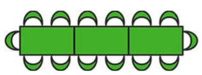
## Student Task Statement

Andre is setting up rectangular tables for a party. He can fit 6 chairs around a single table. Andre lines up 10 tables end-to-end and tries to fit 60 chairs around them, but he is surprised when he cannot fit them all.

1. Write an equation for the relationship between the number of chairs $c$ and the number of tables $t$ when:

   - the tables are apart from each other:

   - the tables are placed end-to-end:



2. Is the first relationship proportional? Explain how you know.
3. Is the second relationship proportional? Explain how you know.

## Student Response

1. When the tables are apart: $c = 6t$ (or $t = \frac{1}{6}c$). When the tables are together: $c = 4t + 2$ (or $t = \frac{1}{4}c - \frac{1}{2}$).
2. This relationship is proportional. Possible reasons:
   - It can be represented with an equation of the form $c = kt$ (or $t = kc$).
   - There are 6 chairs per table no matter how many tables.
3. This relationship is not proportional. Possible reasons:
   - The number of chairs per table changes depending on how many tables there are.
   - The quotient of chairs and tables is not constant.
   - The relationship cannot be expressed with an equation of the form $c = kt$.

As shown below, the number of chairs per table is the same when the tables are apart, but it is not the same if the tables are pushed together.

With tables apart

| tables | chairs | $\frac{\text{chairs}}{\text{tables}}$ |
|---|---|---|
| 1 | 6 | 6 |

| | | |
|---|---|---|
| 2 | 12 | 6 |
| 3 | 18 | 6 |
| 4 | 24 | 6 |
| 10 | 60 | 6 |
| $t$ | $6t$ | 6 |

With tables end-to-end

| tables | chairs | $\frac{\text{chairs}}{\text{tables}}$ |
|---|---|---|
| 1 | 6 | 6 |
| 2 | 10 | 5 |
| 3 | 14 | 4.667 |
| 4 | 18 | 4.5 |
| 10 | 42 | 4.2 |
| $t$ | $4t + 2$ | |

## Student Lesson Summary

If two quantities are in a proportional relationship, then their quotient is always the same. This table represents different values of $a$ and $b$, two quantities that are in a proportional relationship.

| $a$ | $b$ | $\frac{b}{a}$ |
|---|---|---|
| 20 | 100 | 5 |
| 3 | 15 | 5 |
| 11 | 55 | 5 |
| 1 | 5 | 5 |

Notice that the quotient of $b$ and $a$ is always 5. To write this as an equation, we could say $\frac{b}{a} = 5$. If this is true, then $b = 5a$. (This doesn't work if $a = 0$, but it works otherwise.)

If quantity $y$ is proportional to quantity $x$, we will always see this pattern: $\frac{y}{x}$ will always have the same value. This value is the constant of proportionality, which we often refer to as $k$. We can represent this relationship with the equation $\frac{y}{x} = k$ (as long as $x$ is not 0) or $y = kx$.

Note that if an equation cannot be written in this form, then it does not represent a proportional relationship.

## Lesson 8 Practice Problems

1. **Problem 1 Statement**

The relationship between a distance in yards ($y$) and the same distance in miles ($m$) is described by the equation $y = 1\,760\ m$

a. Find measurements in yards and miles for distances by completing the table.

| distance measured in miles | distance measured in yards |
|---|---|
| 1 | |
| 5 | |
| | 3 520 |
| | 17 600 |

b. Is there a proportional relationship between a measurement in yards and a measurement in miles for the same distance? Explain why or why not.

**Solution**

| distance measured in miles | distance measured in yards |
|---|---|
| 1 | 1 760 |
| 5 | 8 800 |
| 2 | 3 520 |
| 10 | 17 600 |

a. There is a proportional relationship. The constant of proportionality is 1 760 yards per mile.

2. **Problem 2 Statement**

Decide whether or not each equation represents a proportional relationship.

a. The remaining length ($L$) of 120-inch rope after $x$ inches have been cut off: $120 - x = L$

b. The total cost ($t$) after 8% VAT is added to an item's price ($p$): $1.08p = t$

c. The number of marbles each sister gets ($x$) when $m$ marbles are shared equally among four sisters: $x = \frac{m}{4}$

d. The volume ($V$) of a rectangular prism whose height is 12 cm and base is a square with side lengths $s$ cm: $V = 12s^2$

**Solution**

a. no

b. yes

c. yes

d. no

3. **Problem 3 Statement**

a. Use the equation $y = \frac{5}{2}x$ to complete the table.

Is $y$ proportional to $x$ and $y$? Explain why or why not.

| $x$ | $y$ |
|---|---|
| 2 | |
| 3 | |
| 6 | |

b. Use the equation $y = 3.2x + 5$ to complete the table.

Is $y$ proportional to $x$ and $y$? Explain why or why not.

| $x$ | $y$ |
|---|---|
| 1 | |
| 2 | |
| 4 | |

**Solution**

| $x$ | $y$ |
|---|---|
| 2 | 5 |
| 3 | $\frac{15}{2}$ |
| 6 | 15 |

Yes, there is a proportional relationship between $x$ and $y$ since $\frac{y}{x} = \frac{5}{2}$ in each row.

| $x$ | $y$ |
|---|---|
| 1 | 8.2 |
| 2 | 11.4 |
| 4 | 17.8 |

No, there is no proportional relationship between $x$ and $y$. In the first row $\frac{y}{x} = 8.2$ but in the second row $\frac{y}{x} = 5.7$.

4. **Problem 4 Statement**

To transmit information on the internet, large files are broken into packets of smaller sizes. Each packet has 1 500 bytes of information. An equation relating packets to bytes of information is given by $b = 1\,500p$ where $p$ represents the number of packets and $b$ represents the number of bytes of information.

a. How many packets would be needed to transmit 30 000 bytes of information?

b. How much information could be transmitted in 30 000 packets?

c. Each byte contains 8 bits of information. Write an equation to represent the relationship between the number of packets and the number of bits.

**Solution**

a. 20 packets

b. 45 000 000 bytes

c. $x = 12\,000p$